Research Article

New Adaptive Numerical Algorithm for Solving Partial Integro-Differential Equations

Rebiha Zeghdane[1]

1. Department of Mathematics, Faculty of Mathematics and Informatics, University of Mohamed El Bachir El Ibrahimi, Bordj Bou Arreridj, Algeria

The paper introduces an accurate numerical approach based on orthonormal Bernoulli polynomials for solving parabolic partial integro-differential equations (PIDEs). This type of equation arises in physics and engineering. Some operational matrices are given for these polynomials and are also used to obtain the numerical solution. By this approach, the problem is transformed into a nonlinear algebraic system. Convergence analysis is given, and some experimental tests are studied to examine the good accuracy of the numerical algorithm. The proposed technique is compared with some other well-known methods.

Corresponding author: Rebiha Zeghdane, rebiha.zeghdane@univ-bba.dz

1. Introduction

Various approximations by orthonormal families of functions have been investigated in the physical sciences, engineering, etc. This type of numerical approximation can also be used in optimal control problems and, in general, to approximate solutions of dynamical systems. Integral equations arise in many physical problems, diffusion problems, concrete problems of physics and mechanics, and some other problems of engineering, different applications of potential theory, synthesis problems, mathematical modelling of economics, population, geophysics, antennas, genetics, communication theory, radiation problems concerning the transport of particles, etc [1][2][3][4][5][6][7]. There are various problems such as differential, integral, and partial integro-differential equations which use polynomial series and orthogonal functions to approximate their numerical solutions [8][9][10][11][12][13][14]. Spectral methods are methods based on polynomial approximation; we can see that the convergence of the approximations is exponential when the functions to be approximated are analytic, which means that the order of convergence is limited by the choice of the regularity of the exact solution. In many scientific fields, systems are described by partial differential equations (PDE). In [15], Fakha et al. used the Legendre collocation technique for solving parabolic PDEs. Radial basis functions are also used for the approximate solutions of nonlinear parabolic type Volterra partial integro-differential equations [14]. In [16], Brunner et al. studied the numerical solution of parabolic Volterra integro-differential equations on unbounded spatial domains. For more applications, we can see [17][18][19]. A method based on the Hermite-Taylor matrix to solve partial integro-differential equations is given in [9]. A matrix method for solving two-dimensional time-dependent diffusion equations is given by Zogheib et al. [20]. In [12], the author presented a new technique by using the Bernoulli operational matrix to solve SDEs. In this article, we use a pseudo-spectral method based on orthonormal Bernoulli polynomials to approximate the following PIDEs

$$u_t(x,t)+\lambda_1 u_{xx}(x,t)=\lambda_2\int_0^t K_1(x,t,s,u(x,s))ds+\lambda_3\int_0^T K_2(x,t,s,u(x,s))ds+g(x,t), \tag{1}$$

with initial and boundary conditions

$$u(x,0)=h_0(x),\ u(0,t)=h_1(t), u(b,t)=h_2(t), \tag{2}$$

where $x\in[0,b], t\in[0,T], \lambda_1, \lambda_2$ and λ_3 are constants, and the functions $g(x,t)$, $k_1(x,t,s,u)$ and $k_2(x,t,s,u)$ are supposed to be sufficiently smooth on $\Gamma:=[0,b]\times[0,T]$ and D where $D:=\{(x,t,s): x\in[0,b], s,t\in[0,T]\}$. The existence and uniqueness of equation (1) are given in [21].

To approximate the integrals appearing in (1), we use the Gauss Legendre quadrature on the interval $[-1,1]$ given by

$$\int_{-1}^1 f(x)dx=\sum_{j=0}^r \omega_j f(x_j), \tag{3}$$

where x_j are the roots of $L_{r+1}(x)$ and $\omega_j=\dfrac{2}{(1-x_j^2)[L'_{r+1}(x_j)]^2}$, $j=0,1,\dots,r$. For Gauss Legendre quadrature on $[a,b]$, we have

$$\int_a^b f(x)dx = \frac{b-a}{2}\sum_{j=0}^{r}\omega_j f\left(\frac{b-a}{2}x_j + \frac{b+a}{2}\right). \tag{4}$$

2. Orthonormal Bernoulli polynomials and approximation

The Bernoulli polynomials $B_n(x)$, are given in [22] and satisfy the following relation

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x) = (n+1)x^n, \quad n = 0, 1, \ldots.$$

The Bernoulli polynomials form a complete basis over the interval $[0,1]$[23]. In this paper, we use the shifted OBPs $P_{i,T}(t)$ and $P_{i,b}(x)$ over $[0,T]$ and $[0,b]$ as follows

$$P_{i,b}(x) = \sqrt{\frac{2i+1}{b}}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{i-k}\binom{i+k}{k}\left(\frac{x}{b}\right)^k,\ i = 0, 1, \ldots \tag{5}$$

$$\Phi_{N,b}(.) = (P_{0,b}(.), P_{1,b}(.), \ldots, P_{N,b}(.))^T, \tag{6}$$

contains the first $N+1$ orthonormal Bernoulli polynomials. Any function $u \in L^2[0,b]$ has a best approximation $\hat{u} \in span\{\Phi_{N,b}(.)\}$ such that

