Research Article

Fermat Polynomials and Extended Fermat's Theorem

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This study discusses the connection between Fermat perfect natural vectors and some specific Fermat polynomials, whose maximal root is a natural number forming part of the Fermat vector radius. Apart from the nature and construction of Fermat's polynomials, some examples of application are given. If found as natural numbers, calculating the maximal roots of Fermat's polynomials constitutes an alternative algorithm to find out Fermat's vectors.

1. Introduction

In recent years, work has been done due to the collaboration with Niño, Muñoz-Caro, and Reyes[1][2][3][4][5][6][7][8], with a later contribution from Castro[9] on studying the extension of the Fermat theorem to larger dimensions and orders. However, the adopted point of view has been mainly based on empirical computational grounds and, thus, is prone to a lack of sufficient natural number sampling extension.

This drawback has stimulated the search for and analysis of new ways to tackle the problem, for example, the study of Fermat's surfaces[7]. The present research can be classified as one on this path.

Here, we discuss the connection of perfect natural vectors with Euclidean and Minkowskian spaces differently from previously discussed[1][4][6]. After this, the next step drives us to consider the construction of reverse perfect natural vectors and polynomials because some can be tightly bound with the Fermat theorem.

The structure of the present study is as follows. First, natural vectors and higher-order Fermat vectors are studied. Then, Fermat polynomials are presented. A discussion of several examples follows. Finally, some additional considerations close the paper.

2. Natural Vectors

2.1. Natural Perfect Vectors

A natural vector $\langle\mathbf{x}| = (x_1, x_2, x_3, \dots x_I, \dots, x_N) \in V_N(\bullet)$ belongs to some N-dimensional semispace $V_N(\bullet)$, defined over the natural number set $(\bullet)$.

Such a vector is named perfect if its elements are non-zero and canonically ordered:

$$0 < x_1 < x_2 < x_3 < \dots < x_I < \dots < x_N. \tag{1}$$

When constructing a $(N+1)$-dimensional natural semispace with vectors now built as:

$$\langle\mathbf{v}| = (\langle\mathbf{x}|, r) \in V_{N+1}(\bullet), \tag{2}$$

then the additional vector element can be called the radius, r, such that, for the augmented vector $\langle\mathbf{v}|$ to be perfect, it has to be constructed as follows: $x_N < r$.

Then, the perfect vector $\langle\mathbf{x}|$ can be called the Euclidean part of the perfect vector $\langle\mathbf{v}|$.

2.2. Natural Vectors in Natural Minkowski Spaces

The natural semispace $V_{N+1}(\bullet)$ can be transformed into a natural Minkowski space simply using the unity vector:

$$\langle\mathbf{1}| = (1, 1, 1, \dots, 1, \dots, 1) \in V_N(\bullet) \tag{3}$$

as the Euclidian part of a Minkowski metric vector form:

$$\langle\mathbf{m}| = (\langle\mathbf{1}|, -1) \in V_{N+1}(\mathbb{Z}). \tag{4}$$

Then, for every vector in the semispace $V_{N+1}(\bullet)$, one can calculate a Minkowski norm written as:

$$\forall\langle\mathbf{v}| \in V_{N+1}(\bullet) : M(\langle\mathbf{v}|) = \langle\langle\mathbf{v}| * \langle\mathbf{m}\ |\rangle, \tag{5}$$

where the inward product in the equation (5) is defined as:

$$\langle\mathbf{v}| * \langle\mathbf{m}| = (\langle\mathbf{x}| * \langle\mathbf{1}|; -r) = (\langle\mathbf{x}|; -r) \in V_{N+1}(\mathbb{Z}), \tag{6}$$

besides, the Minkowski norm uses the complete sum of the vector elements such that:

$$\langle\langle\mathbf{v}\ |\rangle = \sum_{I=1}^{N+1} v_I; \tag{7}$$

therefore, the Minkowski norm of a natural vector can be developed as follows:

$$M(\langle\mathbf{v}|) = \langle\langle\mathbf{v}| * \langle\mathbf{m}\ |\rangle = \langle\langle\mathbf{x}\ |; -r\rangle = \langle\langle\mathbf{x}\ |\rangle - r = \left(\sum_{I=1}^{N} x_I\right) - r \in \mathbb{Z}. (8)$$

Then, the norm $M(\langle\mathbf{v}|)$ in the equation (8) defines the vector space $V_{N+1}(\mathbb{Z})$ as a Banach space, which can be properly called a natural Minkowski space.

