Research Article

Some Results on Maxima and Minima of Real Functions of Vector Variables: A New Perspective

Bibiano Martin Cerna Maguiña[1], Dik Dani Lujerio Garcia[1], Victor Alberto Pocoy Yauri[1], Vladimir Yovanni Rodriguez Sabino[1], Ruben Mario Leiva Bernuy[2]

1. Academic Department of Mathematics, Science Faculty, Universidad Nacional Santiago Antúnez de Mayolo, Peru; 2. Universidad Nacional Santiago Antúnez de Mayolo, Peru

In this work, the extreme points of real vector variable functions are obtained without the use of the classical theory that involves the use of partial derivatives. We illustrate with several theorems and examples a new method that consists of establishing an appropriate link between the function to be optimized, its restrictions and the result, stating that: given n non-zero real numbers $a_1, a_2, \cdots, a_n \in \mathbb{R}$, then there exists a unique $\lambda \in \mathbb{R}$ such that:

This relation is obtained by decomposing the Hilbert space $\mathbb{R}^n$ as the direct sum of a closed subspace and its orthogonal complement. Since the dimension of the space $\mathbb{R}^n$ is finite, this guarantees that any linear functional defined on the space $\mathbb{R}^n$ is continuous, and this guarantees that the kernel of said linear functional is closed in the space $\mathbb{R}^n$, therefore we have that the space $\mathbb{R}^n$ breaks down, as the direct sum of the kernel of the continuous linear functional f and its orthogonal complement, that is: $\mathbb{R}^n - \ker f \bigoplus [\ker f]^{\perp}$, where the dimension of $\ker f - n - 1$ and the dimension of $[\ker f]^{\perp} - 1$.

Adding to the link found new definitions about the hierarchy of one variable in relation to another and the fact that if $x_1^2 + x_2^2 + \ldots + x_n^2 - r^2$ then the $\max\{x_1 + x_2 + \cdots + x_n\} - r\sqrt{n}$ and the $\min\{x_1 + x_2 + \cdots + x_n\} - -r\sqrt{n}$ we solve the optimization problem without using classical theory.

Corresponding authors: B. M.Cerna Maguiña, bcernam@unasam.edu.pe; Dik Dani Lujerio Garcia, dlujerigo@unasam.edu.pe; Victor Alberto Pocoy Yauri, vpocoyy@unasam.edu.pe; Vladimir Yovanni Rodriguez Sabino, vrodriguezs@unasam.edu.pe; Ruben Mario Leiva Bernuy, rleivab@unasam.edu.pe

1. Introduction

In this work, we will find the maxima or minima of real vector variable functions (conditional and unconditional), these will be found without the need to use partial derivatives. To this end, we note that by

 doi.org/10.32388/PMUDR7

solving the problem of $\max(\min) f(x, y, z)$ subject to a condition, there is a hierarchy of one variable over another, depending on how f is defined and its domain of each variable. For example, if our problem is $\max f(x, y, z) = xy^2z^3$ with $x + y + z = 6$ and $0 < x, 0 < y, 0 < z$, we would have to have $z > y > x$, that is, the variable with the highest hierarchy is z, and y has a higher hierarchy than x in the given domain. On the other hand, if we had the problem of $\max f(x, y, z) = xyz$ subject to $x + y + z = a$ where $0 < x < a$, $0 < y < a$, $0 < z < a$, it would be clear that $x = y = z$, that is, these variables have the same hierarchy in the given domain.

In addition, we establish an appropriate link between the optimization problem and the relations

$$\begin{cases} (a_1 + a_2 + \cdots + a_n) = \lambda \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \\ \max\{x_1 + x_2 + \cdots + x_n\} = r\sqrt{n} \\ \min\{x_1 + x_2 + \cdots + x_n\} = -r\sqrt{n}. \end{cases}, \tag{1}$$

where this allows us to obtain desired results. The first relation above was used in other areas of mathematics, see [1][2][3][4]. The verification of these relationships is demonstrated with the following theorems:

Theorem 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$, then the $\max f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i$ subject to the condition $\sum_{i=1}^{n} x_i^2 = r^2$ is $r\sqrt{n}$.*

Proof. Since $\sum_{i=1}^{n} x_i = \langle (x_1, x_2, \ldots, x_n), (1, 1, \ldots, 1) \rangle = |x| \cdot \sqrt{n} \cdot \cos\theta = r\sqrt{n}$, where θ is the angle formed by the vectors $x = (x_1, x_2, \ldots, x_n)$ and the vector $(1, 1, \ldots, 1)$. Here, the maximum and minimum are obtained when $\theta = 0$ and $\theta = \pi$ respectively. □

Theorem 1.2. *Let $a_1, \ldots, a_n$ be any real numbers, then there exists $\lambda \in \mathbb{R}$, such that*

$$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} a_i^2$$

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$, be defined by f is a linear and continuous functional. Therefore

$$\mathbb{R}^n = \ker f \oplus (\ker f)^{\perp}. \tag{2}$$

Thus we have that:

$$\dim \mathbb{R}^n = \dim \operatorname{Im} f + \dim \ker f$$

$$n = 1 + \dim \ker f$$

Therefore $\dim \ker f = n - 1$

Thus, from (2) we have:

$$(1, 1, \ldots, 1) = \sum_{i=1}^{n-1} \lambda_i u_i + \lambda_n u_n \tag{3}$$

where $\{u_1, \ldots, u_n - 1\} \subset \ker f$ and $u_n \in (\ker f)^{\perp}$

From (3) and taking into account that f is a linear functional, we have

$$f(1,\ldots,1)=\lambda f\left(u_n\right)$$

$$\sum_{i=1}^{n} a_i=\lambda \sum_{i=1}^{n} a_i^2$$

$$\text{since } u_n=\left(a_1,\ldots,a_n\right)\in(\ker f)^{\perp}$$

Using this last relation, which must be linked with the function to be maximized and with the given restrictions. Below we show several problems that illustrate the given theory. □

2. Results

Using the aforementioned technique we must standardize the resolution of various problems.

Theorem 2.1. *Let* $H:[a,b]\rightarrow\mathbb{R},\ T:[c,d]\rightarrow\mathbb{R},$ $G:[a,b]\times[c,d]\longrightarrow\mathbb{R}$ *and* $F(x,y)=H(x)+G(x,y)+T(y)$ *be continuous functions. Then we have that* $\operatorname{Max} F(x,y)=\dfrac{3}{\lambda}$ *for some* $\lambda>0$, *where*

$$\operatorname{Max} H(x)=\operatorname{Max} T(y)=\operatorname{Max} G(x,y)=\frac{1}{\lambda} \text{ or}$$

$$Max\,F(x,y)=0, \text{ where } Max\,H(x)=\operatorname{Max} T(y)=\operatorname{Max} G(x,y)=0$$

Proof. Let

$$A=H(x),\quad B=G(x,y),\quad C=T(y) \tag{4}$$

using the relation

$$A+B+C=\lambda\left[A^2+B^2+C^2\right]$$

we obtain the following

$$H(x)+G(x,y)+T(y)=\lambda\left(H^2(x)+G^2(x,y)+T^2(y)\right], \tag{5}$$

where $\lambda=\lambda(x,y)$.

