Qeios

An Added Proof of the "Trace Anomaly Redefinition": Equivalent Wick Rotation Conditions

The Chiral Trace Anomaly cancellation in Gravitational Curved Spaces necessitated, due to the presence of the metric inside the action integral, that it is needed for both the “volume” and the “surface” terms, which can both be explicit functions of time, to be resolved and also sufficiently to be brought down by one more dimension.
This leads into a description through orbits (close to Euclidean), allowing for a generalization of the Wick rotation and confirming the Trace anomaly redefinition already exposed.

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1. Introduction

As the resolution for the redefinition of the Trace, treated in [1], was very abstract and concise, a more proving illustration was needed. And that is a bit described along the concluded necessities and sufficiencies cited in the Appendix.

Since any external line does not exhibit an anomaly (as easily countered by direct duality), the non-trivial (containing the integration weighting metric) task of internal lines propagators, which are a mere description of an internal wave propagation also described by an effective and local Hamiltonian, under the Fourier Transform, becomes apparent.

That is where the interferences occur, and more precisely, coherence should be well-defined; then further, too, at the periphery of time (as an infinity or a fixed mixing phase point), such that the Hilbert space needs so a justification of existence and independence, under these arbitrary waves.

For the decaying at infinity space whose duality matches between the original and the final states, it is proven as necessity to be chosen at the surface terms, via an illustration.

While for the decaying mixed structures that show it is sufficiently existent as chiral from a certain combined flavor-orbital freedom, the chiral from a certain combined flavor-orbital freedom, it is proven as necessity be chosen at the surface terms, via an illustration.

2. Solving a Classical Chiral Orbit Moving up to the Surface

2.1. Working with a Pfaffian Differential Element

An orbital variation under, e.g., a fixed 3+1 space-time (in fact, that is a lift along the 3rd or z-direction, supposed to be a moving-away axis from a projected (x-y) plane), is

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$\theta^k \equiv \sum_{k'} e^{kk'} dx_{k'} = \sum_{k'} \left(\eta^{kk'} + h^{kk'}\right) dx_{k'} \qquad (1)$

Where the tensor $\eta^{kk'}$ $k = 0, 1, 2, 3$ is the usual Minkowskian 4-metric, while h is supposed to be a variation (most probably remaining small) from η.

Despite that θ^k is the same as the Vielbein (a generalization of the 4-dimensional Vierbein to an arbitrary dimension, here it could be very well three), to which it associates a connection such $\omega_k{}^{k'} = \omega_{kl}^{k'} dx^l$ (which we choose the notation for dimensional generality), the following resolution goes without the need of the latest except at verifying end conditions.

The condition for θ to be integrable requires, after dropping the surface term, localization at the end region of x_3, since it is in matrix form made from two space vectors or some constraint tensor, following the form's equations

$$d\theta^k = \sum_{k",l} \frac{\partial e^{kk"}}{\partial x_l} dx_{k"} \wedge dx_l = \sum_{k",l} \frac{\partial h^{kk"}}{\partial x_l} dx_{k"} \wedge dx_l$$
$$\Longrightarrow 0 = \theta^k \wedge d\theta^k = \sum_{k',k",l} \left(\eta^{kk'} + h^{kk'}\right) \frac{\partial h^{kk"}}{\partial x_l} dx_{k'} \wedge dx_{k"} \wedge dx_l \qquad (1')$$

One remarks that the above equation remains true even without the k' and k'' summations, so

$$\overset{\sum_k}{\Longrightarrow} \frac{\partial}{\partial x_l} \left(h^{k'k"} + \frac{1}{2} \sum_k h^{kk'} h^{kk"}\right) dx_{k'} \wedge dx_{k"} \wedge dx_l \longmapsto d\frac{H_l^{k'k"}}{2} \wedge dx_{k'} \wedge dx_{k"} = 0 \qquad (1")$$

