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# A Unified Dynamic Equation of The Classical Field in Local Manifold

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## Abstract

The fluid dynamics, gravitational field dynamics, and electromagnetic field dynamics can be expressed as a unified field equation in a local inertial frame by $D_t\vec{p} = \nabla\mathcal{L}$, where $\vec{p}$ is the momentum vector, and $\mathcal{L}$ is the Lagrangian's density. In a time-freezing configuration (static state), the stored energy density and the mass density in the instantaneous configuration have the relation of $p = \rho c^2$. It can be positive (potential energy) or negative (binding energy), depending on the zero potential energy definition point in the field. Its sign only affects the chirality. Given a slight motion in a local inertial frame, the momentum vector field and potential energy field are combined into a single physical field — a 4-momentum vector field. In general, for a many-particle system, the interactions between particles obey the weak law of action and reaction. The action and interaction forces can be decomposed into two components: one is along the jointing line to consider the linear momentum, and another one is perpendicular to the jointing line to consider the rotational motion. It is suggested that the fluid dynamics equations should include an extra term, a Coriolis-like force term, to consider the spin (or rotational) effect (because of the vorticity field). Electromagnetic fields have no rest frame; they have an intrinsic 4-momentum vector relative to a rest observer. With the 4-momentum vector the Maxwell equations can be deduced. In a "vacuum", each of the electric field and magnetic field contribute half to the total energy. It implies that the linear and rotational kinetic energy equals each other. The Gauss law thinks of an electrical dipole as a "vacuum" space; this implies that photon gas is composed of a mixture of electrical dipoles. Their trajectories will be helical or spiral, as is shown by the circular polarized Electromagnetic waves.

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# 1. Introduction

Classical field theories provide a foundation for understanding and modeling physical phenomena in many areas of physics [1-5]. In classical field theories, fields are mathematical functions that assign a value or intensity (e.g., a scalar field, vector field, or tensor field) to every point in space. For studying the field dynamics, these values are also a function of time. These field values are continuous (maybe differentiable) and defined to spread throughout space and time. the dynamic behavior of these physical phenomena is predicted by field equations, which can be used to describe wave-like and particle-like physical phenomena such as sound (based on the dynamic behavior of the pressure field) and light (based on the electric and magnetic field), or other continuous phenomena such as fluid dynamics. It describes the motion and behavior of fluids using a vector field for velocity and a scalar field for pressure (energy density), and other fluid properties [6].

# 2. Stored potential energy density in the Configuration of a field

Fields in physics can store energy. In general, we have a scalar field of potential energy density that can be specified everywhere in space as a function of position (for the dynamics or time evolution of the field, the potential energy density in the field will depend on time).

In an instantaneous configuration (a snapshot of the field), that means we 'freeze' the time, and no motion occurs. In the language of the relative theory, there is no relative motion with respect to the observer – it is called the co-moving frame.

We consider the first law of thermodynamics for a closed system [7, p.409] in the co-moving frame. An infinitesimal change of the internal energy of the system is:

$$dE = dQ + dW, \tag{1}$$

where $dW$ is work done by the surroundings (external forces) on this system.

$$dW = d(pV). \tag{2}$$

In an elastic fluid, recalling the definition of the speed of sound, it depends on the bulk modulus and density [8]:

$$c^2 = \frac{B}{\rho}. \tag{3}$$

The elastic bulk modulus B can be formally defined by the equation

$$B = -\frac{dp}{\left(\frac{dV}{V}\right)}. \tag{4}$$

Substituting Eq. (3) into Eq. (4), we have

$$-\frac{dp}{\left(\frac{dV}{V}\right)} = \rho c^2. \tag{5}$$

Recalling the mass density definition and re-arranging it, we have

$$dp = -\frac{mc^2}{V^2} dV. \tag{6}$$

Integral from a reference point to the present configuration $(p, V)$:

$$p - p_{ref} = mc^2 \left(\frac{1}{V} - \frac{1}{V_{ref}}\right). \tag{7}$$

We can define an infinitely dilute state as the reference point:

$$reference\ point: \begin{cases} p_{ref} \to 0 \\ V_{ref} \to \infty \end{cases}. \tag{8}$$

Equation (7) thus becomes (volume is compressed from v=∞ to present configuration of V):

$$p = \frac{mc^2}{V} = \rho c^2. \tag{9}$$

We get the equation of the configuration energy density, p, and the mass density, $\rho$.

If there is no other energy exchange, such as heat exchange, between the system and its environment, or it can be ignored, $dQ = 0$, we can rearrange the equation (9) and substitute into eq. (1), we have following expression:

$$E = pV = mc^2. \tag{10}$$

It looks like the mass-energy equivalence. When they are stationary, that is how much net work it takes (work done by external force through a compression process) from the reference point to the present configuration. It is also the amount of net work that we will get back, if we disassemble the present configuration to the reference point through an expansion process.

It contains a square of wave speed. That means the stored energy density in the present configuration (p and V) represents the “wave energy content” in the co-moving frame. When the constraints (imaginary) of this configuration are removed, the disturbance of the field will propagate by a wave through the whole field. Energy will be transported by the wave from one position to another in a wave speed of c.

This stored energy density in the field is called potential energy. In general, it is positive, but potential energy may also be negative, dependent on the zero energy position, the point where the potential energy is assigned to be zero (no any interaction between the researched particle and other particles in the field). We can also call the negative configuration energy as “binding energy”. It refers to the amount of a negative energy needed to disassemble the present configuration into its individual components, where there is no force interaction between individual components.

