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Hamiltonian Chaos and Gravitational Physics

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Abstract

The goal of this paper is to analyze the likely transition from integrability to Hamiltonian chaos in the primordial Universe. The transition is driven by curvature fluctuations and favors the onset of a spacetime endowed with continuous dimensions.

Key words: Complex dynamics, Hamiltonian chaos, curvature perturbations, fractional dynamics, fractal spacetime, continuous spacetime dimensions.

A convenient way to cast General Relativity (GR) in Hamiltonian form is to use the 3+1 decomposition of the metric presented as

$$g_{00} = -\alpha^2 + \gamma^{ij}\beta_i\beta_j \tag{1}$$

$$g_{0i} = \beta_i \tag{2}$$

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$$g_{ij} = \gamma_{ij} \tag{3}$$

The decomposition (1)–(3) replaces the 10 independent metric components with the lapse function $\alpha(x)$, the shift vector $\beta_i(x)$ and the symmetric spatial metric $\gamma_{ij}(x)$, subject to the condition $\gamma^{ik}\gamma_{jk} = \delta^i_j$ [1].

It is known that the early regime of relativistic cosmology may be studied using the *perturbed Friedman-Robertson-Walker (FRW) metric*. The temporal coefficient of the perturbed FRW metric is given by $a^2(t)(-e^{2\Phi})$, where the relationship between α and the expansion parameter $a(t)$ takes the form ((35) in [1]),

$$\alpha = a[1 + \Phi + \frac{1}{2}(\Phi^2 + w^2) + \ldots] \tag{4}$$

Here, Φ and w are treated as small perturbations applied to the standard FRW metric. Assume also that the only form of matter present is a single scalar field with Lagrangian density,

$$L_M = \sqrt{-g}[-\frac{1}{2}g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi)] \tag{5}$$

The field consists of a homogeneous term $\varphi_0(t)$ and a time-independent term as in

$$\varphi \to \varphi_0(t) + \varphi(x) \tag{6}$$

It can be shown that, to avoid extensive complications in formalism, it is convenient to pass from φ to a new variable called *curvature perturbation* defined as ((133) in [1]),

$$\kappa \equiv (\frac{2\eta}{a\dot{\varphi}_0})^2 \frac{\partial}{\partial t}(\frac{a^2}{\eta}\Psi) \tag{7}$$

in which

$$\eta \equiv \frac{\dot{a}}{a} \tag{8}$$

and where Ψ denotes the gauge-invariant spatial curvature perturbation. Introducing

$$z \equiv \frac{a\dot{\varphi}_0}{\eta} \tag{9}$$

and carrying out a long string of tedious manipulations, one arrives at a straightforward formulation of the Hamiltonian $H(q,p,t)$ in terms of a new scalar field given by ((153) in [1]),

$$\boxed{q = \frac{4a}{\dot{\varphi}_0}\Psi} \tag{10}$$

and a new momentum parameter p, canonically conjugate to q. Relationship (10) links q to the expansion parameter $a(t)$, the spatial curvature perturbation Ψ, and the time derivative of the scalar field $\dot{\varphi}_0$. The Hamiltonian $H(q,p,t)$ can be alternatively expressed in terms of the curvature perturbation κ and its conjugate momentum π_κ, namely ((154) – (155) in [1])

$$H[q,p,t] \Leftrightarrow H[\kappa,\pi_\kappa,t] \tag{11}$$

in which

$$\kappa = \kappa[q,p,z,\dot{z}] \tag{12}$$

$$\pi_\kappa = \pi_\kappa[q,p,z,\dot{z}] \tag{13}$$

The field q obeys the classical equation of a free harmonic oscillator, that is,

$$\boxed{\ddot{q}+(\mu_q^2+k^2)q=\ddot{q}+\omega_0^2 q=0} \tag{14}$$

where μ_q^2 and k^2 denote, respectively, the square of the corresponding mass and wavenumber parameters ((156) in [1]).

A reasonable hypothesis is that the primordial Universe evolves in far-from-equilibrium conditions and is unavoidably affected by the signature of *stochastic fluctuations*. The most straightforward way to account for the effect of these fluctuations on either curvature perturbations (7) or scalar field (10) is to use the Hamiltonian model of *periodically kicked oscillator*, extensively discussed in the nonlinear science literature.