Which leads to the definition of $H_l^{k'k"}$, but restricted to two variables considered as the surface terms, so that it satisfies

$$H_l^{k'k"} \equiv 2h^{k'k"} + \sum_k h^{kk'} h^{kk"} = c_l x_l + c_{l'} x_{l'} \qquad l, l' \text{ are different from } k', k" \qquad (2)$$

This equation's solution has the look of an orbit, including the remaining variable, confirming the surface term correlations required in App. a.
Therefore, the metric is defined as

$$g^{kk'} = e^{kk"} e^{k"k'} = \left(\eta^{kk'} + h^{kk"} \eta^{k"k'} + \eta^{kk'} h^{k"k'} + h^{kk"} h^{k"k'}\right)$$
$$= \eta^{kk'} + h^{kk'} + h^{k'k} + h^{kk"} h^{k"k'} = \eta^{kk'} + H^{kk'}$$

And vice versa, since the vierbein is invertible, that leads also to it being a square-root of the metric.

In the product inside $\theta \wedge d\theta$, due to completeness, it can be assumed that the tensor h is either symmetric or anti-symmetric.

Anti-symmetric h with $h^{kk'} + h^{k'k} = 0$ will make the metric second order in $|h| \ll 1$, can be rendered diagonally, but as parts of the orbits, the particles will still pop up after an integration is manifested, however shifted non-commutatively by parallel transports despite being absorbed in the curvature.

The case of symmetry with

$$\frac{\partial h^{k'k"}}{\partial x_l} dx_{k'} \wedge dx_{k"} = 0$$

has a resolvable equation at the boundary similar to (2); then

$$\sum_K \epsilon_{k'k"l} \frac{\partial \left(h^{k'k} h^{kk"} \right)}{\partial x_l} = 0 \implies V_r^{k'} . \left(\nabla \wedge V_l^{k"} \right) = 0 \Longleftrightarrow \nabla . \left(V_r^{k'} \wedge V_l^{k"} \right) = 0 \implies$$

Two solutions:

$$V_l^{k"} \parallel and \neq V_r^{k'} \; for \; k' \neq k" \qquad \text{(2-a)}$$

Where each of the three vectors is such that if it has its row and line associated according to

$$\left. \overrightarrow{V_r^{k'}} \right)_k = h^{k'k} \text{ or } \left. \overrightarrow{V_l^{k"}} \right)_k = h^{kk"}$$

Then, under the duality needed to eliminate anomalies, e.g. [1], a necessary and sufficient condition is to have, instead of the inclusive "or," a decisive "and."

That is, a minimal solution is

$$k' = k \text{ And } V_l^{k"} = 0, \text{ no off-diagonal elements } h^{k'k"} = 0 \qquad \text{(2-a')}$$

The solution, with $h^{k'k} \equiv h_1^{k'k}(x)$, since $h_2^{k_2'k_2}(y) \delta y_{k_2'} \delta y_{k_2} = h_1^{k'k}(x) \delta x_{k_1'} \delta x_{k_1}$, as $\frac{\delta x_{k_1'}}{\delta x_{k_1}}$ becomes different from $\frac{\delta y_{k_2'}}{\delta y_{k_2}}$, is a distortion for the motion.

More, using what was obtained, [2], as such

$$R^{\mu\nu} = \frac{1}{2} \left(\Box H^{\mu\nu} - \partial^\mu \partial^{\mu'} H^{\mu'\nu} - \partial^\nu \partial^{\mu'} H^{\mu'\mu} + \partial^\mu \partial^\nu H^{\rho\rho} \right)$$
$$= \frac{1}{2} \left(\Box H^{\mu\nu} - \partial^\mu \partial^\nu H^{\nu\nu} \right) + \frac{1}{2} \left(\partial^\mu \partial^\nu H^{\rho\rho} - \partial^\nu \partial^\mu H^{\mu\mu} \right)$$