Different from the usual Euclidian norm, by definition, the Minkowski norm $M(\langle\mathbf{v}|)$ might be zero or negative. Therefore, its values belong to the integer set $\mathbb{Z}$. However, for this study, we are interested in zero Minkowski norms.

The zero Minkowski norm vectors are the same as those used in Minkowskian relativistic space-time, where they are named time vectors. Perfect natural vectors $\langle\mathbf{f}|$ with zero Minkowski norms, fulfilling the following equality:

$$\exists\langle\mathbf{f}| \in V_{N+1}(\bullet) \rightarrow M(\langle\mathbf{f}|) = 0 \Rightarrow \left(\sum_{I=1}^{N} f_I\right) - r = 0, \tag{9}$$

for work purposes, they will be called Fermat vectors.

In this Minkowski context, such Fermat natural vectors fulfilling the equation (9), possess a radius equal to the elements' sum of the Euclidean part of the vector:

$$\left(\sum_{I=1}^{N} f_I\right) = r. \tag{10}$$

3. Fermat Vectors of Higher Orders

Until this section, one can consider that we have described first-order Fermat vectors. Additionally, there are several ways to define higher-order Fermat vectors.

3.1. Natural Set Power of p-th Order

Natural power sets are constructed as a start-up technique to reach higher-order Fermat vectors.

The p-th order natural power set is easily computed as:

$$\bullet^{[p]} = \{1, 2^p, 3^p, \ldots, I^p, \ldots\}. \tag{11}$$

3.2. Natural Vectors of p-th Order

Using the natural power set $\bullet^{[p]}$, one can construct a subset $S_{N+1}(\bullet^{[p]})$ of the Minkowski natural semispace $V_{N+1}(\bullet)$:

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$$\forall I = 1, N : s_I^p \in \bullet^{[p]} \wedge r^p \in \bullet^{[p]} \rightarrow$$
$$\forall \left\langle \mathbf{s}^{[p]} \right| = \left(s_1^p, s_2^p, s_3^p, \ldots, s_I^p, \ldots s_N^p, r^p \right) \in S_{N+1}(\bullet^{[p]}) \subset V_{N+1}(\bullet) \tag{12}$$

Then, the p-th order Minkowski norms of these vector subsets can be described in a general manner by:

$$\forall p \in \bullet : M\left(\left\langle \mathbf{s}^{[p]} \right| \right) = \left\langle \left\langle \mathbf{s}^{[p]} \right| * \langle \mathbf{m} \,| \right\rangle$$
$$= \left\langle \left\langle \mathbf{x}^{[p]} \;|; -r^p \right\rangle = \left\langle \left\langle \mathbf{x} \right|^{[p]} \right\rangle - r^p = \left(\sum_{I=1}^{N} x_I^p \right) - r^p \,. \tag{13}$$

3.3. Fermat Vectors of p-th Order

Next, a Fermat vector of p-th order $\left\langle \mathbf{f}^{[p]} \right|$ will be defined as a vector of the subset $S_{N+1}(\bullet^{[p]})$ possessing a zero Minkowski norm.

That is, p-th order Fermat vectors fulfill in general:

$$\forall p \in \bullet : \left\langle \mathbf{f}^{[p]} \right| \in S_{N+1}\left(\bullet^{[p]} \right) \rightarrow M\left(\left\langle \mathbf{f}^{[p]} \right| \right) = 0 \Rightarrow \left(\sum_{I=1}^{N} f_I^p \right) = r^p. \tag{14}$$

4. Polynomial Expression of Fermat's Vectors

4.1. Reverse Fermat Vectors

p-th order Fermat's vectors fulfilling equation (14) possess an alternative description that can be expressed as a polynomial of the radius r.