In a static state (time is “freezing”), this is a conservative energy density; the force density in the field can be expressed as a negative gradient of the potential energy density.

$$\vec{\mathcal{F}} = -\nabla p = -c^2 \nabla \rho, \tag{11}$$

and

$$\begin{cases} -\nabla \times \vec{\mathcal{F}} = \nabla \times (\nabla p) = 0, \\ -\nabla \cdot \vec{\mathcal{F}} = \nabla^2 p = c^2 \nabla^2 \rho. \end{cases} \tag{12}$$

The above equations hold only for a static state — no motion in the field, if there is a relative motion between particles, there exists an extra velocity (momentum) vector field.  Eq. (11) and (12) do not hold anymore. Thus, modifications are needed to reflect the motion effect.

## 3. Dynamic equation of a field using 4 potential

For simplicity, we consider here an isolated field — there exists only one field in the space, thus there are no interaction terms between different fields.

Let it slightly move, the configuration will change. The potential energy product a force to drive the particle to move (exactly to say, the negative gradient of the potential energy is a force). In this case, besides the scalar field of the potential energy, we specify another momentum vector field in space as a function of position and time.

For a system containing a group of particles, in the general case, the internal interaction forces in the field between two particles are equal and opposite, but do not necessarily act along the line joining each other [9,10, 11].

As shown in Fig. 1 (a). We can decompose the internal interaction forces vector as a component of $\vec{\mathcal{F}}_{\parallel}$, along the jointing line (the central force), and a component of $\vec{\mathcal{F}}_{\perp}$, perpendicular to the jointing line (the tangent force). In a stationary state, gravitational and electric forces (negative gradient of the potential energy) are central forces (no relative motion occurs). The tangent forces between the particles may be due to their relative motions and it will affect the angular momentum of the system. That is the angular momentum of a system about the center of mass. In the local manifold, the velocity is an element of tangent space, while the angular velocity is in the normal direction of the tangent space, as shown in Fig. 1(b).

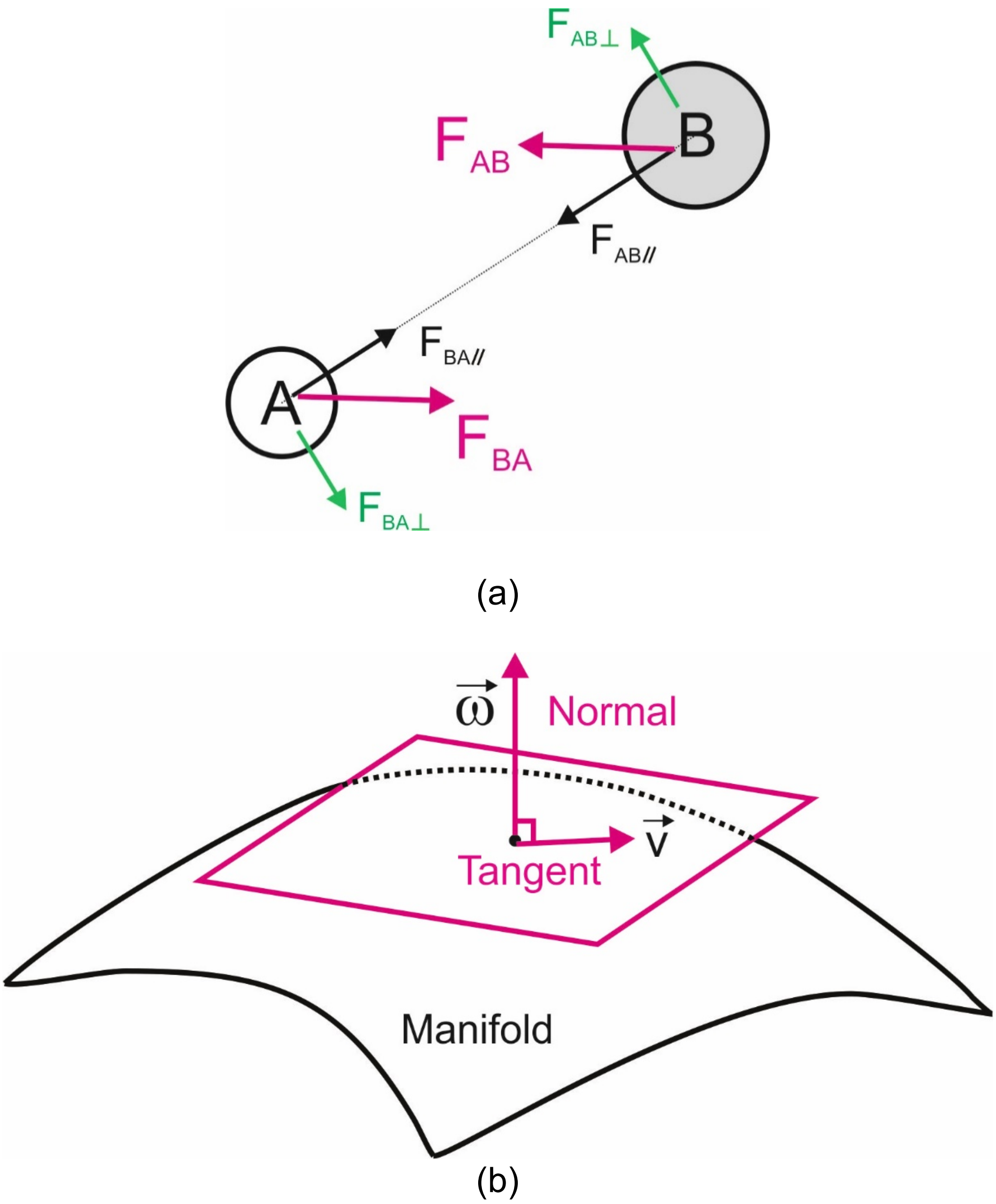


Fig.1(a). The weak law of action and reaction between particles; Fig.1(b), the velocity field is in the tangent space in the local manifold, the vorticity field is the normal direction of the tangent space.