Where the repetitive index sum is valid for non-triple ones only, plus noting the addition of two Ricci tensors $R_1^{\mu\nu} \equiv \frac{1}{2} \left(\Box H^{\mu\nu} - \partial^\mu \partial^\nu H^{\nu\nu} \right)$ and $R_2^{\mu\nu} \equiv \frac{1}{2} \left(\partial^\mu \partial^\nu H^{\rho\rho} - \partial^\nu \partial^\mu H^{\mu\mu} \right)$

From the freedom over the surface orbits, the metric variation is picked such that off-diagonal elements are $\mu \neq \nu; H^{\mu\nu} = 0 \Rightarrow g^{\mu\nu} = 0$.

However, $R^{\mu\nu}$ is still getting off-diagonal elements as

$$\mu \neq \nu \to R^{\mu\nu} = \frac{1}{2}\partial^\mu \partial^\nu \left(\sum_\rho H^{\rho\rho} - H^{\nu\nu} - H^{\mu\mu} \right) = \frac{1}{2}\partial^\mu \partial^\nu \left(\sum_{\substack{\rho \neq \mu \\ \neq \nu}} H^{\rho\rho} \right)$$

$$\mu = \nu \to R^{\mu\mu} = R_1^{\mu\mu} + R_2^{\mu\mu}$$

$$R_1^{00} = -\frac{1}{2}\partial^i \partial^i H^{00} \text{ And } R_1^{ii} = \frac{1}{2}\left(\partial^0 \partial^0 - \sum_{j \neq i} \partial^j \partial^j - 2\partial^i \partial^i \right) H^{ii}$$

$$R_2^{00} = \frac{1}{2}\partial^0 \partial^0 \sum_j H^{jj} \text{ And } R_2^{ii} = \frac{1}{2}\partial^i \partial^i (H^{00} - \sum_{j \neq i} H^{jj}) \qquad \text{(2-b')}$$

So, it can be written to leading order in H as $g_{\mu\nu} \sim \eta_{\mu\nu}$ in the presence of $\times R^{\mu\nu}$

$$R = g_{\mu\nu}\left(R_1^{\mu\nu} + R_2^{\mu\nu} \right) = g_{00} R_1^{00} + g_{ii} R_1^{ii} + g_{00} R_2^{00} + g_{ii} R_2^{ii}$$

$$\left(g_{00} R_1^{00} + g_{ii} R_1^{ii} \right) = R_1^{00} - R_1^{ii} \overset{H \to 0}{=} \frac{1}{2}\left(-\partial^i \partial^i H^{00} \right) - \frac{1}{2}\left[\partial^0 \partial^0 H^{ii} - \partial^j \partial^j H^{ii} \right]$$

$$\left(g_{00} R_2^{00} + g_{ii} R_2^{ii} \right) = R_2^{00} - R_2^{ii} \overset{H \to 0}{=} \frac{1}{2}\partial^0 \partial^0 \sum_j H^{jj} - \frac{1}{2}\left[\partial^i \partial^i \left(H^{00} - \sum_{j \neq i} H^{jj} \right) \right] \qquad \text{(2-b')}$$

$$\Longrightarrow R^2 = R_1^2 + R_2^2 + 2R_1 R_2 =$$

$$\left(R_1^{00} - R_1^{ii} \right)^2 + \left(R_2^{00} - R_2^{ii} \right)^2 + 2(R_1^{00} - R_1^{ii})(R_2^{00} - R_2^{i'i'}) \qquad \text{(2-c)}$$

$$R_{\mu\nu} R^{\mu\nu} = R_{100}\ R_1^{00} + R_{1ii} R_1^{ii} + 2R_{100} R_2^{00} + \ 2R_{1ii} R_2^{ii} + R_{200}\ R_2^{00} + R_{2ii} R_2^{ii}$$

$$+\frac{1}{4}\partial_\mu \partial_\nu \left(\sum_{\substack{\rho \neq \mu \\ \neq \nu}} H_\rho{}^\rho \right) \partial^\mu \partial^\nu \left(\sum_{\substack{\rho \neq \mu \\ \neq \nu}} H_\rho{}^\rho \right) \qquad \text{(2-c')}$$