The equations describing the effect of periodic perturbations on the harmonic oscillator (14) are given by the Hamiltonian [2]

$$H=\frac{1}{2}(p^2+\omega_0^2 q^2)-\frac{\omega_0\Gamma}{T}\cos q\sum\nolimits_n \delta(\frac{t}{T}-n) \tag{15}$$

where $\delta(...)$ represents an infinite train of pointwise ("delta") kicks of period T and amplitude Γ. The equation of the perturbed oscillator derived from (15) is,

$$\ddot{q} + \omega_0^2 q = -\frac{\omega_0 \Gamma}{T} \sin q \sum_n \delta(\frac{t}{T} - n) \tag{16}$$

As shown in [2] (section 5.3), the kicked oscillator can be studied in discrete time $t_n = nT$, leading to the pair of so-called *web-map equations*

$$u_{n+1} = f(u_n, v_n, \Gamma, \lambda) \tag{17a}$$

$$v_{n+1} = g(u_n, v_n, \Gamma, \lambda) \tag{17b}$$

in which the following dimensionless variables are introduced,

$$u = \frac{\dot{q}}{\omega_0} \tag{18a}$$

$$v = -q \tag{18b}$$

$$\lambda = \omega_0 T \tag{18c}$$

The phase-space portrait associated with the web-map (17) is illustrated below in Figs. 1 and 2. It shows a structure consisting of regular orbits (islands of stability), along within regions of chaotic motion. As indicated in [2–4], this phase-space configuration is typical for the transition to Hamiltonian chaos and the emergence of *a fractal spacetime endowed with continuous dimensions.*

References

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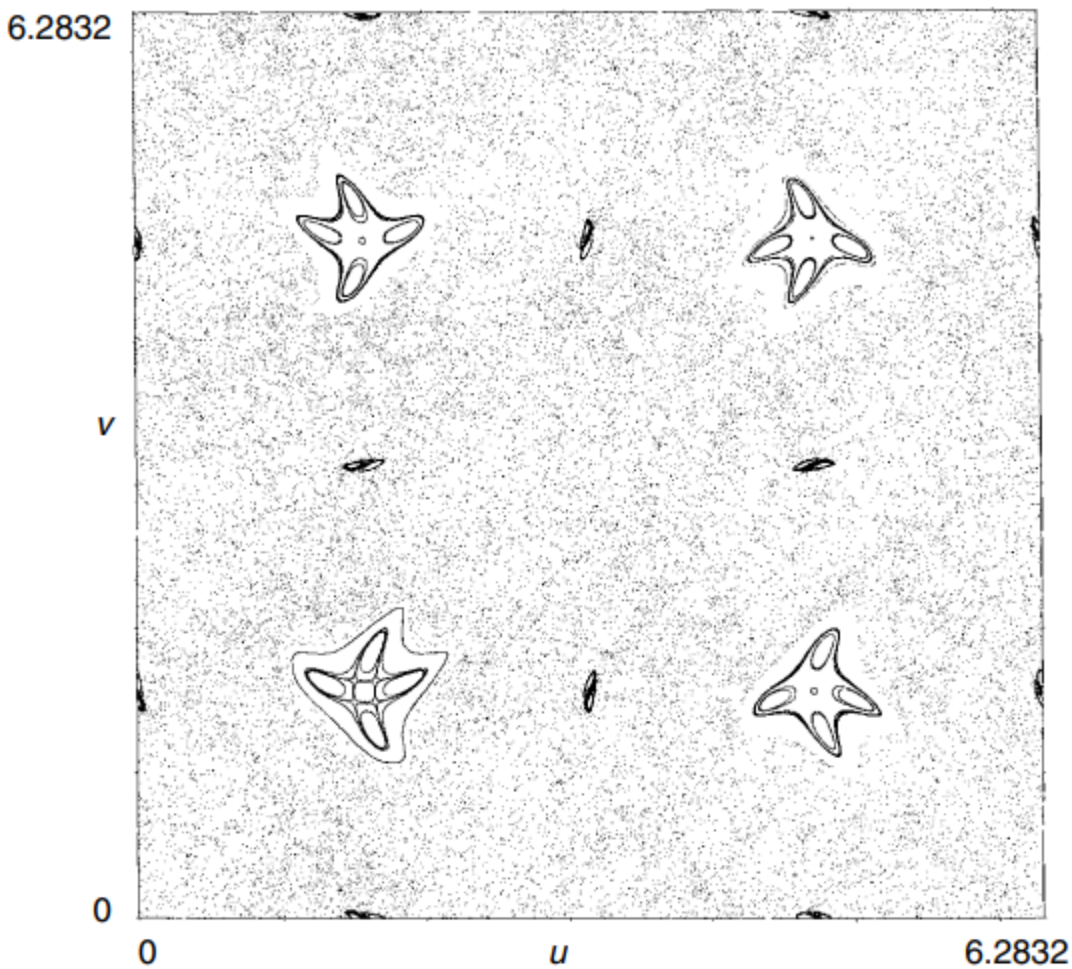