Accordingly, in the space we have now a scalar field and a vector field (or rather to say, a momentum field). It can be assembled as a single physical entity, a 4-momentum flux density per unit volume:

$$A^{\alpha} = \left(\frac{p}{c};\ \rho\vec{v}\right). \tag{13}$$

Its physical unit is $\left[\frac{N\cdot s}{m^3}\right]$ .

In the local manifold, we have the following force density per unit volume:

$$\vec{\mathcal{F}} = \vec{\mathcal{F}}_{\parallel} + \vec{\mathcal{F}}_{\perp}. \tag{14}$$

The central force component points in the same direction as the displacement between the interactions of particles. As mentioned before, because of the extra momentum field, eq. (11) does not hold anymore, it can be modified as:

$$\vec{\mathcal{F}}_{\parallel} = -\nabla p - \frac{\partial \overrightarrow{\rho v}}{\partial t}. \tag{15}$$

The central force components do not affect the total angular momentum of the system. The curl of velocity (momentum) will produce a vorticity field, thus, the tangent component of the interaction force density is expressed as:

$$\vec{\mathcal{F}}_{\perp} = \vec{v} \times (\nabla \times \overrightarrow{\rho v}). \tag{16}$$

We can define a rotational vector field density, similar to the vorticity field:

$$\vec{\omega} = \nabla \times \overrightarrow{\rho v}. \tag{17}$$

With these definitions, then we can get the field dynamic equation for every point in field space:

$$-\nabla p - \frac{\partial \overrightarrow{\rho v}}{\partial t} + \vec{v} \times \vec{\omega} = 0. \tag{18}$$

With the help of the vector identity of

$$\vec{v} \times (\nabla \times \overrightarrow{\rho v}) = \frac{1}{2}\nabla(\vec{v} \cdot \overrightarrow{\rho v}) - (\vec{v} \cdot \nabla)(\overrightarrow{\rho v}), \tag{19}$$

equation (18) can be rewritten as:

$$-\nabla p - \frac{\partial \overrightarrow{\rho v}}{\partial t} + \frac{1}{2}\nabla(\vec{v} \cdot \overrightarrow{\rho v}) - (\vec{v} \cdot \nabla)(\overrightarrow{\rho v}) = 0. \tag{20}$$

Re-arranging the terms:

$$\frac{\partial \overrightarrow{\rho v}}{\partial t} + (\vec{v} \cdot \nabla)(\overrightarrow{\rho v}) = -\nabla p + \frac{1}{2}\nabla(\vec{v} \cdot \overrightarrow{\rho v}). \tag{21}$$

As mentioned by Goldstein et. al., [3, p. 20.], the last term of the right hand side arises from the curvature of the local manifold.

The left hade side of the equation is the total derivate of momentum with respect to time. The right hand side is the Lagrangian's density.

$$\frac{d\overrightarrow{\rho v}}{dt} = \nabla(T - p). \tag{22}$$

It can be written more concisely:

$$D_t \vec{p} = \nabla \mathcal{L}. \tag{23}$$

The above field equation of (23) is more general; actually, no assumptions and restrictions have been made for the derivation of this field equation. If we check it in a deeper sense, it is similar to the general motion equation in classical mechanics for a system with finite degree of freedom, based on the D'Alembert's principle, [3, p.20].

Its physical interpretation is clear, force can be defined either in the time domain or in the spatial domain: if the space is "freezing", force equals the derivative of momentum with respect to time; if the time is "freezing", it will equal the derivative of energy with respect to spatial coordinates. Both definitions should be equivalent to each other. In other words, the space and time is treated on an equal footing.

If above equations are written out explicitly, it will be very lengthy. It is arranged as Appendix A in this work; here we give out the final vector form:

$$\frac{\partial \overrightarrow{\rho \vec{v}}}{\partial t} + (\vec{v} \cdot \nabla)(\overrightarrow{\rho \vec{v}}) = -\nabla p + \frac{1}{2}\bar{\bar{S}}\vec{v} + \frac{1}{2}\vec{v} \times \vec{\omega}. \tag{24}$$

The last term represents a rotation motion; it implies the trajectory of the particle motion in general exhibits helical motion.

If the potential energy density is explicitly given out, combining the relation of eq. (9), we can solve the dynamic behavior of this field. In the following, we consider two typical classical "free" fields, without considering the interaction between different fields.

## 4. Gravitational field: attractive force

In the above derivation of the field equation, we assume the potential energy is positive, as indicated by eq. (9), the zero point is at infinity. Through compression process, we form a positive equation of state of eq. (9). This compression process is similar to a procedure to push electrical charges with equal signs together. The external force must do work to form the present configuration, and the stored energy in the configuration is positive.

In a gravitational field, the stored configuration energy in the field is negative potential energy, namely the external force must do a negative net work to form the present configuration. In other words, from a finite volume of V, the present configuration is pulled away to the infinity of V=∞, where the zero point is defined; the external force must do a negative net work. This process is similar to the process pulling electrical charges with the opposite sign away, as shown in Fig. 2.