1st line of $R_{\mu\nu} R^{\mu\nu} = \left(R_1^{00} + \ R_2^{00} \right)^2 + \left(R_1^{ii} + \ R_2^{ii} \right)^2 = R_{00}^2 + R_{ii}^2$

One can note the appearances of perfect squares which is even more apparent when using

$$R^2 - 2R_{\mu\nu}R^{\mu\nu} = \left(R_1^{00} - R_1^{ii}\right)^2 - 2R_{100}\ R_1^{00} - 2R_{1ii}R_1^{ii} + \left(R_2^{00} - R_2^{ii}\right)^2 - 2R_{200}\ R_2^{00} - 2R_{2ii}R_2^{ii}$$

$$+2\left(R_1^{00} - R_1^{ii}\right)\left(R_2^{00} - R_2^{ii}\right) - 4R_{100}R_2^{00} - 4R_{1ii}R_2^{ii} - \frac{1}{2}\partial^\mu\partial^\nu\left(\sum_{\substack{\rho\neq\mu\\ \neq\nu}} H^{\rho\rho}\right)\partial^\mu\partial^\nu\left(\sum_{\substack{\rho\neq\mu\\ \neq\nu}} H^{\rho\rho}\right)$$

$$\leftrightarrow -\left(R_1^{00} + R_1^{ii}\right)^2 - \left(R_2^{00} + R_2^{ii}\right)^2 - 2\left(R_1^{00} + R_1^{ii}\right)\left(R_2^{00} + R_2^{ii}\right)$$

$$+\sum_i \frac{1}{2}\partial^0\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big)\partial^0\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big) - \sum_{i\neq j}\frac{1}{2}\partial^i\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big)\partial^i\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big)$$

What is resolvable as it leads to the already manifested conformal equation

$$R^2 - 3R_{\mu\nu}R^{\mu\nu} = 0\,0 = R^2 - 3R_{\mu\nu}R^{\mu\nu} = -\left(R^{00} + R^{ii}\right)^2 - \left(R_{00}^2 + R_{ii}^2\ \right)$$
$$-\frac{3}{4}\sum_{i\neq j}\partial^i\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big)\partial^i\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big) + \frac{3}{4}\sum_i \partial^0\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big)\partial^0\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big) \qquad (3)$$

That is to be expected due to the ratio of isotropic 2-surfaces out of 3-spaces in eq. (2) above.

And that is essential for the shuffle of the eigen-frequencies if needed there.

Then, if it has been defined, the surface terms and found to be different from the time coordinate, say, then x_{i_2} and x_{i_3} so they follow linear equation (2).

Then, in the case of an isotropy between the time and one of the surface variables, that turns out to be also for the second and then the 3rd and the 4th terms in eq. (3) eliminate, leading to the equation

$$R^{00} = R^{ii} = 0 \qquad (3')$$

Choose one as the other can be checked to be true

$$0 = -\frac{1}{2}\sum_i \partial^i\partial^i H^{00} + \frac{1}{2}\partial^0\partial^0\sum_i H^{ii} \qquad (3'')$$

2.2. The Generalized Wick Rotation

In the frequency space, the above equation can be written as

$$0 = -\partial_1\partial_1 H^{00} - \partial_2\partial_2 H^{00} - \partial_3\partial_3 H^{00} + \partial_0\partial_0\ \left(H^{11} + H^{22} + H^{33}\right) = -\partial_1\partial_1 H^{00} + \partial_0\partial_0 H^{11} \qquad \text{(3-a'')}$$

So now a generalized Wick rotation can be worked by picking an internal $SU(3)$ invariance under a rescaling of the above orbital definitions using the freedom over $x_l = l = 1, 2$ in $\left(h^{11}, h^{22}\right)$.