From the relationship (5) we obtain:

$$\left(H(x)-\frac{1}{2\lambda}\right)^2+\left(G(x,y)-\frac{1}{2\lambda}\right)^2+\left(T(y)-\frac{1}{2\lambda}\right)^2=\frac{3}{4\lambda^2} \tag{6}$$

From Theorem (1.1) together with the relation (6) we obtain

$$\operatorname{Max}\left(H(x)+G(x,y)+T(y)-\frac{3}{2\lambda}\right)=\frac{3}{2|\lambda|} \tag{7}$$

From the relationship (6) we obtain

doi.org/10.32388/PMUDR7

$$\begin{cases} H(x) - \dfrac{1}{2\lambda} = \dfrac{\sqrt{3}}{2|\lambda|} b_1 \\ G(x,y) - \dfrac{1}{2\lambda} = \dfrac{\sqrt{3}}{2|\lambda|} b_2 \\ T(y) - \dfrac{1}{2\lambda} = \dfrac{\sqrt{3}}{2|\lambda|} b_3 \end{cases} \tag{8}$$

where $b_1 = b_1(x,y),\ b_2 = b_2(x,y),\ b_3 = b_3(x,y),$ and $b_1^2 + b_2^2 + b_3^2 = 1$. The maximum reached in (7) is when $b_1 = b_2 = b_3 = \pm\dfrac{1}{\sqrt{3}}$. For $b_1 = b_2 = b_3 = \dfrac{1}{\sqrt{3}}$ and $\lambda > 0$, we obtain from (8) the following

$$H(x) = G(x,y) = T(y) = \frac{1}{\lambda} \tag{9}$$

For $\lambda < 0$ we obtain

$$H(x) = G(x,y) = T(y) = 0 \tag{10}$$

Similarly for $b_1 = -\dfrac{1}{\sqrt{3}},\ b_2 = b_3 = \dfrac{1}{\sqrt{3}}$ we obtain from (8) the following:

For $\lambda > 0 : H(x) = 0, G(x,y) = \frac{1}{\lambda}, T(y) = \frac{1}{\lambda}$.

For $\lambda < 0 : \quad H(x) = \frac{1}{\lambda}, \quad G(x,y) = 0, \quad T(y) = 0.$

The other cases are similar and the only relation that satisfies the relation (7) are the relations (9) and (10). □

Theorem 2.2. *Let $V = f_1(x)f_2(y)f_3(z)$, where V is constant and $f_i > 0$, then the minimum of the function*

$$S(x,y,z) = f_1(x)f_2(y) + 2f_1(x)f_2(y) + 2f_2(y)f_3(z)$$

is given by

$$\text{Min}\, S = 3\sqrt[3]{4V^2}$$

Proof. Taking into account the expression of V,

$$S = V\left[\frac{1}{f_3(z)} + \frac{2}{f_2(y)} + \frac{2}{f_1(x)}\right]. \tag{11}$$

Linking this equality with the relationship

$$a + b + c = \lambda\left[a^2 + b^2 + c^2\right] \tag{12}$$

where $a = \frac{1}{f_3(z)}, b = \frac{2}{f_2(y)}, c = \frac{2}{f_1(x)}$, the following is obtained from the relationship (12):

$$\left(\frac{1}{f_3(z)} - \frac{1}{2\lambda}\right)^2 + \left(\frac{2}{f_2(y)} - \frac{1}{2\lambda}\right)^2 + \left(\frac{2}{f_1(x)} - \frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2} \tag{13}$$

where $\lambda > 0,\ \lambda = \lambda(x,y,z)$.

Parameterizing the relationship (13) we obtain

$$\frac{1}{f_3(z)} = \frac{\sqrt{3}b_1 + 1}{2\lambda},\ \frac{2}{f_2(y)} = \frac{\sqrt{3}b_2 + 1}{2\lambda},\ \frac{2}{f_1(x)} = \frac{\sqrt{3}b_3 + 1}{2\lambda}, \tag{14}$$

doi.org/10.32388/PMUDR7

where $b_1^2+b_2^2+b_3^2=1$ y $b_i=b(x,y,z)$.

According to the Theorem (1.1), applying to the relation (13) we obtain

$$\text{Max}\left(\frac{1}{f_3(z)}-\frac{1}{2\lambda}+\frac{2}{f_2(y)}-\frac{1}{2\lambda}+\frac{2}{f_1(x)}-\frac{1}{2\lambda}\right)=\text{Max}\left(\frac{1}{f_3(z)}+\frac{2}{f_2(y)}+\frac{2}{f_1(x)}-\frac{3}{2\lambda}\right)=\frac{3}{2\lambda}\ (15)$$

It can be deduced from the relation (14), that the maximum reached in (15) is when $b_1=b_2=b_3=\frac{1}{\sqrt{3}}$. Therefore, we have in (14) the following:

$$f_3(z)=\lambda,\quad f_2(y)=2\lambda,\quad f_1(x)=2\lambda \tag{16}$$

Thus we have $V=4\lambda^3$, which implies that $\lambda=\sqrt[3]{\frac{V}{4}}$. Using this fact and using the relation (16) in (11) we obtain:

$$\text{Min } S=V\left[\frac{1}{\lambda}+\frac{2}{2\lambda}+\frac{2}{2\lambda}\right]=\frac{6V}{2\lambda}=\frac{3V}{\lambda}=3\sqrt[3]{4V^2}$$

□

Theorem 2.3. *Let $g:[a,b]\to\mathbb{R}$ and $F:[a,b]\times[c,d]\to\mathbb{R}$ be continuous functions, there exists $\lambda\in\mathbb{R}$ such that the maximum of the function $H(x,y)=g(x)F(x,y)$ occurs when*

$$|F(x,y)|=1\quad\text{and }\ln|g(x)|=-\frac{1}{\lambda}\text{ or}$$

$$\ln|F(x,y)|=\frac{1}{\lambda}\text{ and }|g(x)|=1$$

Proof. Let $A=\ln|F(x,y)|,\quad B=-\ln|g(x)|$, using the relation:

$$A+B=\lambda\left[A^2+B^2\right]$$

we obtain

$$\ln|F(x,y)|-\ln|g(x)|=\lambda\left[\ln^2|F(x,y)|+\ln^2|g(x)|\right] \tag{17}$$

where $\lambda=\lambda(x,y)$; from the relation (17) we obtain

$$\left(\ln|F(x,y)-\frac{1}{2\lambda}\right)^2+\left(\ln|g(\lambda)|+\frac{1}{2\lambda}\right)^2=\frac{1}{2|\lambda|^2}. \tag{18}$$

Using Theorem (1.1) in (18), we obtain:

$$\text{Max}\left(\ln|F(x,y)|-\frac{1}{2\lambda}+\ln|g(x)|+\frac{1}{2\lambda}\right)=\text{Max}(\ln|F(x,y)\|g(x)|)=\frac{1}{|\lambda|}. \tag{19}$$

From the relationship (17) we obtain:

$$\begin{aligned}\ln|F(x,y)|-\frac{1}{2\lambda}&=\frac{\sqrt{2}}{2\,|\lambda|}b_1\\ \ln|g(x)|+\frac{1}{2\lambda}&=\frac{\sqrt{2}}{2\,|\lambda|}b_2\end{aligned} \tag{20}$$

where $b_1^2+b_2^2=1,\quad b_1=b_1(x,y),\quad b_2=b_2(x,y)$.

The maximum reached in (19) is obtained when

$$b_1 = b_2 = \pm\frac{1}{\sqrt{2}}. \tag{21}$$

For $b_1 = b_2 = +\frac{1}{\sqrt{2}}$ and $\lambda > 0$ we obtain the following from (20)

$$\ln|F(x,y)| = \frac{1}{\lambda} \quad \text{and} \quad \ln|g(x)| = 0 \tag{22}$$

For $b_1 = b_2 = \frac{1}{\sqrt{2}}$ and $\lambda < 0$ we have from relation (15) that

$$\ln|F(x,y)| = 0 \quad \text{and} \quad \ln|g(x)| = \frac{-1}{\lambda} \tag{23}$$

The relations (22) and (23) verify the relation.