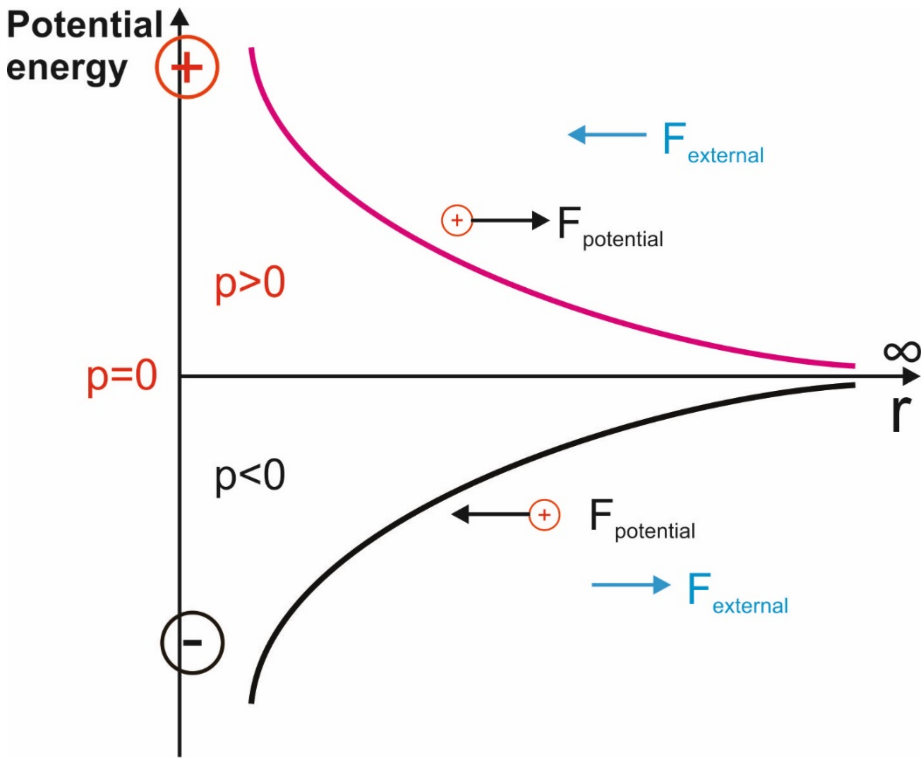


Fig. 2 The configuration-stored energy can be positive or negative.

The Einstein energy-momentum tensor for a perfect fluid is [12, p.140]

$$T^{\mu\nu} = \left(\rho_G + \frac{p_G}{c^2}\right) U^\mu U^\nu - p_G g^{\mu\nu}. \tag{25}$$

The time–time component is the relativistic energy density.

The fluid in the local inertial frame are at rest, the metric tensor degrades to the Minkowski metric tensor for a flat space.

$$g^{\mu\nu} = \eta^{\mu\nu} = (1, -1, -1, -1). \tag{26}$$

The fluid is at rest, the 4-velocity tensor becomes to

$$U^{\mu} = \lim_{\vec{v} \to 0} (U^{\mu}) = (c, 0,0,0). \tag{27}$$

The time-time component of the Einstein energy-momentum tensor in the co-moving frame is just the static energy, this component of the equation (25) becomes

$$T^{00} = -p_G = \rho_G c^2. \tag{28}$$

It has a direct physical interpretation. In the co-moving frame, it is just the stored configuration energy at rest. In case of a perfect fluid this component is expressed as

$$\rho_G = -\frac{p_G}{c^2}. \tag{29}$$

The physical meaning is clear. In the gravitational field, the particles attract each other, just like the interaction between electrical charges with opposite signs, interactions between particles have only attraction forces, (the zero point of potential energy is still defined at x=∞).

Now, let it slightly move, there exists a relative motion with respect to the local observer, in Language of General Relativity, Let us assume that the space is "slightly curved". Then we have 4-vector:

$$A^{\alpha} = \left(\frac{p_G}{c}; \overrightarrow{p_G v}\right). \qquad (30)$$

In the local inertial frame, the relative effect is ignored, namely we assume v/c<<1.

Substituting this 4-momentum vector, combined with equation of (29), into Eq. (24), we can get the dynamic behavior of the gravitational field, observed in the local inertial frame. The sole requirement is to assume the gravitational field wave speed is equal to the electromagnetic wave propagation speed of c, or the gravitational field has its own wave speed.

Comparison of eq. (9) and (29), difference is only the negative or positive sign between the potential energy density and mass density. The positive and negative sign of the potential energy density only affect the rotation direction; it will be shown in section 6 of this work.

## 5. Electromagnetic field — no rest frame

Both sections 3 and 4 have a rest frame (co-moving frame). In the co-moving frame, adding an extra momentum field up to the potential energy density, we can get the dynamic equation of the field, utilizing this 4-momentum vector.

While electromagnetic fields have no rest frame, (to say, relative to Lab frame). A moving charged particle will produce both an electric and a magnetic field. This is because a charged particle always produces an electric field, if the particle is also moving, it will produce a magnetic field in addition to its electric field. The produced magnetic field is perpendicular to the direction of the particle's motion.

Electromagnetic fields (photon gases) carry energy and transport it from one region of space to another at a speed of light. The total energy stored per unit volume [13, p.398] in a region of Electromagnetic space is

$$p_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right). \tag{31}$$

Analog to the pressure density relationship of eq (9), we can define the photon gas mass density using the total energy:

$$\rho_{em} = \frac{p_{em}}{c^2}. \tag{32}$$

Electromagnetic fields have no rest frame; they have an intrinsic 4-potential:

$$A^\alpha = \left(\frac{p_{em}}{c};\ \overrightarrow{\rho_{em}\vec{v}}\right). \tag{33}$$

With this definition, we have electric field density per unit volume:

$$\vec{E} = -\nabla p_{em} - \frac{\partial\left(\overrightarrow{\rho_{em}\vec{v}}\right)}{\partial t} = -\nabla p_{em} - \frac{1}{c^2}\frac{\partial(p_{em}\vec{v})}{\partial t}, \tag{34}$$

and the magnetic field density per unit volume:

$$\vec{B} = \nabla \times \left(\overrightarrow{\rho_{em}\vec{v}}\right) = \frac{1}{c^2}\nabla \times (p_{em}\vec{v}). \tag{35}$$

Substituting these two definitions into eq. (18), We can get the dynamic equation for the electromagnetic field. The role of the $\vec{B}$ field is similar to the vorticity field of eq. (17); it is responsible for the rotational motion of photon particles. Equations (34) and (35) borrow the electrical and magnetic field strength symbols, $\vec{E}$ and $\vec{B}$, but they are interpreted here as field density per unit volume.

