Review of: "The edge rings of compact graphs"

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Let G be a simple graph on n vertices. The edge ring of G over a field k, denoted by $k[G]$, is the subalgebra of $k[x_1,\ldots, x_n]$ generated by x_ix_j with (i,j) is an edge of G. By the result of [H. Ohsugi, T. Hibi, Normal Polytopes Arising from Finite Graphs, J. Algebra, 207 (1998), 406--429], the edge ring $k[G]$ is normal if and only if G satisfies the odd cycle condition. In the paper, the authors study the edge rings of {\it compact graphs}, i.e., graphs satisfying odd cycle condition and have no even cycles. First, they classify all compact graphs. Second, they describe explicitly a Grobner basis with respect to the lexicographic order of the defining ideal of $k[G]$ for each compact graph G. Third, they show that the initial ideal J_G has a natural Betti splitting. Fourth, they compute the total Betti numbers of J_G and the regularity of J_G. By the result of [A. Conca and M. Varbaro,

Square-free Gr\"obner degenerations, Invent. Math. 221 (2020), 713--730], they then deduce the formula for the regularity and projective dimension of $k[G]$. Note that the result of Conca and Varbaro says that the extremal Betti numbers of I_G and J_G are also equal. Finally, they use the Danilov-Stanley description of the canonical module of $k[G]$ to compute its Cohen-Macaulay type. The results are nice and the presentation is clear. I have a few comments as follow.

1. On page 2, in the definition of free resolution, the full stop (.) should be within the equation.
2. It is easier to follow if they provide some examples for each compact graph type, where they write down explicitly the equation of I_G and its initial ideal J_G (underlying the initial term).
3. On page 15, when they displace the minimal generators of J_G of type two compact graph, they may use an align environment to avoid overfull equation.
4. When they ask a question on the equality of all Betti numbers of I_G and J_G, it is nice to have some explit examples to verify the correctness of this question.
5. Since the Betti splittings give rise to recursive formula for the graded Betti numbers as well, I am wondering if they could find explicit formula for all the graded Betti numbers of the initial ideal of I_G.