Now one has to choose the left and the right unitary variations such that

$$(dt, dx_1) \to \left(dt^L = \frac{1}{2}dt + i\frac{\sqrt{3}}{2}dx_1, dx_{123}^L = \frac{1}{2}dx_1 - \frac{i\sqrt{3}}{2}dt\right) \qquad \text{(3-a”’)}$$

$$(dt, dx_1) \to (dt^R = \frac{1}{2}dt - i\frac{\sqrt{3}}{2}dx_1, dx_{123}^R = \frac{1}{2}dx_1 + (i\sqrt{}3)/2\ dt) \qquad \text{(3-b”’)}$$

So one can proceed, also in the Fourier projection space, as

$$0 = -\ \left(k_1^{0L}k_1^{0R} - 3\omega_0^L\omega_0^R\right) H^{00} \ + \left(\omega_1^L\omega_1^R - 3k_1^{1L}k_1^{1R}\right) H^{11} \qquad (4)$$

To deduce from the above duality the new expression for the generalized Wick’s frequencies (or Hamiltonian) squared

$$0 = -\ \left(k_1^{0^2} - 3\omega_0^2\right) H^{00} + \left(\omega_1^2 - 3k_1^{1^2}\right) H^{11} \qquad \text{(4')}$$

One confirms the result of ref. [1] in view of the factors since they also meet the used 3-isotropy. Since it has been proved that multiplying a differential element by the unit modular factors F or F* will not change its unitarity properties.

Appendix

a. The Necessary Lorentzian Surface Term

The simplest common example is the Dirac equation on a covariant curved space, whose details were worked out in [3].

So one can deduce a convenient representation for $\gamma^\mu(x) = b_a{}^\mu(x)\gamma^a$ where $b_a{}^\mu$ is the vierbein as the metric is $g^{\mu\nu} \equiv b_a{}^\mu b_b{}^\nu \eta^{ab}$, while $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$. More, the connection is

$$\Gamma_\mu = -\frac{1}{4}\gamma_a\gamma_b b^a{}_\nu g^{\nu\lambda} \underset{\lambda;\mu}{\overset{a}{b}} + iqA$$

A can be set to zero, since arbitrary in this context.

The equation, with covariant derivatives $\nabla_\mu \equiv \partial_\mu - \Gamma_\mu$ via Γ_μ the spinor affine connection, is

$$\left(\gamma^\mu(x)\nabla_\mu + m\right)\psi(x) = 0 \qquad \text{(a-1)}$$

Where $\gamma^{\mu}(x)$ is the coordinate-dependent Dirac matrices whose covariant derivative is given by

$$\nabla_{\mu}\gamma_{\nu}(x) = \partial_{\mu}\gamma_{\nu}(x) - \Gamma^{\lambda}_{\mu\nu}\gamma_{\lambda}(x) - \Gamma_{\mu}\gamma_{\nu}(x) + \gamma_{\nu}(x)\Gamma_{\mu} = 0$$

Where here $\Gamma^{\lambda}_{\mu\nu}$ is the Christoffel symbol, it differs from the gauge invariance correction Γ_{μ}, noted also ω_{μ} as it deals with tensors with tensors.

The Hamiltonian, defined as $i\frac{\partial}{\partial t}\psi = H\psi$, should be regulated as

$$\frac{i}{2}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\right)\psi = H_{reg}\psi \Rightarrow 2H_{reg} = \left(2H + i\left(\frac{\partial}{\partial t}\right)\right) \qquad \text{(a-2)}$$

to include the time twist in it. One can see this equation in the Hilbert Space of ψ's as

$$(\psi_j, H_{reg}\psi_i) = \frac{i}{2}\int d^3x < {\psi_j}^{\dagger}(x,t)\left[\gamma^0(x)\right]^{-1}\frac{\partial}{\partial t}\left[\gamma^0(x)\psi_i(x,t)\right]$$
$$-\frac{\partial}{\partial t}\left\{{\psi_j}^{\dagger}(x,t)\left[\gamma^0(x)\right]^{-1}\right\}\gamma^0(x)\psi_i(x,t) > \qquad \text{(a-2')}$$

One can deduce that the Hamiltonian becomes Hermitian, since in the space of frequencies the time derivative transforms into frequencies while $\left[\gamma^0(x)\right]^{-1}\gamma^0(x) = 1$.