For $b_1 = \frac{1}{\sqrt{2}}$, $b_2 = -\frac{1}{\sqrt{2}}$ and $\lambda < 0$ we obtain from (20) the following:

$$\ln|F(x,y)| = \frac{1}{\lambda} \quad \text{and} \quad \ln|g(x)| = -\frac{1}{\lambda} \tag{24}$$

For $b_1 = \frac{1}{\sqrt{2}}$, $b_2 = -\frac{1}{\sqrt{2}}$ and $\lambda < 0$, we obtain from (20) the next:

$$\ln|F(x,y)| = 0 \quad \text{and} \quad \ln|g(x)| = 0 \tag{25}$$

Of the relations (24) and (25) none verify the relation (19). Similar analysis for the other cases □

Theorem 2.4. *Let $F(x,y,z) = f(x)g^3(y)h^2(z)$ be a continuous function, where f, g and h are real functions of a real variable. If $f(x) + g(y) + h(z) = a$ where $a > 0$, $f(x) > 0$, $g(y) > 0$ and $h(z) > 0$, then the maximum of the function $F(x,y,z)$ is $\text{Max}\, F(x,y,z) = \frac{a^6}{432}$, and is reached when $f(x) = \frac{a}{6}$, $g(y) = \frac{a}{2}$ and $h(z) = \frac{a}{3}$.*

Proof. Let $A = f(x)$, $B = g(y)$, $C = h(z)$. Using the relationship $A + B + C = \lambda[A^2 + B^2 + C^2]$ we obtain

$$f(x) + g(y) + h(z) = \lambda[f^2(x) + g^2(y) + h^2(z)], \tag{26}$$

where $\lambda > 0$, $\lambda = \lambda(x,y,z)$ and $f(x) < h(z) < g(y)$ in order to obtain the desired maximum of F.

From the relationship (26) and the problem data we obtain:

$$a = \lambda\left[a^2 - 2f(x)g(y) - 2(a - f(x) - g(y))(f(x) + g(y))\right]. \tag{27}$$

Let

$$U(x,y) = \frac{f(x) + g(y)}{2}, \quad V(x,y) = \frac{g(y) - f(x)}{2}. \tag{28}$$

Replacing (28) into (27) we have

$$\frac{a}{2\lambda} - \frac{a^2}{6} = V^2(x;y) + 3\Big(U(x;y) - \frac{a}{3}\Big)^2. \tag{29}$$

From the relation (26) we obtain

$$\text{Max}\, \lambda = \frac{3}{a} \tag{30}$$

From the relation $f(x) < h(z) < g(y)$ and the restriction $f(x) + g(y) + h(z) = a$ we obtain

$$f(x) < \frac{a}{3} \quad \text{and} \quad g(y) > \frac{a}{3}. \tag{31}$$

On the other hand, from (29) we obtain $\text{Max}\, V^2(x,y)$ when

$$U(x,y) = \frac{a}{3} \tag{32}$$

Replacing this last equality in (28) we have

$$f(x) + g(y) = \frac{2a}{3}, \tag{33}$$

and from here together with the relationship $f(x) + g(y) + h(z) = a$ we obtain that

$$h(z) = \frac{a}{3}. \tag{34}$$

From the relation (33) we obtain the following based on the relation (31)

$$g(y) - \frac{a}{3} = \frac{a}{3} - f(x) \tag{35}$$

Using Theorem 2.3 we see that the maximum of the function $f(x)\left[-1 + \frac{a}{3f(x)}\right]$ is given by

$$|f(x)| = 1 \text{ or } \left|-1 + \frac{a}{3f(x)}\right| = 1. \tag{36}$$

De la relacion (36) se obtiene $-1 + \frac{a}{3f(x)} = 1$, de esta ultima relacion se obtiene:

$$f(x) = \frac{a}{6}. \tag{37}$$

From the relation (35) and (37) we obtain

$$g(y) = \frac{a}{2}. \tag{38}$$

Therefore, from the relations (34), (37) and (38) we obtain

$$\text{Max}\ F(x,y,z) = \frac{a}{6} \times \frac{a^3}{8} \times \frac{a^2}{9} = \frac{a^6}{432}$$

□

Remark 2.1. *In (35), note that* $g(y) - \frac{a}{3} = \frac{g(y)}{3}\left(3 - \frac{a}{g(y)}\right)$*, then we can apply the Theorem 2.3 and thus, the*

$$\text{Max}\left\{\frac{g(y)}{3}\left(3 - \frac{a}{g(y)}\right)\right\}$$

occurs when $\left|\frac{g(y)}{3}\right| = 1$ *or* $\left|3 - \frac{a}{g(y)}\right| = 1$*, obtaining* $g(y) = \frac{a}{2}$*. The expression* $\frac{a}{4}$ *is discarded, since* $g(y) > \frac{a}{3}$*.*

Theorem 2.5. *For* $\overline{x} \in \mathbb{R}^3$*, the extrema of function* $F(\overline{x}) = f_1^2(x) + f_2^2(y) + f_3^2(z)$ *where* $f_1(x)$*,* $f_2(y)$ *and* $f_3(z)$ *are continuous functions, subject to the condition*

$$\begin{cases} \frac{f_1^2(x)}{A^2} + \frac{f_2^2(y)}{B^2} + \frac{f_3^2(z)}{C^2} = 1, \ 0 < A < B < C \\ f_1(x) + f_2(y) = f_3(z) \end{cases}$$

are given by

$$\text{Max}\, F(\bar{x}) = \frac{2A^2B^2\left(3C^2+1\right)}{4A^2B^2 + B^2(C-1)^2 + A^2(C+1)^2}, \ \text{Min}\, F(\bar{x}) = \frac{26A^2B^2C^2}{16B^2C^2 + 9A^2C^2 + A^2B^2}$$

Proof. Let

$$\tilde{f}_1(x) = \frac{f_1(x)}{A}, \ \tilde{f}_2(y) = \frac{f_2(y)}{B}, \ \tilde{f}_3(z) = \frac{f_3(z)}{C}. \tag{39}$$

From the expression (39) in the conditions of the problem we have

$$F(\bar{x}) = A^2\tilde{f}_1^2(x) + B^2\tilde{f}_2^2(x) + C^2\tilde{f}_3^2(z) \tag{40}$$

and the constraints are written as

$$\tilde{f}_1^2(x) + \tilde{f}_2^2(x) + \tilde{f}_3^2(z) = 1 \text{ y } A\tilde{f}_1(x) + B\tilde{f}_2(y) = C\tilde{f}_3(z). \tag{41}$$

Let's assume that $f_1(x), f_2(y), f_3(z)$ are positive, from the relation (41) suppose that

$$A\tilde{f}_1(x) < B\tilde{f}_2(y) \tag{42}$$

From the relation (41) and (42) we obtain

$$B\tilde{f}_2(y) - \frac{c}{2}\tilde{f}_3(z) = \frac{c}{2}\tilde{f}_3(z) - A\tilde{f}_1(x). \tag{43}$$