Equations (33)-(35) automatically fulfills the two homogeneous Maxwell equations:

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}; \\ \nabla \cdot \vec{B} = 0. \end{cases} \tag{36}$$

With some algebra manipulations, it is easy to show that we can also get the Gauss's law and the Ampère/Maxwell law with source terms for the non-homogeneous Maxwell equations, see D. J. Griffith [13, p.437]. With a similar procedure we can also get the pressure wave equation for fluid dynamics, see Appendix B.

Here we are interested on the "vacuum" state.

### 5.1 In vacuum, "no source term" of the charge particles

In "vacuum", the Gauss's law reads:

$$\nabla \cdot \vec{E} = 0 \tag{37}$$

Substituting eq. (34) into the eq. (37), we have

$$\nabla \cdot \left[ -\nabla p_{em} - \frac{\partial (\overrightarrow{\rho_{em} v})}{\partial t} \right] = 0. \tag{38}$$

In a local inertial frame, we have the mass conservation law (it is exactly the mass conservation law for the compressible fluid):

$$\nabla \cdot (\overrightarrow{\rho_{em} v}) = -\frac{\partial \rho_{em}}{\partial t} = -\frac{1}{c^2} \frac{\partial p_{em}}{\partial t}. \tag{39}$$

This condition is also similar to the Lorenz-invariant gauge condition for the Maxwell's equation in the Electromagnetic fields.

Manipulating the eq. (38) a bit and using the definition of eq. (32) and the mass conservation condition of (39), we get a wave equation for electromagnetic wave energy:

$$\nabla^2 p_{em} - \frac{1}{c^2}\frac{\partial^2 p_{em}}{\partial t^2} = 0. \tag{40}$$

Similarly, substituting the definition (34) and (35) into the Ampère/Maxwell law, we have:

$$\left[\nabla \cdot (\overrightarrow{\rho_{em}v}) + \frac{1}{c^2}\frac{\partial p_{em}}{\partial t}\right] - \nabla^2(\overrightarrow{\rho_{em}v}) + \frac{1}{c^2}\frac{\partial^2(\overrightarrow{\rho_{em}v})}{\partial t^2} = 0. \tag{41}$$

In the process, we have used the following vector calculus identity:

$$\nabla \times \left(\nabla \times (\overrightarrow{\rho_{em}v})\right) = \nabla[\nabla \cdot (\overrightarrow{\rho_{em}v})] - \nabla^2(\overrightarrow{\rho_{em}v}). \tag{42}$$

and the relation:

$$\frac{1}{c^2} = \mu_0\varepsilon_0. \tag{43}$$

Again, using the energy-mass equivalence relation of (32) and the mass conservation law of equation (39), we get another wave equation for the photon particle momentum, and it propagates in a wave shape at the speed of light in vacuum.

$$\nabla^2(\overrightarrow{\rho_{em}v}) - \frac{1}{c^2}\frac{\partial^2(\overrightarrow{\rho_{em}v})}{\partial t^2} = 0. \tag{44}$$

Both equation (40) and (44) show that the energy and the momentum of the photon gas propagates in electromagnetic field in a speed of light in vacuum. Physically it is clear, because of the existence of a magnetic field (produced by other moving charged particles), the particle will perform a rotational motion in addition to its linear motion.

Observing the total electromagnetic energy density of equation (31), we saw that the $p_e$ $(J/m^3)$ stored in a static electric field E is

$$p_e = \frac{1}{2}\varepsilon_0 \vec{E}^2. \tag{45}$$

The energy density $p_m$ $(J/m^3)$ stored in a magnetic field $\vec{B}$ is given by

$$p_m = \frac{1}{2}\frac{\vec{B}^2}{\mu_0}. \tag{46}$$

In equation (31), E and B represent the electric and magnetic field density of the wave at any instant in a small region of space. we can either write eq. (31) in terms of E field only using the relation of B=E/C and the wave speed relation of (43), or we can write the energy density in terms of the B field only, thus [7, p. 623],

$$p_{em} = 2p_e = 2p_m. \tag{47}$$

Noticed that the energy density associated with the B field equals that due to the static E field, and each contributes half to the total energy.

Comparison of eq. (32) and (47), we have

$$p_{em} = 2p_e = 2p_m = \rho_{em}c^2. \tag{48}$$

Photons exist as moving particles (at least for the observer in the Lab frame). The Planck-Einstein relation says that the total energy of a photon depends on its frequency. It is directly proportional to the frequency [14, 15].

$$E_{tot} = \hbar\omega. \tag{49}$$

where $\hbar$ is the reduced Planck constant, and $\omega$ is the angular frequency of a photon wave.

$$p_e = p_m = \frac{1}{2}\hbar\omega. \tag{50}$$

Accordingly, we have the following relation:

$$p_e = p_m = \frac{1}{2}\rho_{em}c^2. \tag{51}$$

It is well known that photon gas particle travels in vacuum in the speed of light, the right-hand side represents the linear kinetic energy of photon gas, and it amounts to half of the total photon gas energy. The magnetic part represents the rotational motion of particles, it can be deduced that the rotating kinetic energy equals the linear kinetic energy. Thus, it means the particle motion trajectory of the photon particle is a helical motion (rotational plus linear); both the linear and rotational kinetic energy contribute to half of the total energy.

