Review of: "Riemann Hypothesis on Grönwall's Function"

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Potential competing interests: No potential competing interests to declare.

This short note builds on elementary approaches to the Riemann Hypothesis. The central idea is Robin's Theorem, which states that the Riemann Hypothesis is equivalent to the inequality $\sigma(n) < e^\gamma n \log \log n$ for all $n > 5040$. It is common to transform Robin's Theorem into other elementary-seeming ideas equivalent to RH.

The ideas in this note are most closely related perhaps to Nazardonyavi and Yakubovich's*Extremely Abundant Numbers and the Riemann Hypothesis*, published in the Journal of Integer Sequences (2014). Some of the terminology and aims seem to have been adopted from there.

The “Central Lemma” in this note is unfortunately ambiguously phrased and not quite correct as written. The lemma states that if x is sufficiently large and $y > x$, then $\frac{y}{x} \gg \frac{\log y}{\log x}$, where $\gg$ here means “much greater than”. Note that the inequality rearranges to $\frac{y}{\log y} \gg \frac{x}{\log x}$. Replacing $\gg$ with a mere inequality for a moment, we look at $\frac{y}{\log y} > \frac{x}{\log x}$ for $y > x$. This is true for $y > x \geq e$, which follows immediately from the fact that the function $F(x) = \frac{x}{\log x}$ is an increasing function for $x \geq e$. I also note that this leads to a shorter proof of the underlying (not with $\gg$) inequality than the one given.

But $F(x)$ is also continuous. This means that for any $\epsilon > 0$, no matter how large x is, for y close enough to x we will have $\lvert F(y) - F(x) \rvert < \epsilon$. Specifically, no matter how large x is, one can choose y such that $y > x$ and $\frac{y}{x}$ is arbitrarily close to $\frac{\log y}{\log x}$, disproving the central lemma.

There are certain convexity-type arguments that can be made, as $F''(x) > 0$, but $F''(x)$.

I did not read any of the remaining proofs closely.