From the relation (43) and using the Theorem 2.3 we have

$$2A\tilde{f}_1(x) = (C-1)\tilde{f}_3(z), \ C \neq 1. \tag{44}$$

From the relation (44) and (41) obtenemos

$$\tilde{f}_2(y) = \tilde{f}_3(z)\frac{[C+1]}{2B} \tag{45}$$

From the relation (44) and (45) in (41) we obtain

$$\tilde{f}^2(z) = \frac{4A^2B^2}{B^2(C-1)^2 + A^2(C+1)^2 + A^2B^2} \tag{46}$$

From the relations (44), (45) and (46) in (40) we obtain

$$\text{Max}\ F(\bar{x}) = \frac{2A^2B^2\left(3C^2+1\right)}{4A^2B^2 + B^2(C-1)^2 + A^2(C+1)^2} \tag{47}$$

Of all the possible variants on the sign of the functions $\tilde{f}_1(x), \tilde{f}_2(y), \tilde{f}_3(z)$ the following is deduced; for $\tilde{f}_1(x) > 0$, $\tilde{f}_2(y) < 0$ and $\tilde{f}_3(z) > 0$ we have the relation (41)

$$A\tilde{f}_1(x) = C\tilde{f}_3(z) - B\tilde{f}_2(y) \tag{48}$$

doi.org/10.32388/PMUDR7

From the relation (48) assuming that $C\tilde{f}_3(z) < -B\tilde{f}_2(y)$ we obtain from (48) the next

$$\frac{A}{2}\tilde{f}_1(x) > C\tilde{f}_3(z) \tag{49}$$

From the relation (48) and (49) we have

$$\frac{A}{2}\tilde{f}_1(x) - C\tilde{f}_3(z) = -\frac{A}{2}\tilde{f}_1(x) - B\tilde{f}_2(y), \tag{50}$$

on the other hand the relation (50) can also be written as follows

$$C\tilde{f}_3(z)\left[-1 + \frac{A\tilde{f}_1(x)}{2C}\right] = -\frac{A}{2}\tilde{f}_1(x) - B\tilde{f}_2(y) \tag{51}$$

Applying Theorem 2.3 to the product $C\tilde{f}_3(z)\left[-1 + \frac{A\tilde{f}_1(x)}{2C}\right]$ we get

$$\tilde{f}_1(x) = \frac{4C}{A}\tilde{f}_3(z) \tag{52}$$

From the relation (52) and (48) we obtain

$$\tilde{f}_2(y) = -\frac{3C}{B}\tilde{f}_3(z) \tag{53}$$

Replacing the relations (52), (53) into (41) we obtain

$$\tilde{f}_3(z) = \frac{A^2B^2}{16B^2C^2 + 9A^2C^2 + A^2B^2} \tag{54}$$

Finally, using the relations (52), (53) and (54) in (40) gives us

$$\operatorname{Min} F(\bar{x}) = \frac{26A^2B^2C^2}{A^2B^2 + 9A^2C^2 + 16B^2C^2}.$$

The other assumption $C\bar{f}_3(z) \geq -B\bar{f}_2(y)$, does not lead to optimal values of $F(\bar{x})$. □

Various examples are illustrated below, which are applications of these Theorems. Other problems not related to these Theorems are solved following the ideas described in these theorems presented.

Remark 2.2. *To find the extremes of a function that is not continuous, it is still possible to use the technique shown in the various theorems and examples. Below we present an example that could be a starting point for such a study.*

Example 2.1. [5] *Find the maximum of the function $f(x,y) = [x] + \left[y^2\right] + [x]\left[y^2\right]$ subject to the condition*

$$2x + y = 6, \quad 1 \leq x, \quad 0 < y. \tag{55}$$

Solution.

It is *clear that $1 \leq x < y$. Suppose that $2x < y$, from here, together the relation (55) we obtain*

$$y - 3 = 3 - 2x \tag{56}$$

We use the Theorem in 2.3 in the relation (56)

$$\operatorname{Max}(y-3) = \operatorname{Max}(3-2x) \tag{57}$$

doi.org/10.32388/PMUDR7

From the relation (57) we obtain $y-3=\pm 1$, which implies that $y=4$ or $y=2$. Replacing in (56) we have the following points $P_1=(1,4)$, $P_2=(2,2)$. We discard P_2, since $2\cdot 2\not< 2$.

Another way to write $y-3$ in (56) is $y-3=3\left(-1+\frac{y}{3}\right)$, and so on

$$\operatorname{Max} 3\left(-1+\frac{y}{3}\right)=\operatorname{Max}(3-2x) \tag{58}$$

From (58) together with Theorem 2.3, we obtain $-1+\frac{y}{3}=\pm 1$, which implies $y=6$, which is false, since $x=0$ is not possible due to the relation (55).

Of all the possible variants we have $y=4$ and $x=1$. The other assumption is $y<2x$, which does not lead to optimal values of y. Therefore $\operatorname{Max} f(x,y)=33$.

Example 2.2. [5] *Find the extrema of the function $\operatorname{Max}\left(\frac{x}{a}+\frac{y}{b}\right)$ subject to the constraint: $x^2+y^2=1$.*

Solution.

By Theorem 1.2, we have

$$x+y=\lambda\left[x^2+y^2\right] \tag{59}$$

Using the condition of the problem, we have

$$x+y=\lambda \tag{60}$$

Completing squares in (59) we have

$$x-\frac{1}{2\lambda}=\frac{b_1}{\sqrt{2}|\lambda|} \tag{61}$$

$$y-\frac{1}{2\lambda}=\frac{b_2}{\sqrt{2}|\lambda|} \tag{62}$$

where $b_1^2+b_2^2=1$.

From the relation (61) and (62), for $\lambda>0$ we obtain

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{2\lambda}\left[\frac{\sqrt{2}b_1+1}{a}+\frac{\sqrt{2}b_2+1}{b}\right]. \tag{63}$$

The relation (63) can be written as

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{2\lambda}\left((\sqrt{2}b_1+1,\sqrt{2}b_2+1)\cdot\left(\frac{1}{a},\frac{1}{b}\right)\right) \tag{64}$$

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{2\lambda}\sqrt{4+2\sqrt{2}\,(b_1+b_2)}\cdot\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\cos\theta \tag{65}$$

Also from the relation (61) and (62) we obtain:

$$\lambda^2-1=\frac{b_1+b_2}{\sqrt{2}} \tag{66}$$

From the relations (65) and (66) we obtain:

doi.org/10.32388/PMUDR7

$$\text{Max}\left(\frac{x}{a}+\frac{y}{b}\right)=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} \tag{67}$$

Example 2.3. [6] *Find the Maxima and Minima of* $h(x;y)=x^2+y^2+z^2$*, subject to the conditions:*

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \quad \text{and} \quad x+y=z$$

Solution.

We use Theorem 2.5 with $f_1(x)=x$*,* $f_2(y)=y$*,* $f_3(z)=z$*,* $A=2, B=\sqrt{5}, C=5$ *with which we have*

$$\text{Max}\, F(\overline{x})=10$$

$$\text{Min}\, F(\overline{x})=4.4520547945$$

According to Lagrange's method the $\max\left(x^2+y^2+z^2\right)=10$ *and the* $\min\left(x^2+y^2+z^2\right)=\frac{75}{17}$.

Example 2.4. [6] *A rectangular box without a lid must have a volume of 32 cubic units, what must be the dimensions so that the total surface area is minimal?.*

Solution.

To solve this problem we follow what is described in Theorem 2.2.