$$\forall v \in span\{\Phi_{N,b}(.)\},\ \|u-\hat{u}\| \leq \|u-v\|,$$

and

$$u(x) \simeq \hat{u} = \sum_{i=0}^{N} u_i P_{i,b}(x) = U^T\Phi_{b,N}(x), \tag{7}$$

where $U = (u_0, u_1, u_2, \ldots, u_N)^T$, and u_i can be computed by the formula

$$u_i = \int_0^b u(x)P_{i,b}(x)dx,\ i = 0, 1, \ldots, N. \tag{8}$$

Any function $u(x,t)$ defined over $[0,b]\times[0,T]$ can be approximated by shifted OBPs as follows:

$$u_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} u_{ij}P_{i,b}(x)P_{j,T}(t) = \Phi_{b,N}(x)^T U \Phi_{T,M}(t), \tag{9}$$

where $U = [u_{ij}]$ is a matrix of order $(N+1)\times(M+1)$ with

$$u_{ij} = \int_0^b\int_0^T u(x)P_{i-1,b}(x)P_{j-1,T}(x)dtdx,\ i = 1, 2, \ldots N+1,\ j = 1, 2, \ldots, M+1. \tag{10}$$

3. Pseudo-spectral method for Solving PIDE

In this section, we describe our numerical technique t to solve PIDEs (1). Let the solution of (1) be approximated by the polynomial $u_{N,M}(x,t)$ such that

$$u_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} u_{ij}P_{i,b}(x)P_{j,T}(t) = \Phi_{b,N}(x)^T U \Phi_{T,M}(t). \tag{11}$$

In this work, we take $N = M$. It is easy to approximate the derivatives of the approximate solution of (1). In view of (11), one can write

$$\frac{\partial}{\partial t}[u_{N,M}(x,t)] = \sum_{i=0}^{N}\sum_{j=0}^{M} u_{ij}P_{i,b}(x)\frac{\partial}{\partial t}[P_{j,T}(t)] = \Phi_{b,N}(x)^T U\frac{\partial}{\partial t}[\Phi_{T,M}(t)], \tag{12}$$

$$\frac{\partial^2}{\partial x^2}[u_{N,M}(x,t)] = \sum_{i=0}^{N}\sum_{j=0}^{M} u_{ij}\frac{\partial^2}{\partial x^2}[P_{i,b}(x)][P_{j,T}(t)] = \frac{\partial^2}{\partial x^2}[\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)], \tag{13}$$

Now we give some relations for the derivatives of the shifted OBPs. We use a technique used by various researchers for solving different kinds of integral equations [15][24][25]. Let, given the vector defined in (6), we can write

$$\Phi_{b,N}(x) = T_{b,N}\Psi_N(x), \tag{14}$$

where

$$\Psi_N(x) = [1, x, x^2, \ldots, x^{N-1}, x^N]^T, \tag{15}$$

with $T_{b,N}$ being a lower triangular square matrix of order $N+1$ with entries

$$[T_{b,N}]_{ij} = \begin{cases} (-1)^{i-j}\sqrt{2i-1}\frac{1}{b^{j-1/2}}\binom{i-1}{i-j}\binom{i+j-2}{j-1}, & \text{if } i \geq j, \\ 0, & \text{if } i < j. \end{cases}$$

The matrix $T_{b,N}$ is invertible. From equation (14), we have

$$\frac{\partial}{\partial x}\Phi_{b,N}(x) = T_{b,N}\frac{\partial}{\partial x}\Psi_N(x) = T_{b,N}A_N\Psi_N(x) = T_{b,N}A_N T_{b,N}^{-1}\Phi_{b,N}(x), \tag{16}$$

where A_N is a square matrix of order $N+1$ given by

$$A_N = \begin{Bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 & 0 & 0 \\ . & . & . & \dots & 0 & 0 & 0 & 0 \\ . & . & . & \dots & N-2 & 0 & 0 & 0 \\ . & . & . & \dots & 0 & N-1 & 0 & 0 \\ . & . & . & \dots & 0 & 0 & N & 0 \end{Bmatrix},$$

and

$$\frac{\partial^2}{\partial x^2}\Phi_{b,N}(x) = T_{b,N}\frac{\partial^2}{\partial x^2}\Psi_N(x) = T_{b,N}A_N^1\Psi_N(x) = T_{b,N}A_N^1 T_{b,N}^{-1}\Phi_{b,N}(x), \tag{17}$$

where A_N^1 is a square matrix of order $N+1$ given by

$$A_N^1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ (3-1)(3-2) & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & (4-1)(4-2) & 0 & \dots & 0 & 0 & 0 & 0 \\ . & . & . & \dots & 0 & 0 & 0 & 0 \\ . & . & . & \dots & 0 & 0 & 0 & 0 \\ . & . & . & \dots & (N-1)(N-2) & 0 & 0 & 0 \\ . & . & . & \dots & 0 & N(N-1) & 0 & 0 \end{pmatrix}.$$