To obtain such a situation, one must be aware that the Fermat vectors of any order are considered before some other property, perfect natural vectors, their elements fulfilling:

$$0 < f_1^p < f_2^p < f_3^p < \ldots < f_I^p < \ldots f_N^p < r^p \tag{15}$$

therefore, there exists a set of natural numbers that can be expressed in the form of an alternative natural vector: $\{a_I \mid I = 1, N\} = \langle \mathbf{a}| \in V_N(\bullet)$, satisfying:

$$\forall I = 1, N : (r - a_I)^p = f_I^p \Leftarrow r - a_I = f_I. \tag{16}$$

The vector $\langle \mathbf{a}|$ can be considered perfect in reverse mode because of the ordering nature of the Fermat perfect vector elements and the radius:

$$r > a_1 > a_2 > a_3 > \ldots > a_I > \ldots > a_N > 0. \tag{17}$$

Because of this reversal of the canonical ordering mode in the vector $\langle \mathbf{a}|$, such a vector, when connected with a Fermat vector, can be named a reverse Fermat vector.

4.2. Polynomial representation of Fermat Vectors

Also, one can use the binomial Newton development for each power in the equation (16), then it can be written:

$$\forall I = 1, N : (r - a_I)^p = \sum_{k=0}^{p} (-1)^k \binom{p}{p-k} r^{p-k} a_I^k, \tag{18}$$

and one can express the Minkowski zero norm condition for Fermat's vectors using equations (16) and (18):

$$\begin{aligned} r^p &= \sum_{I=1}^{N} (r - a_I)^p = \sum_{I=1}^{N} \sum_{k=0}^{p} (-1)^k \binom{p}{p-k} r^{p-k} a_I^k \\ &= \sum_{k=0}^{p} (-1)^k \binom{p}{p-k} \left(\sum_{I=1}^{N} a_I^k r^{p-k} \right) = \sum_{k=0}^{p} A_k r^{p-k} = N r^p + \sum_{k=1}^{p} A_k r^{p-k} \end{aligned}, \tag{19}$$

where the polynomial coefficients on the right side of the equation (19) can be written as:

$$A_0 = N \wedge \forall k = 1, p : A_k = (-1)^k \binom{p}{p-k} \sum_{I=1}^{N} a_I^k = (-1)^k \binom{p}{p-k} \left\langle \left\langle \mathbf{a}^{[k]} \mid \right\rangle \right., \tag{20}$$

hence, reverse Fermat's vectors might fulfill the equation:

$$0 = (N-1) r^p + \sum_{K=1}^{p} A_k r^{p-k} \tag{21}$$

Therefore, knowing the set of vector elements $\langle \mathbf{a}| = \{a_I \mid I = 1, N\}$, the radius defined for the Fermat vectors will be a natural root of the polynomial (21).

If a natural root of the polynomial doesn't exist, then the vector $\langle \mathbf{a}|$ does not correspond to a reverse Fermat vector.

Therefore, what can be called a Fermat polynomial can be written as:

$$F_{N,p}(r) = (N-1) r^p + \sum_{k=1}^{p} A_k r^{p-k}. \tag{22}$$

Note that because Fermat polynomials are generated with the elements of the equation (16), there is a possibly infinite number of polynomials fulfilling the Fermat condition. This is because a Fermat vector remains as such when multiplied by any natural scalar factor; see, for example, reference[4].

Then, the attached Fermat polynomials possess an infinite number of roots obtained by multiplying the original root by any natural number.