If x, y, z *are the edges, we have*

(i) Box volume $V=xyz=32$

(ii) Box surface $S=xy+2xz+2yz$

Of these relations, we have:

$$S=V\left[\frac{1}{z}+\frac{2}{y}+\frac{2}{x}\right] \tag{68}$$

In the relation (68) we use the technique

$$a+b+c=\lambda\left[a^2+b^2+c^2\right]$$

In this case take $a=\frac{1}{z}, b=\frac{2}{y}, c=\frac{2}{x}$*, therefore we have*

$$\frac{1}{z}+\frac{2}{y}+\frac{2}{x}=\lambda\left[\frac{1}{z^2}+\frac{4}{y^2}+\frac{4}{x^2}\right]$$

Completing squares we have

$$\left(\frac{1}{z}-\frac{1}{2\lambda}\right)^2+\left(\frac{2}{y}-\frac{1}{2\lambda}\right)^2+\left(\frac{2}{x}-\frac{1}{2\lambda}\right)^2=\frac{3}{4\lambda^2}$$

For this last relation we have that $\lambda>0$ *and,*

$$\frac{1}{z}-\frac{1}{2\lambda}=\frac{\sqrt{3}}{2\lambda}b_1$$

$$\frac{2}{y} - \frac{1}{2\lambda} = \frac{\sqrt{3}}{2\lambda} b_2 \tag{69}$$

$$\frac{2}{x} - \frac{1}{2\lambda} = \frac{\sqrt{3}}{2\lambda} b_3$$

where

$$b_1^2 + b_2^2 + b_3^2 = 1$$

From the relation (69) we have that z, y, x *must be minimum, therefore,* b_1,b_2,b_3 *and must have maximum values simultaneously. This happens when* $b_1 = b_2 = b_3 = \frac{1}{\sqrt{3}}$. *Then in (69) we have*

$$z = \lambda, \quad y = 2\lambda, \quad x = 2\lambda$$

Substituting these last relations in (i) we have $4\lambda^3 = 32$*, that is,* $\lambda = 2$.

Then the Minimum surface in (ii) is:

$$S = 32\left[\frac{1}{2} + \frac{2}{4} + \frac{2}{4}\right]$$

$$S = 32 \times \frac{3}{2} = 48$$

Example 2.5. [6] *What is the maximum volume of the rectangular parallelepiped that can be inscribed in the ellipsoid* $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{36} = 1$?

Solution.

The volume of the parallelepiped is:

$$V = 8xyz \tag{70}$$

where (x, y, z) belongs to the ellipsoid.

Let

$$x = 3\tilde{x}, y = 4\tilde{y}, \quad z = 6\tilde{z} \tag{71}$$

After (70) and (71) we obtain:

$$V = 72 \times 8\tilde{x}\tilde{y}\tilde{z} \tag{72}$$

where $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1$.

Now, using the relationship

$$\tilde{x} + \tilde{y} + \tilde{z} = \lambda\left[\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2\right]$$

is obtained

$$\left(\tilde{x} - \frac{1}{2\lambda}\right)^2 + \left(\tilde{y} - \frac{1}{2\lambda}\right)^2 + \left(\tilde{z} - \frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2}$$

So, you have to

$$\tilde{x} - \frac{1}{2\lambda} = \frac{\sqrt{3}b_1}{2\lambda}$$

$$\tilde{y} - \frac{1}{2\lambda} = \frac{\sqrt{3}}{2\lambda}b_2$$

$$\tilde{z} - \frac{1}{2\lambda} = \frac{\sqrt{3}}{2\lambda}b_3$$

where

$$b_1^2 + b_2^2 + b_3^2 = 1 \tag{73}$$

, and also

$$\tilde{x} = \frac{\sqrt{3}b_1 + 1}{2\lambda}, \tilde{y} = \frac{\sqrt{3}b_2 + 1}{2\lambda}, z = \frac{\sqrt{3}b_3 + 1}{2\lambda}, \tag{74}$$

It is observed that $\tilde{x}, \tilde{y}$ and $\tilde{z}$ sare maximum real values if b_1, b_2 and b_3 are maximum and that happens when $b_1 = b_2 = b_3 = \frac{1}{\sqrt{3}}$. Therefore

$$\tilde{x} = \frac{1}{\lambda}, \tilde{y} = \frac{1}{\lambda}, \tilde{z} = \frac{1}{\lambda} \tag{75}$$

From the relation (73) and (75) we have that

$$\frac{3}{\lambda^2} = 1$$

and replacing this in (72) we have that the maximum volume is

$$V_{max} = 72 \times 8.\frac{1}{\lambda^3} = \frac{72 \times 8v}{3\sqrt{3}} = \frac{24 \times 8\sqrt{3}}{3} = 64\sqrt{3}u^3$$

Example 2.6. [6] *Find the distance from point $P_0(a, b, c)$ to the plane of equation $P : Ax + By + Cz = D$*

Solution.

We *have to*

$$d(P_0, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}, \text{ where } Q = (x, y, z) \in P. \tag{76}$$

Defining $f : \mathbb{R}^3 \to \mathbb{R}$ as

$$f(x_1, x_2, x_3) = (x-a)x_1 + (y-b)x_2 + (z-c)x_3 \tag{77}$$

we have that f Defining $f : \mathbb{R}^3 \to \mathbb{R}$ as

$$f(x_1, x_2, x_3) = (x-a)x_1 + (y-b)x_2 + (z-c)x_3 \tag{78}$$

we have that f is continuous and also

$$\mathbb{R}^3 = \ker f \oplus [\ker f]^{\perp}. \tag{79}$$

Since $(1, 1, 1) \in \mathbb{R}^3$ we have from (77) and (79)

doi.org/10.32388/PMUDR7

$$(1,1,1) = \lambda_1 f + \lambda_2 = u + v, u \in \ker f, v \in [\ker f]^{\perp}$$

$$f(1,1,1) = f(u) + f(v) = \lambda \left[(x-a)^2 + (y-b)^2 + (z-c)^2 \right]. \tag{80}$$

That is

$$(x-a) + (y-b) + (z-c) = \lambda \left[(x-a)^2 + (y-b)^2 + (z-c)^2 \right]. \tag{81}$$

From the relation (81) we obtain

$$\begin{aligned} x - a - \tfrac{1}{2\lambda} &= \tfrac{\sqrt{3}}{2|\lambda|} b_1 \\ y - b - \tfrac{1}{2\lambda} &= \tfrac{\sqrt{3}}{2|\lambda|} b_2 \\ z - c - \tfrac{1}{2\lambda} &= \tfrac{\sqrt{3}}{2u1} b_3 \end{aligned} \tag{82}$$

where $b_1^2 + b_2^2 + b_3^2 = 1$*, from the relation (64) is obtained*

$$\begin{aligned} Ax - aA - \tfrac{A}{2\lambda} &= \tfrac{\sqrt{3}}{2|\lambda|} b_1 A \\ By - bB - \tfrac{B}{2\lambda} &= \tfrac{\sqrt{3}}{2|\lambda|} b_2 B \\ Cz - cC - \tfrac{C}{2\lambda} &= \tfrac{\sqrt{3}}{2|\lambda|} b_3 C \end{aligned} \tag{83}$$

From the relation (83) we have

$$Ax + By + Cz - aA - bB - cC = \frac{\sqrt{3}b_1 A + \sqrt{3}b_2 B + \sqrt{3}b_3 C + A + B + C}{2\lambda} \tag{84}$$

This is if $\lambda > 0$*. Since* $Q \in P$*, we have*

$$Ax + By + Cz = D \tag{85}$$

From (84) and (85) we obtain:

$$\lambda = \frac{A\left(\sqrt{3}b_1 + 1\right) + B\left(\sqrt{3}b_2 + 1\right) + C\left(\sqrt{3}b_3 + 1\right)}{2(D - aA - bB - cC)} \tag{86}$$