Then, we get

$$\frac{\partial}{\partial t}[u_{N,M}(x,t)] = \sum_{i=0}^{N}\sum_{j=0}^{M}u_{ij}P_{i,b}(x)\frac{\partial}{\partial t}[P_{j,T}(t)] = \Phi_{b,N}(x)^T U\frac{\partial}{\partial t}[\Phi_{T,M}(t)] = \Phi_{b,N}(x)^T U T_{T,M}A_M T_{T,M}^{-1}\Phi_{T,M}(t), \tag{18}$$

and

$$\frac{\partial^2}{\partial x^2}[u_{N,M}(x,t)] = \sum_{i=0}^{N}\sum_{j=0}^{M}u_{ij}\frac{\partial^2}{\partial x^2}[P_{i,b}(x)][P_{j,T}(t)] = \frac{\partial^2}{\partial x^2}[\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)] = [T_{b,N}A_N^1 T_{b,N}^{-1}\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)]. \tag{19}$$

Thus, by substituting (18) and (19) into equation (1) we get

$$\begin{aligned} \Phi_{b,N}(x)^T U T_{T,M}A_M T_{T,M}^{-1}\Phi_{T,M}(t) + \lambda_1\left[[T_{b,N}A_N^1 T_{b,N}^{-1}\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)]\right] &= \lambda_2\int_0^t k_1(x,t,s,u(x,s))ds \\ &+ \lambda_2\int_0^T k_2(x,t,s,u(x,s))ds + g(x,t), \end{aligned} \tag{20}$$

with $x \in [0,b]$ and $t \in [0,T]$, We collocate equation (20) at points $t_j,\ j = 0,1,\dots,r_1$ which are the Gauss-Legendre noeuds on the interval $[0,T]$ defined by (3), then we obtain

$$\begin{aligned} \Phi_{b,N}(x)^T U T_{T,M}A_M T_{T,M}^{-1}\Phi_{T,M}(t_j) + \lambda_1\left[[T_{b,N}A_N^1 T_{b,N}^{-1}\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t_j)]\right] &= \lambda_2\int_0^{t_j} K_1(x,t_j,s,u(x,s))ds \\ &+ \lambda_3\int_0^T K_2(x,t_j,s,u(x,s))ds + g(x,t_j), \end{aligned} \tag{21}$$

The integrals appeared in equation (21) are approximated by using Gauss-Legendre quadrature as follows

$$\int_0^{t_j} K_1(x,t_j,s,u(x,s))ds = \sum_{k=0}^{r_1}\beta_k K_1(x,t_j,l_k,u(x,l_k)), \tag{22}$$

where $l_k = \dfrac{t_j + t_j x_k}{2}$ and $\beta_k = \dfrac{t_j}{(1-x_k)^2[L'_{r_1+1}(x_k)]^2}$, $k = 0,1,2,\dots,r_1$. are Gauss Legendre nodes and weight on $[0,t_j]$.

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$$\int_0^T K_2(x,t_j,s,u(x,s))ds = \sum_{k=0}^{r_2} \omega_k K_2(x,t_j,s_k,u(x,s_k)), \tag{23}$$

where $s_k = \dfrac{T+Tx_k}{2}$ and $\omega_k = \dfrac{T}{(1-x_k)^2[L'_{r_2+1}(x_k)]^2}$, $k=0,1,2,\ldots,r_2$ are nodes and weight of Gauss Legendre quadrature on $[0,T]$. Substituting equations (22) and (23) in equation (21) we get

$$\Phi_{b,N}(x)^T U T_{T,M} A_M T_{T,M}^{-1} \Phi_{T,M}(t_j) + \lambda_1 \left[[T_{b,N} A_N^1 T_{b,N}^{-1} \Phi_{b,N}(x)]^T U [\Phi_{T,M}(t_j)] \right] = \sum_{k=0}^{r_1} \beta_k K_1(x,t_j,l_k,u(x,l_k))$$

$$+ \sum_{k=0}^{r_2} \omega_k K_2(x,t_j,s_k,u(x,s_k)) + g(x,t_j).$$

Let $RES(x,t_j)$ the residual function given by

$$RES(x,t_j) = \Phi_{b,N}(x)^T U T_{T,M} A_M T_{T,M}^{-1} \Phi_{T,M}(t_j) + \lambda_1 \left[[T_{b,N} A_N^1 T_{b,N}^{-1} \Phi_{b,N}(x)]^T U [\Phi_{T,M}(t_j)] \right] \tag{24}$$

$$-\lambda_2 \sum_{k=0}^{r_1} \beta_k K_1(x,t_j,l_k,u(x,l_k)) - \lambda_3 \sum_{k=0}^{r_2} \omega_k K_2(x,t_j,s_k,u(x,s_k)) - g(x,t_j) = 0.$$

Using relation (9) and the initial and boundary conditions given in (6), we get

$$\begin{cases} \Phi_{b,N}(x)^T U \Phi_{T,M}(0) - h_0(x) = 0, \\ \Phi_{b,N}(0)^T U \Phi_{T,M}(t) - h_1(t) = 0, \\ \Phi_{b,N}(b)^T U \Phi_{T,M}(t) - h_2(t) = 0. \end{cases}$$

Now, we extract the below $(N+1)\times(M+1)$ algebraic system

$$\begin{cases} RES(s_i,t_j) = 0,\ 2 \le i \le N,\ 2 \le j \le M+1, \\ \Phi_{b,N}(x)^T U \Phi_{T,M}(0) - h_0(s_i) = 0,\ 1 \le i \le N+1, \\ \Phi_{b,N}(0)^T U \Phi_{T,M}(t_j) - h_1(t_j) = 0,\ 2 \le j \le M+1 \\ \Phi_{b,N}(b)^T U \Phi_{T,M}(t_j) - h_2(t_j) = 0,\ 2 \le j \le M+1, \end{cases}$$

where $s_i = \dfrac{b+bx_i}{2}$ and $t_j = \dfrac{T+Tx_j}{2}$ are respectively the nodes of Gauss-Legendre quadrature on $[0,b]$ and $[0,T]$, with x_i as the corresponding nodes on $[-1,1]$.