However, it has been dropped above the surface term $\int d^3x\partial/\partial t\left[{\psi_j}^{\dagger}(x,t)\psi_i(x,t)\right]$.

b. The Expansion' al Popping up of an Independent Neutrino Flavor Wave Vector

One can proceed with a simplifying illustration that has a physical implication on flavor physics. Recalling that the amplitude of one neutrino generation, say c, out of ν_a $a = e, \mu, \tau$, is given to first order by an expression $\widehat{a}_{cc} = \sum_{ab}(M_{cc})_{ab}(a_L)_{ab}$, when expanded, [4], as such

$$\widetilde{a}_{cc} = \sum_{\substack{a \neq c \\ b \neq c}} (M_{cc})_{ab}(a_L)_{ab} + (M_{cc})_{ec}(a_L)_{ec} + (M_{cc})_{\mu c}(a_L)_{\mu c} + (M_{cc})_{c\tau}(a_L)_{c\tau} \qquad \text{(b-1)}$$

While using in plus of $\sum_{a'} U^{*}_{a'a}U_{a'b} = \delta_{ab}$,

$$(M_{cc})_{ab} = \sum_{a'b'}\tau_{a'b'}U^{*}_{a'c}U_{a'a}U^{*}_{b'b}U_{b'c}$$

Where the matrix amplitude τ derives from the plane waves of oscillations taken at the boundary $t \to 0 \ or \ t \ll \frac{1}{E_{scale}}$.

Since in the above it was derived that the boundary should contain at least a relation between two boundary variables, which leads to the conclusion that the radial distance can be parameterized in terms of time. So

$$\tau_{a'b'} = \begin{cases} \exp\left(-iE_{b'}t\right) & E_{a'} = E_{b'} \\ \frac{\exp\left(-iE_{a'}t\right) - \exp\left(-iE_{b'}t\right)}{-i\left(E_{a'} - E_{b'}\right)t} & E_{a'} \neq E_{b'} \end{cases} \tag{b-2}$$

Since a kink in the region subject to an energy $E_{a'} = E_{b'}$ eliminates,

We are interested in $E_{a'} \neq E_{b'}$.

The fact of arbitrariness of the difference of energy in the case of the oscillation leads to taking one of the energies to be zero, say $E_{b'} \equiv 0$.

Plus, due to a doubling that can occur in the cusp case only, which and so $E_{a'}t \ll 1$, one has then,

$$\tau_{a'b'} \to \frac{1}{2i}\left(E_{a'} + E_{b'}\right)t \Rightarrow$$

$$\left(M_{cc}\right)_{ab} \approx \sum_{a'b'}\left(1 + \frac{1}{2i}E_{a'}t\right)U^*_{a'c}U_{a'a}U^*_{b'b}U_{b'c} = \sum_{a'}\left(1 + \frac{1}{2i}E_{a'}t\right)U^*_{a'c}U_{a'a}\delta_{bc} \tag{b-1'}$$

One sees that the summation over $b^{'}$ is totally decoupled and it cannot be reduced (to a Kronecker δ), unless the orthonormal eigen proper basis indexed by b has no mixing with the orthonormal eigen proper basis indexed by $b^{'}$.

Therefore, the flavor indexing is independent from any external currents that may link it, then, through color or charge, as seen in [1].

With the existence of a Wave Vector under which the radial parameterization is directly related to the time through a linear relation, it leads to the velocity being well-defined and, therefore, its Helicity being non-zero and unique; then, its Chiral nature.

References

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