From the relations given in (82) we have

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = \frac{\left(\sqrt{3}b_1 + 1\right)^2 + \left(\sqrt{3}b_2 + 1\right)^2 + \left(\sqrt{3}b_3 + 1\right)^2}{4\lambda^2} \tag{87}$$

Therefore of (87) and (76) we obtain

$$d\left(P_0, Q\right) = \frac{\left[\left(\sqrt{3}b_1 + 1\right)^2 + \left(\sqrt{3}b_2 + 1\right)^2 + \left(\sqrt{3}b_3 + 1\right)^2\right]^{1/2}}{2|\lambda|} \tag{88}$$

From the relation (88) and (86) we have

$$d\left(P_0, Q\right) = \frac{\left[\left(\sqrt{3}b_1 + 1\right)^2 + \left(\sqrt{3}b_2 + 1\right)^2 + \left(\sqrt{3}b_3 + 1\right)^2\right]^{1/2} |D - aA - bB - cC|}{\left|A\left(\sqrt{3}b_3 + 1\right) + B\left(\sqrt{3}b_2 + 1\right) + C\left(\sqrt{3}b_3 + 1\right)\right|}. \tag{89}$$

Where

doi.org/10.32388/PMUDR7

$$\cos\theta = \frac{\left\langle (A,B,C), \left(\sqrt{3}b_1+1, \sqrt{3}b_2+1, \sqrt{3}b_3+1\right)\right\rangle}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{\left(\sqrt{3}b_1+1\right)^2+\left(\sqrt{3}b_2+1\right)^2+\left(\sqrt{3}b_3+1\right)^2}} \tag{90}$$

from the relation (89) and (90) we obtain

$$d\left(P_0, Q\right) = \frac{|D - aA - bB - cC|}{\sqrt{A^2+B^2+C^2}|\cos\theta|} \tag{91}$$

Example 2.7. [6] *Find the maximum of xy^2z^3, $si x+y+z=6,\ 0<x, 0<y, 0<z$.*

Solution.

Using the Theorem 2.4 with $f(x)=x, g(y)=y, h(z)=z, a=6$ we obtain

$$\operatorname{Max} F(x,y,z) = \frac{6^6}{432} = 108.$$

Example 2.8. [5] *Find the extremes of the function $f(x,y,z)=xyz$ subject to the conditions $x^2+y^2+z^2=1$ and $x+y=2z$.*

Solution.

Using the technique $a+b=\lambda\left[a^2+b^2\right]$ we get $a=x, b=y$

$$x+y=\lambda\left[x^2+y^2\right] \tag{92}$$

From the data and from (92) we obtain

$$2z=\lambda\left(1-z^2\right) \tag{93}$$

From the relation (93) we obtain

$$\left(z+\frac{1}{\lambda}\right)^2-\frac{1}{\lambda^2}=1 \tag{94}$$

From the relation (94) it is easy to see that

$$\operatorname{Min} z = 1-\frac{1}{\lambda} \quad y \quad \operatorname{Max} z = -\frac{1}{\lambda}-1, \quad \lambda\neq 0.$$

This relation leads to nothing since $\lambda=\infty$ is an absurdity in (92). Using the relation

$$x+y+z=\lambda\left[x^2-y^2+z^2\right] \tag{95}$$

we get from the data $3z=\lambda$, which implies that $z=\frac{1}{3}$. Then we have

$$\begin{aligned} x^2+y^2 &= 1-\tfrac{\lambda^2}{9} \\ x+y &= \tfrac{2\lambda}{3}. \end{aligned} \tag{96}$$

From the relation (96) we obtain:

$$\operatorname{Max}(x+y) = \sqrt{1-\frac{\lambda^2}{9}}\cdot\sqrt{2} = \frac{2\lambda}{3} \tag{97}$$

Solving we get $\lambda = \pm\sqrt{3}$.

- *If* $\lambda = +\sqrt{3}$ *we get* $z = \frac{1}{\sqrt{3}}, \quad x = \frac{1}{\sqrt{3}}, \quad y = \frac{1}{\sqrt{3}}$. *Therefore* $\max\{xyz\} = \frac{1}{3\sqrt{3}}$.
- *If* $\lambda = -\sqrt{3}$, *we get* $x = -\frac{1}{\sqrt{3}}, \quad y = -\frac{1}{\sqrt{3}}, \quad z = -\frac{1}{\sqrt{3}}$. *Therefore* $\min\{xyz\} = \frac{-4}{3\sqrt{3}}$.

Example 2.9. [5] *Find the highest point on the surface*

$$z = \frac{8}{3}x^3 + 4y^3 - x^4 - y^4.$$

Solution.

To solve this problem we use Theorem 2.3.

It is observed that

$$z = x^3\left(\frac{8}{3} - x\right) + y^3(4 - y) \tag{98}$$

Since x *and* y *are independent variables we have from (98)*

$$0 < x \leqslant \frac{8}{3} \quad y \quad 0 \leqslant y \leqslant 4. \tag{99}$$

Applying the technique, we have

$$\ln y^3 + \ln(4 - y) = \lambda\left[\left(\ln y^3\right)^2 + \ln(4 - y)\right] \tag{100}$$

From the relation (100) we obtain

$$\left(\ln y - \frac{1}{2\lambda}\right)^2 + \left(\ln(4 - y) - \frac{1}{2\lambda}\right)^2 = \frac{1}{2\lambda^2}. \tag{101}$$

From the relation (101) for $\lambda > 0$ *we obtain*

$$\ln y^3 - \frac{1}{2\lambda} = \frac{\sqrt{2}b_1}{2\lambda} \tag{102}$$

and so

$$\ln(4 - y) - \frac{1}{2\lambda} = \frac{\sqrt{2}b_2}{2\lambda}, \text{ where } b_1^2 + b_2^2 = 1, \lambda = \lambda(y) \tag{103}$$

The maximum value of $\ln y^3 + \ln(4 - y) = \ln(4 - y)y^3$ *is obtained when* $b_1 = b_2 = \pm\frac{1}{\sqrt{2}}$. *Of the four possibilities we obtain* $b_2 = -\frac{1}{\sqrt{2}}, b_1 = \frac{1}{\sqrt{2}}$. *This is true, since* y^3 *and* y *must have different hierarchies in* $y \in \langle 0, 4]$.

Therefore, $\ln(4 - y) = 0 \Leftrightarrow 4 - y = 1 \Leftrightarrow y = 3$. *For the variable* x *we have*

$$\ln x^3 + \ln\left(\frac{8}{3} - x\right) = \tau\left[\left(\ln x^3\right)^2 + \ln^2\left(\frac{8}{3} - x\right)\right]. \tag{104}$$

From the relation (104) we have for $\tau > 0$

$$
\begin{aligned}
\ln x^3 - \frac{1}{2\tau} &= \frac{\sqrt{2}b_1}{2\tau} \\
\ln\left(\frac{8}{3} - x\right) - \frac{1}{2\tau} &= \frac{\sqrt{2}b_2}{2\tau}
\end{aligned} \tag{105}
$$

The maximum value of $\ln x^3 + \ln\left(\frac{8}{3} - x\right)$ *is obtained when*

$$\tilde{b}_1 = \tilde{b}_2 = \pm\frac{1}{\sqrt{2}}$$

For $\tilde{b}_1 = \frac{1}{\sqrt{2}}, \quad \tilde{b}_2 = -\frac{1}{\sqrt{2}}$ *we get* $\frac{8}{3} - x = 1$*, which implies that* $x = \frac{5}{3}$*. Therefore, in (98) we get*

$$\max z = \frac{12.5}{27}\left(\frac{8}{3} - \frac{5}{3}\right) + 27(4-3) = 27 + 4.62962 = 31.629 \tag{106}$$

The maximum applying the superior calculus theory, we obtain:

$$\text{Max}\, z = 32.333$$

The error that is made is 0.7

Remark 2.3. *If instead of the equation (104) we put the following expression:*

$$\ln\frac{2}{3}x^3 + \ln\left(4 - \frac{3}{2}x\right) = \tau\left[\ln^2\frac{2}{3}x^3 + \ln^2\left(4 - \frac{3}{2}x\right)\right].$$

From this relationship, similar to what was done in (104) is obtained $x = 2$*. So we have*

$$max\{z\} = \frac{16}{3} + 27 = 32.333$$

Example 2.10. [5] *Determine the absolute maximum and minimum of the function*

$$z = \sin x + \sin y + \sin(x+y)$$

where $0 \leqslant x \leqslant \pi/2, \quad 0 \leqslant y \leqslant \pi/2$.