4. Error bound of the present method

In [26], the error estimates for some orthogonal systems are given in the norms of the Sobolev spaces $H^\mu(\Omega)$, with $\Omega = I^d \subset \mathbb{R}^d$ and I being a bounded open interval of $\mathbb{R}$. In this section, we consider orthogonal approximations in multiple dimensions. Let $k=(k_1,k_2;\ldots,k_d)$, $|k| = \sum_{i=1}^d k_i$, k_i be any non-negative integers, and $\partial_x^k \Phi = \dfrac{\partial^{|k|} u}{\partial_{x_1}^{k_1} \ldots \partial_{x_d}^{k_d}}$. For $\mu \ge 0$, we define the Sobolev space

$$H^\mu(\Omega) = \left\{ \Phi,\ \partial_x^k \Phi \in L^2(\Omega),\ 0 \le |k| \le \mu \right\}, \tag{25}$$

with the norm

$$\|\Phi\|_\mu^2 = \sum_{k\in\mathbb{N}^d, k_1+k_2+\ldots k_d<\mu} \int_\Omega \left| \left(\prod_{j=1}^d D_j^{k_j} \right) \Phi \right|^2 dx, \tag{26}$$

where $D_j = \dfrac{\partial}{\partial x^{(j)}}$. If $\{\Phi_k\}_{k=0}^\infty$ is the system of orthonormal in $L^2(I)$ with $deg\Phi_k = k$ then the system $\{\Phi_k\}_{k\in N^d}$, where $\Phi_k(x) = \prod_{j=1}^d \Phi_{k_j}(x^{(j)}$ is complete and orthonormal in $L^2(\Omega)$ and any $u \in L^2(\Omega)$ is as follows

$$u = \sum_{k\in\mathbb{N}^d} u_k \Phi_k,\ u_k = \langle u, \Phi_k \rangle, \tag{27}$$

with $\|u\|_0^2 = \sum_{k\in\mathbb{N}^d} |u_k|^2$. Setting,

$$S_N = S_N(\Omega) = \{\text{span}\}\{\Phi_k : k \in \mathbb{N}^d, \max(k) < N\}, \tag{28}$$

S_N is the set of all polynomials of degree at most N in each variable $x^{(j)},\ j=1,\ldots,d$. Let $P_N : L^2(\Omega) \longrightarrow S_N$ be the orthogonal projection on S_N in $L^2(\Omega)$.

Theorem 1. [26] *For any real $\mu > 0$, there exists a constant C such that*

$$\|u - P_N u\|_0 \le C N^{-\mu} \|u\|_\mu,\ \forall u \in H^\mu(\Omega). \tag{29}$$

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For the error estimation of $u - P_N$ in the Sobolev space, we need the following lemmas.

Lemma 1. [26] *For any real μ and r such that $0 < r < \mu$, there exists a constant C such that*

$$\|u\|_\mu \leq CN^{2(r-\mu)}\|u\|_r, \, \forall u \in S_N. \tag{30}$$

Lemma 2. *For the two real r and μ with $0 \leq r \leq \mu - 1$ there exists a constant C such that for $j = 1, \ldots, d$*

$$\|(P_N D_j - D_j P_N)u\|_r \leq CN^{(2r-\mu+3/2)}\|u\|_\mu, \, \forall u \in H^\mu(\Omega). \tag{31}$$

The following theorems give the error estimation of the approximation of u

Theorem 2. *For the two real μ and r with $0 \leq r \leq \mu$, we can get a constant C such that*

$$\|u - P_N u\|_r \leq CN^{e(r,\mu)}\|u\|_\mu, \, \forall u \in H^\mu(\Omega), \tag{32}$$

where

$$e(r,\mu) = \begin{cases} 2r - \mu - 1/2, & r \geq 1 \\ 3r/2 - \mu, & 0 \leq r \leq 1 \end{cases}$$

Proof 1. *For $r = 0$, 32 reduces to 29. Now suppose that 32 holds for any integer $r \leq m - 1$ by inductive hypothesis. Then*

$$\|u - P_N u\|_m \leq \sum_{j=1}^{d} \|D_j u - D_j P_N u\|_{m-1}$$

$$\leq \sum_{j=1}^{d} \|D_j u - P_N D_j u\|_{m-1} + \sum_{j=1}^{d} \|P_N D_j u - D_j P_N u\|_{m-1},$$

by taking the inductive hypothesis for $D_j \in H^{\mu-1}$ and by using 31, we get

$$\|u - P_N u\|_m \leq C' N^{e(m-1,\mu-1)} \sum_{j=1}^{d} \|D_j u\|_{\mu-1} + c'' N^{e(m,\mu)}\|u\|_\mu,$$

we have $e(m-1, \mu-1) < e(m,\mu)$, then we get the result 32.