In “vacuum”, Maxwell’s equation for electrostatic field states that the divergence of the electrical field equals zero, as expressed by eq. (37). However, if we are not careful, to choose a Gaussian integral surface that is not small enough, so that exits an electrical dipole in this small region, the Gaussian integral region will contain an electrical dipole, eq. (37) still hold for this case. Consequently, we are of the opinion that the space is a “vacuum”. Under this circumstance, the electrical dipole can be entangled together to propagate in “vacuum” space in the form of an electromagnetic wave. Based on the above arguments, we can deduce that a photon is a helically entangled electrical dipole in free space; both the linear and rotational kinetic energy contribute half to the total energy.

## 6. **Chirality and potential energy sign (attractive or repulsive force)**

### 6.1 The classical fluid dynamic equation (repulsive force)

Eq. (24) contains a rotational term of $\frac{1}{2}\vec{v} \times \vec{\omega}$**.** Physically it represents half of the tangent force to account for the rotational motion:

$$\frac{1}{2}\vec{v} \times \vec{\omega} = \frac{1}{2}\vec{\mathcal{F}}_{\perp}. \tag{52}$$

Substituting the mass density and potential energy density equivalence of eq. (9) into eq. (53), we get

$$\vec{\mathcal{F}}_{\perp} = \vec{v} \times (\nabla \times \overline{\rho\vec{v}}) = \vec{\beta} \times \left[\nabla \times \left(p\vec{\beta}\right)\right], \tag{53}$$

where $\vec{\beta}$ is the ratio of particle velocity to the wave speed:

$$\vec{\beta} = \frac{\vec{v}}{c}. \tag{54}$$

6.2 Gravitational field (attractive force)

For perfect fluid of gravitational field, substituting the mass density and potential energy density relation of the eq. (29), into the Eq. (53), we have

$$\vec{\mathcal{F}}_{\perp} = \vec{v} \times (\nabla \times \overline{\rho_G \vec{v}}) = -\vec{\beta} \times \left[\nabla \times \left(p_G \vec{\beta}\right)\right]. \tag{55}$$

In another word, the vorticity field, in general, can be expressed as:

$$\vec{\omega} = \nabla \times (\overline{\rho \vec{v}}) = \frac{1}{c^2} \nabla \times \overline{p \vec{v}} = \frac{1}{c} \nabla \times \left(\overline{p \vec{\beta}}\right). \tag{56}$$

Comparison of eq. (53) and (55), It can be seen that the potential energy sign will determine the rotational motion directions.

6.3 Electromagnetic field — observed in an inertial frame

As mentioned before, the photon gases have no rest mass, they travel along the left and right light cone surface in Minkowski space at the speed of light in the vacuum, relative to a rest observer, e.g. relative to Lab frame. The world line is just the light cone surface, as shown in Fig.3.

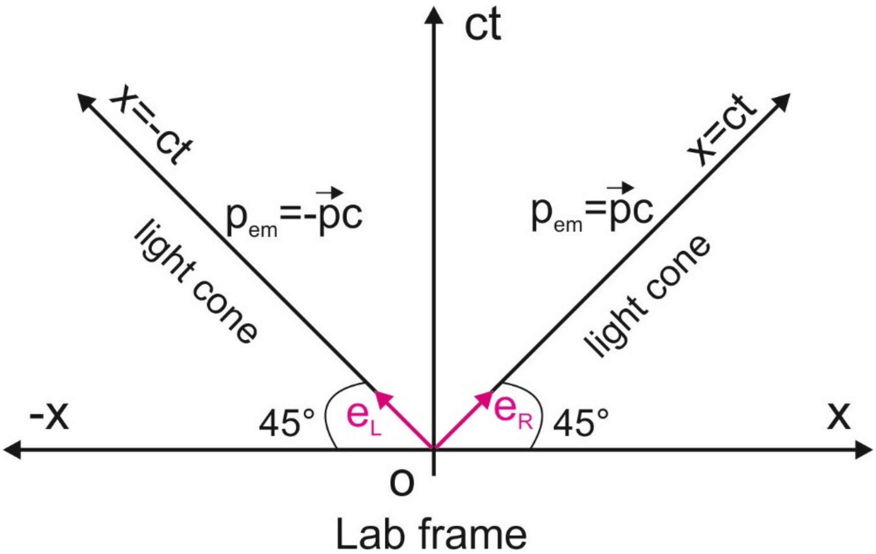


Fig. 3 the photon travels along the left and right light cone at a speed of light in vacuum, observed in Lab frame.

On the light cone, we have the following relation (the relation between photon momentum and photon total energy, “off mass-shell”):

$$p_{em}^2 = \vec{p}^2 c^2. \tag{57}$$

where, $\vec{p}$ is the linear momentum of photon gas, relative to a rest observer. Taking square root for both side,

$$p_{em} = \pm \vec{p} c; \quad or \quad \vec{p} = \pm \frac{p_{em}}{c} = \pm \frac{1}{c} \hbar\omega \tag{58}$$

Namely, the photon gas energy can be either positive or negative, depending on the momentum direction relative to the observer.

$$\vec{\mathcal{F}}_\perp = \vec{v} \times \vec{B} = \pm \vec{\beta} \times \left[\nabla \times \left(p_{em}\vec{\beta}\right)\right]. \tag{59}$$

Comparison eq. (53), (55) and (59). It can be concluded that the potential energy density sign (or rather to say, the interactions between particles are either through attractive forces or through repulsive forces), will affect the particle rotation direction in the local manifold, or to say, depends on the perspective views of the observer. The electromagnetic waves have no rest frame; they can be either positive or negative, and the positive or negative sign depends on the traveling direction between the photon gasses and the observer.