Solution.

Following what is described in Theorem 2.1 we have:

Using the relationship

$$A + B + C = \lambda\left[A^2 + B^2 + C^2\right] \tag{107}$$

where $A = \text{sen}\, x, B = \text{sen}\, y, C = \text{sen}(x+y), \quad \lambda = \lambda(x,y)$*. After replacing these values in the relation (107) we have*

$$\sin x + \text{sen}\, y + \text{sen}(x+y) = \lambda\left[\sin^2 x + \sin^2 y + \text{sen}^2(x+y)\right] \tag{108}$$

From the relation (108) we have

$$\sin x - \frac{1}{2\lambda} = \frac{\sqrt{3}b_1}{2\lambda}$$

doi.org/10.32388/PMUDR7

$$\sin y - \frac{1}{2\lambda} = \frac{\sqrt{3}b_1}{2\lambda} \tag{109}$$

$$\sin(x+y) - \frac{1}{2\lambda} = \frac{\sqrt{3}b_3}{2\lambda}$$

where $\lambda > 0$ *y* $b_1^2 + b_2^2 + b_3^2 = 1$.

Then

$$\text{Max}(\text{Sen}\, x + \text{Sen}\, y + \text{Sen}(x+y)) = \text{Max}\,\frac{\text{Cos}\,\theta + 3}{2\lambda} = \frac{3}{\lambda} \tag{110}$$

and this value is reached when $b_1 = b_2 = b_3 = \frac{1}{\sqrt{3}}$, *therefore from the relation (109) we have*

$$\text{Sen}\, x = \frac{1}{\lambda}, \quad \text{Sen}\, y = \frac{1}{\lambda}, \quad \text{Cos}\, y + \text{Cos}\, x = 1 \tag{111}$$

From the relation(111) we obtain

$$\frac{\sqrt{\lambda^2 - 1}}{\lambda} + \frac{\sqrt{\lambda^2 - 1}}{\lambda} = 1. \tag{112}$$

Solving the relation obtained in (112) we have

$$\lambda = \pm \frac{2}{\sqrt{3}} \tag{113}$$

From the relation (110) y (113) we have that $maxz = \frac{3\sqrt{3}}{2}$. *Since* $x \in [0, \pi/2]\,, y \in [0, \pi/2]$, *for* $\lambda \leqslant 0$, *we have that* $maxz = 0$, *since* $\operatorname{sen} x \geqslant 0, \operatorname{sen} y \geqslant 0, \operatorname{sen}(x+y) \geqslant 0$.

Note: The maximum given in (110) is correct, since x *and* y *have the same hierarchy in the interval* $[0, \pi/2]$

Example 2.11. [5] *Find the Maxima and Minima of the function* $z = x^3 + y^3 - 3xy, \quad 0 \leqslant x \leqslant 2, \quad -1 \leqslant y \leqslant 2.$

Solution.

We use Theorem 2.3 as indicated below. It is had that $z = x^3 + y\left(y^2 - 3x\right)$

Then we will use the following

$$\ln|y| + \ln\left|y^2 - 3x\right| = \lambda\left[\ln^2|y| + \ln^2\left|y^2 - 3x\right|\right]$$

$$\left(\ln|y| - \frac{1}{2\lambda}\right)^2 + \left(\ln\left|y^2 - 3x\right| - \frac{1}{2\lambda}\right)^2 = \frac{1}{2\lambda^2} \tag{114}$$

From the relation (114) obtain me for $\lambda > 0$

$$\ln|y| - \frac{1}{2y} = \frac{\sqrt{2}b_1}{2\lambda}$$

$$\ln\left|y^2 - 3x\right| - \frac{1}{2\lambda} = \frac{\sqrt{2}b_2}{2\lambda} \tag{115}$$

where $b_1^2 + b_2^2 = 1$. *Of the relation (115)* $\ln|y|\left|y^2 - 3x\right| = \ln y\left(y^2 - 3x\right) = \ln|y| + \ln\left|y^2 - 3x\right|$, *and so we have*

$$\text{Max}\ \ln y\left(y^2 - 3x\right) = \frac{1}{\lambda} \tag{116}$$

doi.org/10.32388/PMUDR7

The maximums or minimums reached are at the points $b_1 = b_2 = \pm\frac{1}{\sqrt{2}}$*, then we have* $\ln|y| = 0$*, therefore* $y = \pm 1$*. In the relation (115) we have:*

$$\ln^2\left|y^2 - 3x\right| = \frac{1}{\lambda} = \ln|1 - 3x| \tag{117}$$

From this last equality

$$\text{Max } \frac{1}{\lambda} = \ln 5 \tag{118}$$

So we have (118) and (116)

$$\text{Max } \ln y\left(y^2 - 3x\right) = \ln 5 \tag{119}$$

In addition $\text{Min } \ln|y|\left(y^2 - 3x\right)| = 0$ *occurs when* $y = -1$, $\left|y^2 - 3x\right| = 1, \quad x = 0.$

Therefore, $\text{Min } z = -1.$

Example 2.12. [5] *At what point of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*, the line tangent to this line forms the triangle of minor area?*

Solution.

We know that the tangent line to a circle of equation $x^2 + y^2 = 1$ *at the point* $P = (x_0, y_0)$ *is given by* $L_T : \sin y = -\frac{x_0}{y_0}x + B$*; where the slope is* $-\frac{x_0}{y_0}$*. As* $(x_0, y_0) \in L_T$ *we obtain* B*, that is to say*

$$L_T : \quad y = -\frac{x_0}{y_0}x + \frac{y_0^2 + x_0^2}{y_0} \tag{120}$$

Por lo tanto, la recta tangente a la elipse en el punto $Q = (x_0, y_0)$ *puede ser hallado usando la transformacion*

$$x = a\tilde{x}, \quad y = b\tilde{y} \tag{121}$$

Therefore, using the transformation (121) we obtain the circumference:

$$\tilde{x}^2 + \tilde{y}^2 = 1$$

From the relation (120) and (121) we obtain that the equation of the tangent line to the ellipse is given by

$$\tilde{y} = \frac{-\tilde{x}_0}{\tilde{y}_0}\tilde{x} + \frac{\tilde{x}_0^2 + \tilde{y}_0^2}{y_0} \tag{122}$$