Also, one can consider the following sentences:

- Every reverse-perfect vector $\langle \mathbf{a}| \in V_N(\bullet)$ can be associated with a polynomial of type (22) and be subject to an equation like (21).
- Many polynomials, even those with the equation's (22) form, possess only real or imaginary roots and thus cannot be truly associated with Fermat's vectors.
- Only Fermat's polynomials have a maximal natural root. Thus, one can use the name Fermat polynomial for polynomials with the structure of the equation (22) but having a maximal natural root.
- Fermat's polynomials correspond one-to-one with Fermat's vectors.
- Seeking maximal natural roots of the equation (21) constitutes the backbone of an algorithm to search for general Fermat vectors in $(N+1)$-dimensional Minkowski spaces.

5. The Last Fermat's Theorem as Second-order (2+1) Dimensional Case

Here, we discuss some aspects of Fermat's polynomial theory, which was previously developed in this study, providing some cases as application examples.

5.1. Last Fermat's Theorem

The last Fermat theorem is related to a Minkowski space of dimension (2+1) and second-order vectors. In this case, the Fermat polynomials will have a simple structure like:

$$r^2 = (r-a_1)^2 + (r-a_2)^2 = 2r^2 - 2(a_1+a_2)r + \left(a_1^2+a_2^2\right), \tag{23}$$

which reduces to the equation:

$$r^2 - 2A_1 r + A_2 = 0 \Leftarrow A_1 = (a_1+a_2); A_2 = \left(a_1^2+a_2^2\right), \tag{24}$$

so the roots can be easily written as:

$$r_\pm = A_1 \pm \sqrt{A_1^2 - A_2} = (a_1+a_2) \pm \sqrt{2a_1a_2}. \tag{25}$$

The square root minus sign can be discarded because the radius has to have a maximal value.

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Also, the square root value has to be a natural number if the equation (25) corresponds to a Fermat polynomial.

Consequently, if this is the case, the product within the square root must be written as:

$$\exists\alpha \in \bullet : a_1 a_2 = 2\alpha^2 \rightarrow \alpha = \sqrt{\frac{a_1 a_2}{2}} \Rightarrow 2a_1 a_2 = 4\alpha^2, \tag{26}$$

this guarantees that the square root argument in the equation (25) will be a squared natural number yielding a natural number, and thus, one can write:

$$r_+ = (a_1 + a_2) + 2\alpha \in \bullet. \tag{27}$$

This last relationship can also be associated with a Pythagorean triple and a Fermat second-order (2+1)-dimensional vector constructed as:

$$\langle \mathbf{f}| = (r_+; (r_+ - a_1); (r_+ - a_2)). \tag{28}$$

5.2. Reformulating Fermat's Theorem

Then, the last Fermat theorem can be reformulated, admitting that natural roots cannot be found for Fermat polynomials of orders higher than the second.

For instance, the third-order Fermat polynomials:

$$F_{3,3}(r) = r^3 - 3A_1 r^2 + 3A_2 r - A_3 \tag{29}$$

with the set of coefficients obtained from any 2-dimensional reverse perfect natural vector $\langle \mathbf{a}| = (a_1, a_2) \in V_2(\bullet)$:

$$\left\{A_1 = (a_1 + a_2); A_2 = \left(a_1^2 + a_2^2\right); A_3 = \left(a_1^3 + a_2^3\right)\right\} \tag{30}$$

cannot have a maximal natural root, according to Fermat's theorem.

5.3. The Root Structure of Third-order Fermat Polynomials

The root structure of third-order polynomials has generally been deeply studied from the old times. They are connected to Diophantine equations, already described in the 3rd century AD, and Fermat's theorem.

A recent account of Diophantine equations can be found on the website[10]. Also, references[11][12] can provide more information on the subject. Durand published an exhaustive review of polynomial root computing[13].

The coefficients $\{A_1, A_2, A_3\}$ in the polynomial (29), as constructed in the equation (30), constitute cases where one should expect one real root and two complex conjugate ones. The real root might be expressed as:

$$r = a_1 + a_2 + \phi(a_1; a_2) = A_1 + \phi(a_1; a_2), \tag{31}$$

where the function $\phi(a_1; a_2)$ corresponds to a complicated expression involving the natural parameters $\{a_1, a_2\}$ and their powers; also, square and cubic roots appear in the function via direct and inverse summands. A constant factor: $\sqrt[3]{2}$, is also included in direct and inverse formats. Such a cubic root element might be the first signal indicating the difficulty of obtaining a natural root from the two natural variable components $\{a_1, a_2\}$ polynomial (29).