## 7. Conclusion

The configuration stored energy density and mass density have the relation of $p = \rho c^2$. This energy density and momentum vector forms a 4-vector potential. The scalar energy field and momentum vector field are entangled with each other through the mass density. Based on the weak law of action and reaction, a unified dynamic equation of the classical field in the local manifold is derived. This field equation reads $D_t\vec{p} = \nabla\mathcal{L}$. For the derivation of this field equation, no further assumptions and restrictions have been applied. The configuration-stored energy density can be positive or negative, depending on the zero energy definition position in the field. Its sign will affect the rotational direction (vorticity field) of the particle motions. The gradient of the kinetic energy density contains all the possible deformations of the infinitesimal element. It can be decomposed into two parts: a symmetric part, which is mainly responsible for the expansion (contraction) and shear deformation; and an antisymmetric part, which is responsible for a pure rotational motion. The classical Navier-Stokes equation models the symmetric part as viscosity stress based on the Stokes hypothesis, while the antisymmetric part is ignored. This antisymmetric part will result in a Coriolis-like force. A moving charged particle produces an electric field and a magnetic field. The magnetic field will force other charged particles to make a rotational motion. The rotating kinetic energy equals the linear kinetic energy, both contribute to half of the total energy of the electromagnetic field. It can be deduced that the electromagnetic wave is a helically entangled electrical dipole. The moving trajectory of the photon particle will be helical or spiral and the circular polarized lights confirm this phenomenon.

## Appendix A. The fluid dynamics equations

The right hand side of the equation (22) contains a gradient of the kinetic energy; it arises from the curvature of the local manifold.

If the kinetic energy is written explicitly in local manifold using Cartesian coordinate, it reads:

$$T = \frac{1}{2}(\overrightarrow{\rho v} \cdot \vec{v}) = \frac{1}{2}(\rho uu + \rho vv + \rho ww). \tag{A1}$$

Thus the gradient of the kinetic energy is:

$$\nabla T = J_{\overrightarrow{\rho v}}^{T} \vec{v}. \tag{A2}$$

where $J_{\overrightarrow{\rho v}}^{T}$ is the transpose of the Jacobian Matrix of momentum.

$$J_{\vec{p}}^{T} = \begin{bmatrix} \frac{\partial \rho u}{\partial x} & \frac{\partial \rho v}{\partial x} & \frac{\partial \rho w}{\partial x} \\ \frac{\partial \rho u}{\partial y} & \frac{\partial \rho v}{\partial y} & \frac{\partial \rho w}{\partial y} \\ \frac{\partial \rho u}{\partial z} & \frac{\partial \rho v}{\partial z} & \frac{\partial \rho w}{\partial z} \end{bmatrix}. \tag{A3}$$

Any square matrix can be decomposed into its symmetric and antisymmetric parts. This decomposition is often referred to as the "symmetric part" and "skew-symmetric part".

$$J_{\rho\vec{v}}^{T} = \frac{1}{2}\left(J_{\vec{p}}^{T} + J_{\vec{p}}\right) + \frac{1}{2}\left(J_{\vec{p}}^{T} - J_{\vec{p}}\right) = \frac{1}{2}\bar{\bar{S}} + \frac{1}{2}\bar{\bar{A}}. \tag{A4}$$

**Symmetric part:**

The Symmetric part of the gradient of the kinetic energy, thus, is:

$$\frac{1}{2}\bar{\bar{S}}\vec{v} = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{xy} & s & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{A5}$$

where

$$\bar{\bar{S}} = \begin{bmatrix} \frac{\partial \rho u}{\partial x} & \frac{1}{2}\left(\frac{\partial \rho v}{\partial x} + \frac{\partial \rho u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial \rho w}{\partial x} + \frac{\partial \rho u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial \rho u}{\partial y} + \frac{\partial \rho v}{\partial x}\right) & \frac{\partial \rho v}{\partial y} & \frac{1}{2}\left(\frac{\partial \rho w}{\partial y} + \frac{\partial \rho v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial \rho u}{\partial z} + \frac{\partial \rho w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial \rho v}{\partial z} + \frac{\partial \rho w}{\partial y}\right) & \frac{\partial \rho w}{\partial z} \end{bmatrix}. \tag{A6}$$

It can be interpreted as the classical strain-rate tensor, if the density in the local manifold is regards as a constant value.

Recalling the definition of the stress tensor in the Navier-Stokes equation by Stokes hypothesis, we have the flowing relation:

$$\frac{1}{2}\bar{\bar{S}}\vec{v} = 2\mu S_{ij}. \tag{A7}$$

**Anti-Symmetric part:**

Another part is the anti-symmetric term:

$$\frac{1}{2}\bar{\bar{A}}\vec{v} = \frac{1}{2}\begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{A8}$$

Where, the rotational tensor is expressed as:

$$\bar{\bar{A}} = \begin{bmatrix} 0 & \left(\frac{\partial \rho v}{\partial x} - \frac{\partial \rho u}{\partial y}\right) & -\left(\frac{\partial \rho u}{\partial z} - \frac{\partial \rho w}{\partial x}\right) \\ -\left(\frac{\partial \rho v}{\partial x} - \frac{\partial \rho u}{\partial y}\right) & 0 & \left(\frac{\partial \rho w}{\partial y} - \frac{\partial \rho v}{\partial z}\right) \\ \left(\frac{\partial \rho u}{\partial z} - \frac{\partial \rho w}{\partial x}\right) & -\left(\frac{\partial \rho w}{\partial y} - \frac{\partial \rho v}{\partial z}\right) & 0 \end{bmatrix}. \tag{A9}$$