As $x_0 = a\tilde{x}_0, \quad y_0 = b\tilde{y}_0$*, we obtain from (122) the following*

$$\frac{y}{b} = \frac{-x_0 b}{a y_0}\frac{x}{a} + \frac{\frac{y_0^2}{b^2} + \frac{x_0^2}{a^2}}{\frac{y_0}{b}} \tag{123}$$

and from the equation (123)

$$y = -\frac{b^2}{a^2}\frac{x_0}{y_0}x + \frac{b^2}{y_0} \tag{124}$$

So from the equation (124) we have that the area limited by the tangent line and the coordinate axes is

doi.org/10.32388/PMUDR7

$$S\left(x_0, y_0\right)=\frac{a^2 b^2}{2 x_0 y_0} \tag{125}$$

Using the described technique, we obtain that

$$A+B=\lambda\left[A^2+B^2\right]. \tag{126}$$

Let's put

$$A=\frac{x_0}{a}, B=\frac{y_0}{b}. \tag{127}$$

From the relation (126) and (127) we have

$$\frac{x_0}{a}+\frac{y_0}{b}=\lambda. \tag{128}$$

From (128) we obtain squaring $1+\frac{2 x_0 y_0}{2 b}=\lambda^2$*, which implies that*

$$x_0 y_0=\frac{a b}{2}\left(\lambda^2-1\right) \tag{129}$$

From the relation (126) and (127) we obtain

$$\begin{aligned} \frac{x_0}{2}-\frac{1}{2 \lambda}=\frac{\sqrt{2} b}{2 \lambda}, \lambda>0 \\ \frac{y_0}{b}-\frac{1}{2 \lambda}=\frac{\sqrt{2}}{2 \lambda} b_2 \end{aligned} \tag{130}$$

where $b_1^2+b_2^2=1$*. From the relation (130) and (128) we have*

$$\lambda-\frac{1}{\lambda}=\frac{\sqrt{2}}{2 \lambda}\left(b_2+b_1\right) \tag{131}$$

From the relation (131) we obtain

$$\operatorname{Max}\left(\lambda^2-1\right)=1 \tag{132}$$

From the relation (132) and (129) we obtain

$$\operatorname{Max} x_0 y_0=\frac{a b}{2}$$

Example 2.13. [5] *The courses of two Rivers (within the limits of a determined region) represent approximately a parabola,* $y=x^2$*, and a straight line,* $x-y-2=0$*. It is necessary to unite these rivers by means of a rectilinear channel that has the shortest possible length. For what points will it be necessary to draw them?*

Solution.

Let $P=(x, y)$ *be a point on the parabola and* $Q=(Z, w)$ *be a point on the line. The distance from* P *to* Q *is given by*

$$d(P, Q)=\sqrt{(x-z)^2+(y-w)^2}. \tag{133}$$

Therefore, the function to Mainimize is the one given by the equation (133) subject to the condition

$$y=x^2 \quad 1 \quad z-w-2=0 \tag{134}$$

doi.org/10.32388/PMUDR7

Let $a = z - x, \quad b = y - w$, *using the relation*

$$a + b = \lambda\left[a^2 + b^2\right] \tag{135}$$

we obtain

$$2 + x^2 - x = \lambda\left[(z - x)^2 + (y - \omega)^2\right]. \tag{136}$$

Also

$$\min\sqrt{(z - x)^2 + (y - \omega)^2} = \min\sqrt{\frac{(z + x^2 - x)\cdot 2}{2\lambda}} \tag{137}$$

From the relation (134) we have

$$\left(z - x - \frac{1}{2\lambda}\right)^2 + \left(y - w - \frac{1}{2\lambda}\right) = \frac{1}{2\lambda^2} \tag{138}$$

From the relation (138) we have that

$$\begin{aligned} z - x &= \frac{1 + \sqrt{2}b_1}{2\lambda} \\ y - w &= \frac{1 + \sqrt{2}b_2}{2\lambda} \end{aligned} \tag{139}$$

where $b_1^2 + b_2^2 = 1$. *From the relations in (139) we obtain*

$$2 + x^2 - x = \frac{2 + \sqrt{2}\,(b_1 + b_2)}{2\lambda} \tag{140}$$

After the relations (138) and (140) we obtain

$$\text{Min}\sqrt{(y - x)^2 + (y - w)^2} = \text{Min}\,\frac{\left(2 + x^2 - x\right)\sqrt{2}}{\sqrt{2 + \sqrt{2}\,(b_1 + b_2)}} \tag{141}$$

The minimum in (141) is given when $b_1 = \frac{1}{\sqrt{2}}, b_2 = \frac{1}{\sqrt{2}}$. *Therefore*

$$\begin{aligned} \text{Min}\sqrt{(z - x)^2 + (y - w)^2} &= \text{Min}\left(\frac{\left(2 + x^2 - x\right)}{2}\right)^{\sqrt{2}} \\ &= \text{Min}\,\frac{\left(\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}\right)}{2}\sqrt{2} \\ &= \frac{7}{8}\sqrt{2} \end{aligned} \tag{142}$$

and this happens when $x = \frac{1}{2}$; *therefore* $y = \frac{1}{4}$. *To calculate* (z, w) *in (142) we obtain*

$$\sqrt{\left(z - \frac{1}{2}\right)^2 + \left(\frac{1}{4} - (z - 2)\right)^2} = \frac{7}{8}\sqrt{2} \tag{143}$$

Solving this equation we get z.

doi.org/10.32388/PMUDR7

3. Conclusions

The different theorems obtained and illustrated examples show that the relationship $(a_1 + a_2 + \cdots + a_n) = \lambda\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)$. where $a_1, a_2, \cdots, a_n \in \mathbb{R}$ and is valid for some $\lambda \in \mathbb{R}$. Linking this relationship with the problem under study and using the Theorem 1.1, it is possible to obtain the desired results, taking into account the hierarchy of a variable over the other variables. It should be noted that the hierarchy of a variable depends on the correspondence rule of the function and its given domain.

We believe that the Theorems and examples shown are a starting point to create a general theory that allows us to find the conditional maxima and minima of real functions of a vector variable without said functions being differentiable and without placing emphasis on the given domain.

Acknowledgments

The authors thank God, the family, UNASAM and CONCYTEC for the partial financial support and also for providing us with a pleasant work environment.

Other References

- Groetsch, Charles W. "Elements of Applicable Functional Analysis," 2nd Printing Edition, Vol. 55. Marcel Dekker, 1980, págs. 1-300.

References

1. ^ *B. M. Cerna Maguiña, Dik D. Lujerio García, Héctor F. Maguiña, "Some Results on Number Theory and Differential Equations," Mathematics and Statistics, Vol. 9, No. 6, pp. 984-993, 2021. DOI: 10.13189/ms.2021.090614.*
2. ^ *B. M. Cerna Maguiña, Janet Mamani Ramos, "Some Results on Integer Solutions of Quadratic Polynomials in Two Variables," Mathematics and Statistics, Vol. 9, No. 6, pp. 931-938, 2021. DOI: 10.13189/ms.2021.090609.*
3. ^ *Maguiña, B. M. Cerna, H. Blas, and V. H. López Solís, "Some results on natural numbers represented by quadratic polynomials in two variables," In Journal of Physics: Conference Series, vol. 1558, no. 1, p. 012011. IOP Publishing, 2020. DOI: 10.1088/1742-6596/1558/1/012011.*
4. ^ *Maguiña, B. M. Cerna, "Some results on prime numbers," International Journal of Pure and Applied Mathematics 118, no. 3, págs. 845-851, 2018. DOI: 10.12732/ijpam.v118i3.29.*
5. a, b, c, d, e, f, g, h *Demidovich, B. P. "Problemas y ejercicios de análisis matemático." España: Ediciones Paraninfo, 1980.*
6. a, b, c, d, e *Spiegel, Murray R. "Cálculo Superior," México D.F: McGraw-Hill, 2004.*

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.