Wolfram Alpha AI system[14] has provided the formula of the real root offered in a raw form. After simplifying and rearranging terms, one can write it as:

$$r = a_1 + a_2 + 2a_1 a_2 \frac{\sqrt[3]{2}}{S} + \frac{1}{\sqrt[3]{2}} S \tag{32}$$

using:

$$S = \sqrt[3]{a_1 a_2 \left[3(a_1 + a_2) + \sqrt{9(a_1 - a_2)^2 + 4a_1 a_2}\right]}. \tag{33}$$

Fourth- and higher-order polynomials can also be candidates for not having a natural root. However, the discussion of higher-order polynomials is left for further study and development.

5.4. Fermat's Second-Order Polynomials in Higher-dimensional Minkowski Semispaces

Also, higher-dimensional natural semispaces and the associated Minkowski extensions provide even more complicated Fermat polynomials, where the chance of obtaining a natural root might vanish.

Another interesting fact is that second-order Fermat vectors correspond to those contained in natural vector semispaces of arbitrary dimensions, a well-known occurrence that was recently studied[4]. A computational search up to dimension 200 has found many second-order Fermat vectors without problems.

The polynomials associated with this situation are related to Fermat's reverse natural vectors $\langle \mathbf{a}| \in V_N(\bullet)$. The related second-order polynomials can be easily written similarly to the equation (24):

$$(N-1)r^2 - 2A_1 r + A_2 = 0 \Leftarrow A_1 = \left\langle \langle \mathbf{a} \,| \rangle; A_2 = \left\langle \left\langle \mathbf{a}^{[2]} \,\right| \right\rangle, , \tag{34}$$

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where:

$$A_1 = \langle\langle \mathbf{a} \,|\rangle = \sum_{I=1}^{N} a_I \quad A_2 = \left\langle\left\langle \mathbf{a}^{[2]} \,|\right\rangle\right. = \sum_{I=1}^{N} (a_I)^2, \tag{35}$$

and therefore, the possible maximal natural root can be easily rewritten, extending the equation (25), after simplifying a factor 2:

$$r_+ = (N-1)^{-1}\left(A_1 + \sqrt{A_1^2 - (N-1)A_2}\right) = (N-1)^{-1}\left(A_1 + \sqrt{\Delta}\right) \tag{36}$$

where the discriminant Δ can be written in terms of the reverse Fermat vector elements as:

$$\begin{aligned} \Delta &= 2\sum_{I=1}^{N}\sum_{J=1}^{N}\delta(I>J)a_I a_J - (N-2)\sum_{I=1}^{N}a_I^2 \\ &= \sum_{I=1}^{N}a_I\left(\left(2\sum_{J=1}^{N}\delta(I>J)a_J\right) - (N-2)a_I\right) \end{aligned} \tag{37}$$

wherever $\delta(I > J)$ is a logical Kronecker's delta[1].

To admit that a Fermat vector with a maximal natural root has been obtained, the discriminant must be a squared natural number, that is: $\Delta \in \bullet^{[2]}$.

Computationally, there seems to be no limit to the dimension of the reversed Fermat's vector to obtain second-order Fermat's polynomials. Based on empirical grounds only, such a statement must be considered a conjecture.

6. Further Considerations

Previously, in this paper, the problem of finding Fermat's vectors has been transformed into the computation of a maximal natural root of a polynomial, which can be constructed for each reverse perfect natural vector. Therefore, looking for an algorithm to obtain the maximal root of Fermat's polynomials is worthwhile.

In this line of thought, a paper by Davenport and Mignotte[15] defines obtaining a bound of maximal polynomial roots.

The procedure is related to an old one described in the 19th century by Dandelin, Lobachevski, and Graeffe (DLG), developed in detail by Durand[13] and also in[16].