Theorem 3. *Suppose that $u(x,t) \in H^\mu(\Omega)$ and $\bar{u}(x,t) \in H^\mu(\Omega)$ with $\mu \geq 0$ be the exact and the numerical solution of equation (1), respectively. Also, suppose K_1 and K_2 satisfy the following uniform Lipschitz conditions*

$$|K_1(x,t,s,u_1) - K_1(x,t,s,u_2)| \leq l_1|u_1 - u_2|, \quad |K_2(x,t,s,u_1) - K_2(x,t,s,u_2)| \leq l_2|u_1 - u_2|. \tag{33}$$

Then, for any real μ and r such that $0 \leq r < \mu$ the error bound E_N of the present method is given by

$$\|E_N\|_{H^\mu(\Omega)} \leq \Big[\Big(C + \lambda_1 C\Big)N^{(2r-\mu+3/2)} + \Big(l_1\lambda_2 + l_2\lambda_3\Big)cN^{e(r,\mu)}\Big]\|u\|_{H^\mu(\Omega)}, \tag{34}$$

where

$$e(r,\mu) = \begin{cases} 2r - \mu - 1/2, & r \geq 1 \\ 3r/2 - \mu, & 0 \leq r \leq 1. \end{cases}$$

Proof 2. *Using equation 1, we get*

$$\begin{aligned} ||E_N||_{H^r(\Omega)} &= ||u_t(x,t) + \lambda_1 u_{xx}(x,t) - \lambda_2 \int_0^t K_1(x,t,s,u(x,s))ds - \lambda_3 \int_0^T K_2(x,t,s,u(x,s))ds \\ &- \Big(\bar{u}_t(x,t) + \lambda_1 \bar{u}_{xx}(x,t) - \lambda_2 \int_0^t K_1(x,t,s,\bar{u}(x,s))ds - \lambda_3 \int_0^T K_2(x,t,s,\bar{u}(x,s))ds\Big)||_{H^r(\Omega)} \\ &\leq ||u_t(x,t) - \bar{u}_t(x,t)||_{H^r(\Omega)} + \lambda_1||u_{xx}(x,t) - \bar{u}_{xx}||_{H^r(\Omega)} \\ &+ \lambda_2|| \int_0^t \Big(K_1(x,t,s,u(x,s)) - K_1(x,t,s,\bar{u}(x,s))\Big)ds||_{H^r(\Omega)} + \lambda_3|| \int_0^T \Big(K_2(x,t,s,u(x,s)) - K_2(x,t,s,\bar{u}(x,s))\Big)ds||_{H^r(\Omega)} \\ &\leq ||u_t(x,t) - \bar{u}_t(x,t)||_{H^r(\Omega)} + \lambda_1||u_{xx}(x,t) - \bar{u}_{xx}||_{H^r(\Omega)} \\ &+ \lambda_2 \int_0^t ||K_1(x,t,s,u(x,s)) - K_1(x,t,s,\bar{u}(x,s))||_{H^r(\Omega)}ds + \lambda_3 \int_0^T ||K_2(x,t,s,u(x,s)) - K_2(x,t,s,\bar{u}(x,s))||_{H^r(\Omega)}ds, \end{aligned} \tag{35}$$

since K_1 and K_2 satisfied Lipschitz conditions, then we have

$$\begin{aligned} ||E_N||_{H^r(\Omega)} &\leq ||u_t(x,t) - \bar{u}_t(x,t)||_{H^r(\Omega)} + \lambda_1||u_{xx}(x,t) - \bar{u}_{xx}||_{H^r(\Omega)} \\ &+ l_1\lambda_2 \int_0^t ||u(x,s)) - \bar{u}(x,s))||_{H^r(\Omega)}ds + l_2\lambda_3 \int_0^T ||u(x,s)) - \bar{u}(x,s))||_{H^r(\Omega)}ds, \end{aligned} \tag{36}$$

by using Lemma 31 and theorem 2, we get

$$
\begin{aligned}
||E_N||_{H^r(\Omega)} \leq\ & CN^{(2r-\mu+3/2)}||u||_{H^\mu(\Omega)} + \lambda_1 CN^{(2r-\mu+3/2)}||u||_{H^\mu(\Omega)} \\
&+ l_1\lambda_2 \int_0^t cN^{e(r,\mu)}||u||_{H^\mu(\Omega)} ds + l_2\lambda_3 \int_0^T cN^{e(r,\mu)}||u||_{H^\mu(\Omega)} ds \\
&\leq \left[\left(C + \lambda_1 C \right) N^{(2r-\mu+3/2)} + \left(l_1\lambda_2 + l_2\lambda_3 \right) cN^{e(r,\mu)} \right] ||u||_{H^\mu(\Omega)},
\end{aligned} \tag{37}
$$

where

$$
e(r,\mu) = \begin{cases} 2r-\mu-1/2, & r \geq 1 \\ 3r/2-\mu, & 0 \leq r \leq 1. \end{cases}
$$

Then if u is infinitely smooth, then $\|\|E_N\|_{H^r(\Omega)} \longrightarrow 0$ as $N \longrightarrow \infty$.