Fig. 1: Phase space portrait of the web-map (17)

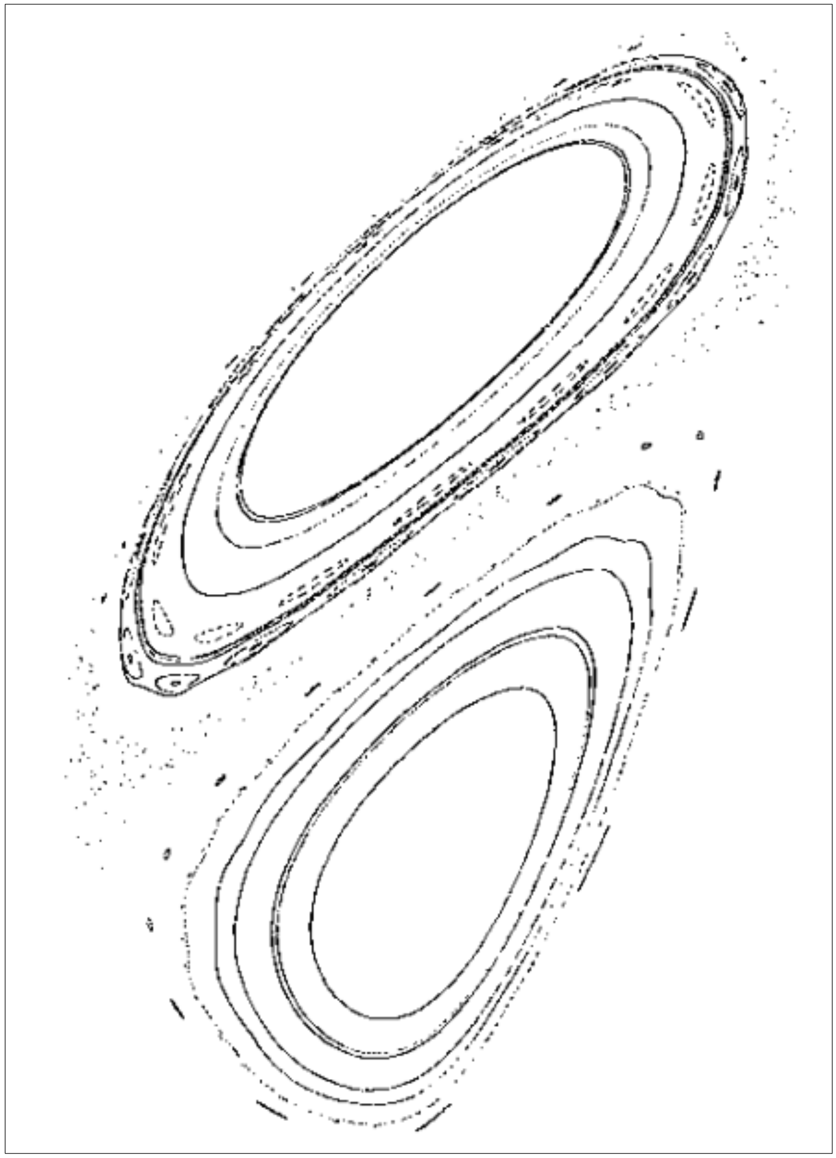

Fig. 2: Magnified detail of Fig. 1 showing Hamiltonian chaos