Knowing if a maximal natural root can be attached to a given reverse perfect vector is interesting enough because this knowledge might connect a tested vector with its Fermat nature.

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Details of the DLG procedure can be obviated because it can be retrieved from references[13][16]. The basic technique refers to constructing a new polynomial whose roots are powers of the original ones. This is made by iterating the polynomial coefficients at each increasing root power until a stable set of root values is reached for a given precision.

6.1. DLG method

Just after the first iteration of the DLG method, one can obtain an approximate value of the maximal root as:

$$r_{\max} < \frac{9A_1^2 - 6(N-1)A_2}{(N-1)^2} = 9\left((N-1)^{-1}A_1\right)^2 - 6(N-1)^{-1}A_2, \tag{38}$$

which using:

$$B_1 = \left(3(N-1)^{-1}A_1\right)^2 \wedge B_2 = 3(N-1)^{-1}A_2 \tag{39}$$

transforms into a simple expression:

$$r_{\max} < B_1 - 2B_2 \tag{40}$$

The third-order (2+1)-dimensional problem might serve to test the possibility of obtaining an upper bound to the radius of a given reverse perfect vector, as then one can write the maximal root using:

$$\begin{aligned} r_{\max} &< 3\left[\frac{3}{4}(a_1+a_2)^2 - \left(a_1^2+a_2^2\right)\right] \\ &= \frac{3}{4}\left[4a_1a_2 - (a_1-a_2)^2\right] \end{aligned} \tag{41}$$

considering that the expression on the right is positive for the pairs of natural numbers $\{a_1, a_2\}$.

6.2. Knuth Method

The paper of reference[15] also mentions the Knuth criterion for obtaining the maximal root of a polynomial. One can write, in our case, with Fermat's polynomials:

$$r_{\max} \le 2\max\left\{\sqrt[k]{|A_{k-1}|} \mid k = 1, p+1\right\}, \tag{42}$$

which constitutes another possible evaluation of the maximal root, which is more involved as the whole coefficients must be known. For the case of third-order (2+1)-dimensional Fermat vectors, it reduces to:

$$r_{\max} \le 2\max\left\{1; |A_1|; \sqrt{|A_2|}; \sqrt[3]{|A_3|}\right\}, \tag{43}$$

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remembering that one can write taking into account the Newton formulation factors:

$$|A_1| = 3\left(a_1 + a_2\right) \wedge |A_2| = 3\left(a_1^2 + a_2^2\right) \wedge |A_3| = \left(a_1^3 + a_2^3\right), \tag{44}$$

then, the equation (43) can be easily rewritten in the present case as:

$$r_{\max} \leq 2\,|A_1| = 6\left(a_1 + a_2\right). \tag{45}$$

7. Conclusion

Search for Fermat's vectors is equivalent to setting up a Fermat polynomial with integer coefficients using a reverse perfect Fermat vector. Fermat's polynomials have a natural number root that coincides with the radius of the associated Fermat vector.

Succinctly: To test a reverse perfect natural vector $\langle\mathbf{a}| \in V_N(\bullet)$ as a Fermat vector, search for a natural maximal root of the corresponding Fermat's polynomial (22); then, if this natural root exists, r_F say, one can construct a Fermat vector of the form:

$$\begin{aligned} &\left\langle\mathbf{f}^R\right| = \left(r_F; \left(r_F\langle\mathbf{1}| - \langle\mathbf{a}|\right)\right) \rightarrow \\ &\langle\mathbf{f}| = \left(\left(r_F\langle\mathbf{1}| - \langle\mathbf{a}|\right)^R; r_F\right) = \left(\left(r_F\langle\mathbf{1}| - \left\langle\mathbf{a}^R\right|\right); r_F\right) \in V_{(N+1)}(\bullet) \end{aligned}, \tag{46}$$

here the reversal operator *R*, as seen in reference[17], has been used, to indicate an order reversal of the elements of the original vector.

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Footnotes

[1] A logical Kronecker's delta, in this case, yields one if I>J and zero if I<J.

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Declarations

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