5. Numerical implementation of the proposed algorithm

In this section, some numerical test equations are considered to show the accuracy of the presented algorithm, where we have calculated the maximum absolute errors at different times. In these examples, the linear and nonlinear algebraic systems are solved by the Newton iterative method and using MATLAB software.

Example 1. *Consider the PIDEs*

$$
u_t(x,t) - u_{xx}(x,t) = g(x,t) - \int_0^t e^{s-t} u(x,s) ds, \tag{38}
$$

with initial and boundary conditions $u(x,0) = x$, $u(0,t) = 0$, $x \in [0,1], u(1,t) = e^{-t}$, $t \in [0,1]$, and $g(x,t) = (2t - x^2 - t^2 x)\exp(-xt) + \dfrac{x(\exp(-t) - \exp(-xt))}{x-1}$. The analytical solution for this example is $u(x,t) = xe^{-xt}$. The numerical experiments are given in table (1).

t	$N=3$	$N=4$	$N=5$	$N=6$
0.0625	2.4918 E-4	1.6010 E-5	7.4614 E-7	2.5947 E-8
0.1250	3.1382 E-4	1.4463 E-5	3.7456 E-7	6.8578 E-9
0.1875	2.6376 E-4	6.1076 E-6	1.2642 E-7	1.3100 E-8
0.2500	1.5539 E-4	2.3487 E-6	3.5723 E-7	9.0718 E-9
0.3125	6.4499 E-5	7.4193 E-6	2.7735 E-7	2.9712 E-9
0.3750	1.4299 E-5	8.0728 E-6	1.1176 E-7	1.0793E-8
0.4375	2.5590E-4	6.6364 E-5	2.1644 E-7	9.4010E-9
0.500	3.5960 E-4	1.1409 E-5	3.4232 E-7	8.3644 E-9
0.5625	4.6127 E-4	1.8037 E-5	6.0631 E-7	1.6629 E-8
0.6250	6.2660 E-4	2.6913 E-5	1.0077 E-6	3.0685 E-8
0.6875	8.2174 E-4	3.8470 E-5	1.5912 E-6	5.3212E-8
0.7500	1.0486 E-3	5.3175 E-5	2.4089 E-6	8.7621 E-8
0.8125	1.3090 E-3	7.1535 E-5	3.5395E-6	2.1009 E-7
0.8750	1.6050 E-3	9.4094 E-5	5.0403 E-6	3.0975 E-7
0.9375	1.9385 E-3	1.2143 E-4	6.9854 E-6	4.4474 E-7

Table 1. Errors using the OBP method for test (1).

Example 2. *Let given the nonlinear equation*

$$
u_t(x,t) = u_{xx}(x,t) + g(x,t) - \int_0^1 u^2(x,s) ds, \tag{39}
$$

with conditions $u(x,0) = x^2$, $u(0,t) = t^2$, $x \in [0,1]$,$u(1,t) = t^2 + 1$, $t \in [0,1]$. The function $g(x,t)$ is obtained from the analytical solution $u(x,t) = t^2 + x^2$. The numerical results are presented in figure 1.

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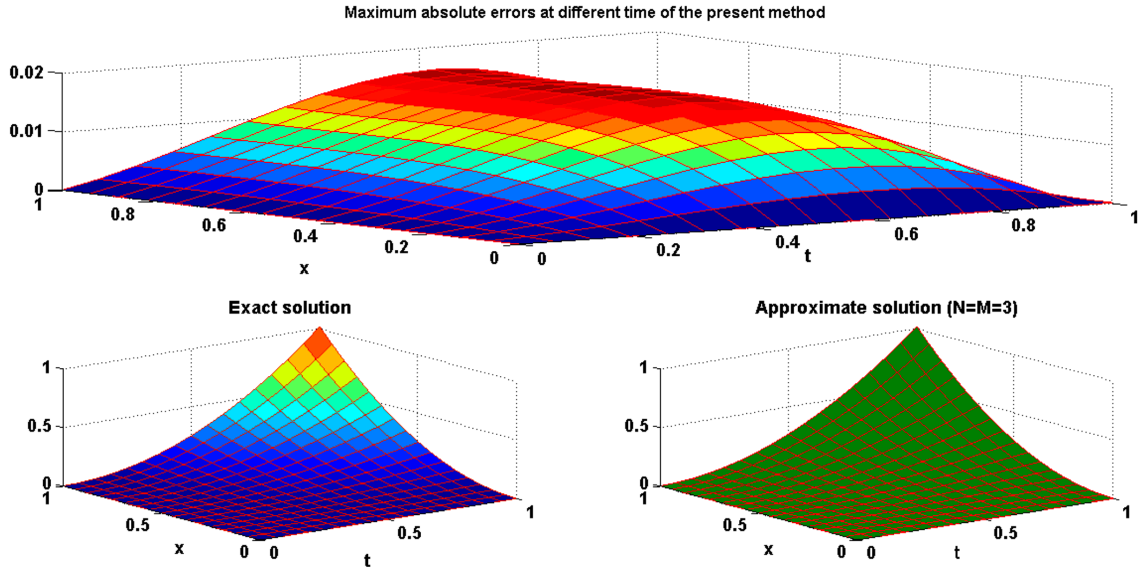

Figure 1. Exact (left) and approximate (right) solutions for example 2 for ($N = M = 3$).