Changing the positive sign, accordingly:

$$\bar{\bar{A}} = -\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = -\vec{\omega}. \tag{A10}$$

then, the rotational part can be expressed as:

$$\frac{1}{2}\bar{\bar{A}}\vec{v} = -\frac{1}{2}\vec{\omega} \times \vec{v} = \frac{1}{2}\vec{v} \times \vec{\omega}. \tag{A11}$$

finally, we get the field equation as expressed by eq. (24):

$$\frac{\partial \overrightarrow{\rho v}}{\partial t} + (\vec{v} \cdot \nabla)(\overrightarrow{\rho v}) = -\nabla p + \frac{1}{2}\bar{\bar{S}}\vec{v} + \frac{1}{2}\vec{v} \times \vec{\omega}. \tag{A12}$$

The symmetric part, $\left(\frac{1}{2}\bar{\bar{S}}\vec{v}\right)$, performs shear; reflection and expansion (dilation) deformation of the infinitesimal element in the field; the antisymmetric part, $\left(\frac{\mathbf{1}}{\mathbf{2}}\vec{\boldsymbol{v}} \times \vec{\boldsymbol{\omega}}\right)$, performs a pure rotation of the element.

The second term in the left hand side of eq. (A12) is the directional derivative of momentum along the velocity vector v at a given point x,

$$(\vec{v} \cdot \nabla)(\overline{\rho\vec{v}}) = \nabla_{\vec{v}}(\overline{\rho\vec{v}}). \qquad \text{(A13)}$$

It represents the instantaneous rate of change of the momentum, moving through a point in space of x with a velocity specified by v. Geometrically, it represents momentum gradient projection onto the velocity vector field at a given point of x.

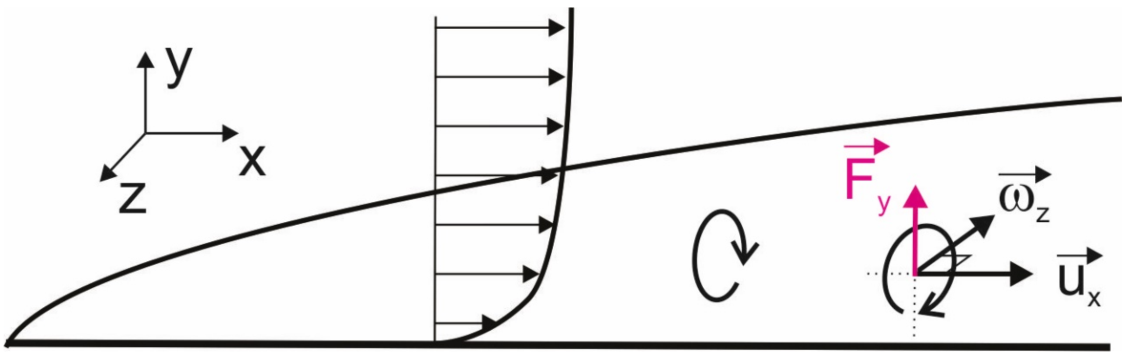


Fig. A1. The antisymmetric term produce a lift force, which leads the boundary layer to become thicker along the flow direction.

The antisymmetric term, $\left(\frac{\mathbf{1}}{\mathbf{2}}\vec{\boldsymbol{v}} \times \vec{\boldsymbol{\omega}}\right)$, will produce an extra force, which leads the shear boundary layer to become thicker along the flow direction, as shown in Fig. A1. Assuming it is one dimensional shear flow along the x-direction, $\vec{v} = (u, 0,0)$. The produced vorticity field is, thus, $\vec{\omega} = (0,0, -\partial(\rho u)/\partial y)$. This extra force term is then:

$$\vec{F} = \frac{1}{2}\vec{v} \times \vec{\omega} = \begin{bmatrix} 0 \\ u\dfrac{\partial(\rho u)}{\partial y} \\ 0 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}. \tag{A14}$$

It can be seen that interaction between the velocity and vorticity in the shear boundary layer will produce an extra lift force, $F_y = u\frac{\partial(\rho u)}{\partial y}$, which depends on the magnitude of the velocity and the gradient of the momentum in the boundary layer.

## Appendix B. From the field equation to a pressure wave equation

Rearranging the equation (18):

$$-\nabla p - \frac{\partial \overrightarrow{\rho v}}{\partial t} = -\vec{v} \times (\nabla \times \overrightarrow{\rho v}). \tag{B1}$$

Taking divergence of both side:

$$-\nabla^2 p - \frac{\partial}{\partial t}(\nabla \cdot \rho\vec{v}) = -\nabla \cdot [\vec{v} \times (\nabla \times \overrightarrow{\rho v})]. \tag{B2}$$

Mass conservation law in the local coordinate frame states:

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho\vec{v}) = 0 \ \ \rightarrow \ \ (\nabla \cdot \rho\vec{v}) = -\frac{\partial \rho}{\partial t}. \tag{B3}$$

Substituting equation B3 into eq. B2:

$$-\nabla^2 p + \frac{\partial^2 \rho}{\partial t^2} = -\nabla \cdot [\vec{v} \times (\nabla \times \overrightarrow{\rho v})]. \tag{B4}$$

Recalling the potential energy density and mass density relation of the eq. (9):

$$p = \rho c^2 \; \rightarrow \; \rho = \frac{p}{c^2}. \tag{B5}$$

Substituting eq. B5 into B4, thus we get the pressure wave equation for compressible fluid in the local inertial frame.

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla \cdot (\vec{v} \times \vec{\omega}). \tag{B6}$$

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