Example 3. [27] *Let consider the following partial integro-differential equation*

$$u_t(x,t) - u_{xx}(x,t) = g(x,t) - \int_0^t e^{x(t-s)} u(x,s)ds \tag{40}$$

with conditions

$$u(x,0) = 0, \quad x \in [0,1], \, u(0,t) = \sin(t), \quad u(1,t) = 0, \quad t \in [0,1],$$

With $g(x,t) = (1-x^2)\cos(t) + 2\sin(t) + \dfrac{(x^2-1)\cos(t) + x\sin(t) - e^{xt}}{x^2+1}$. *The exact solution is given by* $u(x,t) = (1-x^2)\sin(t)$. *The numerical results of example 3 are summarized in table 2 and figure 2. Table 3 gives a comparison between the proposed method in* [27] *and Cardinal Chebyshev functions* [28]. *Better accuracy than the other methods.*

Example 4. *In this example* [15]*, we take a diffusion problem as*

$$u_t(x,t) = u_{xx}(x,t) + g(x,t) - \int_0^t \frac{t-s+1}{x+1} u(x,s)ds, \quad x,t \in [0,1], \tag{41}$$

where $g(x,t)$ *is determined such that the solution is* $u(x,t) = \dfrac{1-x^2}{1+t^2}$. *The numerical results for this example are summarized in Figure 3. Our numerical tests are better than that given by the Legendre multi-wavelets collocation method* [27].

t	$N = 3$	$N = 4$	$N = 5$	$N = 6$
0.0625	2.2589 E-4	4.2169 E-5	1.4692 E-6	1.5646 E-7
0.1250	3.2716 E-4	4.9342 E-5	1.4308 E-6	1.2011 E-7
0.1875	3.3538 E-4	3.9105 E-5	8.5875 E-7	5.3754 E-8
0.2500	2.8047 E-4	2.2820 E-5	3.0315 E-7	1.2997 E-8
0.3125	1.9333 E-4	1.0381 E-5	1.9737 E-7	4.1951 E-9
0.3750	9.6781 E-5	1.1357 E-5	1.4119 E-7	1.5299 E-8
0.4375	1.0161 E-4	8.9784 E-6	1.6239 E-7	1.4767 E-8
0.500	1.2012 E-4	6.8816 E-6	2.3785 E-7	1.0269 E-8
0.5625	1.1537 E-4	6.8278 E-6	2.0673 E-7	1.2452 E-8
0.6250	1.0561 E-4	1.0505 E-6	1.3352 E-7	1.4616 E-8
0.6875	7.0710 E-5	9.1550 E-6	2.0449 E-7	1.2901 E-8
0.7500	9.5995 E-5	6.0418 E-6	2.5585 E-7	1.1463 E-8
0.8125	1.4199 E-4	6.3528 E-6	2.2374 E-7	1.5682 E-8
0.8750	1.1942 E-4	1.2261 E-5	2.2730 E-7	5.8400 E-9
0.9375	7.4179 E-5	8.711 E-6	2.8581 E-7	2.0667 E-8

Table 2. Errors of the present method using the OBP method for test (3).

	LMW Collocation Method [27]			Chebyshev CF [28]	Present Method
t	$N = 8$	$N = 16$	$N = 32$	$N = 8$	$N = 8$
0.0625	7.4383 E-5	4.6240 E-6	1.2106 E-5	2.2070 E-8	2.5266 E-10
0.1250					1.2593 E-10
0.1875	7.5155 E- 5	1.2275 E-5	2.4685E-5	1.1514 E- 9	4.6585 E-11
0.2500					3.1922 E-11
0.3125	1.4643 E-4	2.5696 E-5	3.5745 E- 5	4.8570 E-8	2.8864 E-11
0.3750					1.3091 E-11
0.4375	7.5929 E-5	4.2169 E-5	4.5563 E-5	1.4616 E- 9	1.2449 E-11
0.500					1.0695 E-11
0.5625	1.2180 E-4	6.0743 E-5	5.3926 E- 5	1.7855 E-9	1.2261 E-11
0.6250					9.3796 E-12
0.6875	1.0567 E-4	8.1933 E-5	6.0499 E- 5	1.0870 E-7	6.6454 E-12
0.7500					1.3368 E-11
0.8125	4.7215 E- 5	1.0738 E- 4	6.4915 E-5	5.3619 E- 9	1.1561 E-11
0.8750					1.5320 E-11
0.9375	2.1869 E-4	1.3833 E-4	6.6396 E-5	3.8717 E-7	1.2016 E-11

Table 3. Errors of the present method using the OBBP method for test (3).

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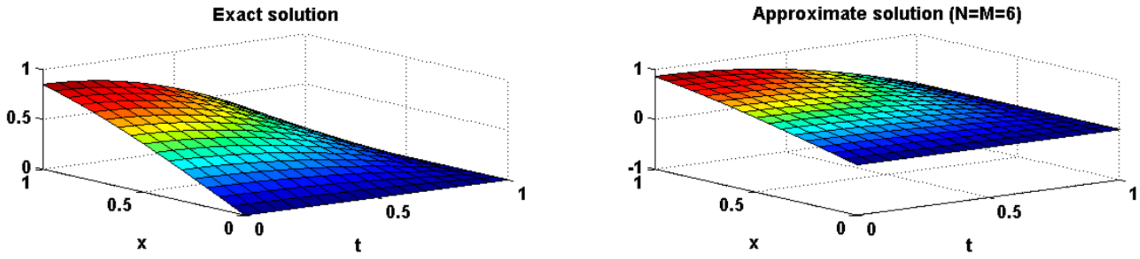

Figure 2. Exact (left) and approximate (right) solutions for example 3 for (N = M = 6).

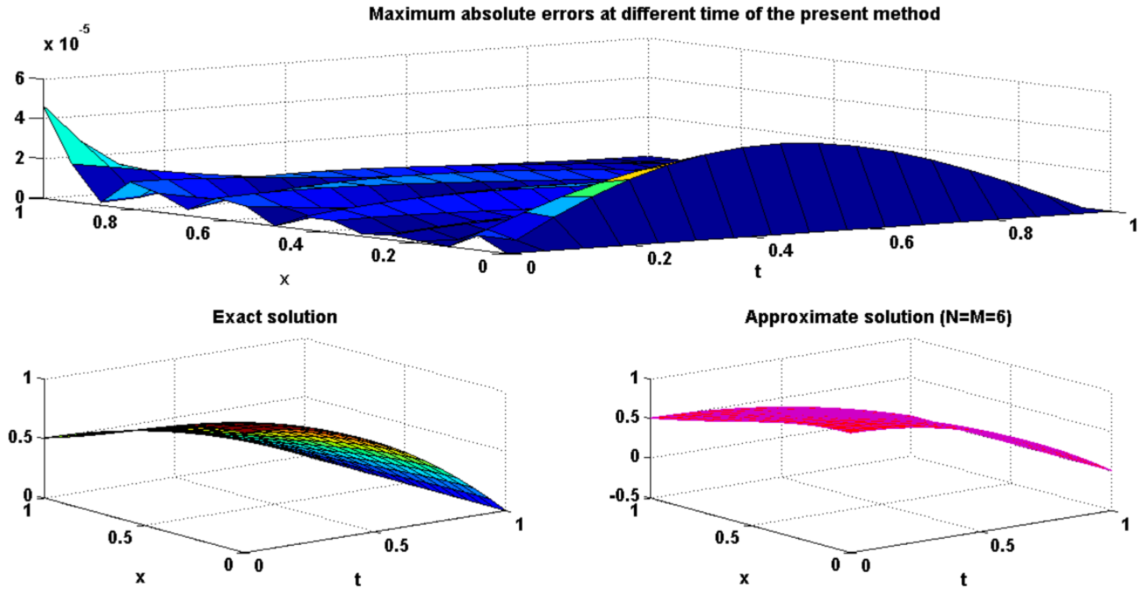

Figure 3. Errors, Exact (left) and approximate (right) solutions for example 4 for (N = M = 6).

Example 5. *We are given a linear problem as follows* [27]

$$u_t(x,t) = u_{xx}(x,t) + g(x,t) - \int_0^t u(x,s)ds, \quad x,t \in [0,1], \tag{42}$$

subject to the following conditions: $u(x,0) = \frac{1-x^2}{2}$, $u(0,t) = \frac{\cosh(t)}{2+\sinh^2(t)}$ *and* $u(1,t) = 0$, *where* $g(x,t)$ *is determined such that the analytical solution is* $u(x,t) = \frac{(1-x^2)\cosh(t)}{2+\sinh^2(t)}$. *The results for this example are given in Tables 4-5 and Figure 4. The numerical experiments obtained for this example are better than that given by the Legendre multi-wavelets collocation method* [27].

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t	N=4	N=5	N=8
0.0625	2.3407E-4	5.1581 E-6	3.1127 E-7
0.1250	2.7731 E-4	3.7329 E-6	1.5271 E-7
0.1875	2.2190 E-4	2.1161 E-6	5.2969 E-8
0.2500	1.2981E-4	1.5113 E-6	4.0246 E-8
0.3125	5.9121 E-5	1.3529 E-6	4.0669 E-8
0.3750	6.5468E-5	9.7856 E-7	1.8271 E-8
0.4375	5.3368 E-5	2.8124 E-7	1.6847 E-8
0.500	4.0608 E-5	1.0203 E-6	1.7874 E-8
0.5625	4.0269E-5	1.6929 E-6	1.7609 E-8
0.6250	6.2131 E-5	1.3648 E-6	1.7117 E-8
0.6875	5.4164 E-5	7.2680 E-7	1.0399 E-8
0.7500	3.7198 E-5	1.8706 E-6	2.1713 E-8
0.8125	3.7310 E-5	2.2765 E-6	1.8944 E-8
0.8750	7.1458E-5	1.6466 E-6	2.6507 E-8
0.9375	4.7623 E-5	2.8961 E-6	2.1335 E-8

Table 4. Errors of the present method using the OBBP method for test (5).

Example 6. *Here, we take the following PIDE*

$$u_t(x,t) + u_{xx}(x,t) = g(x,t) - \int_0^t e^{s-t} u(x,s) ds \tag{43}$$

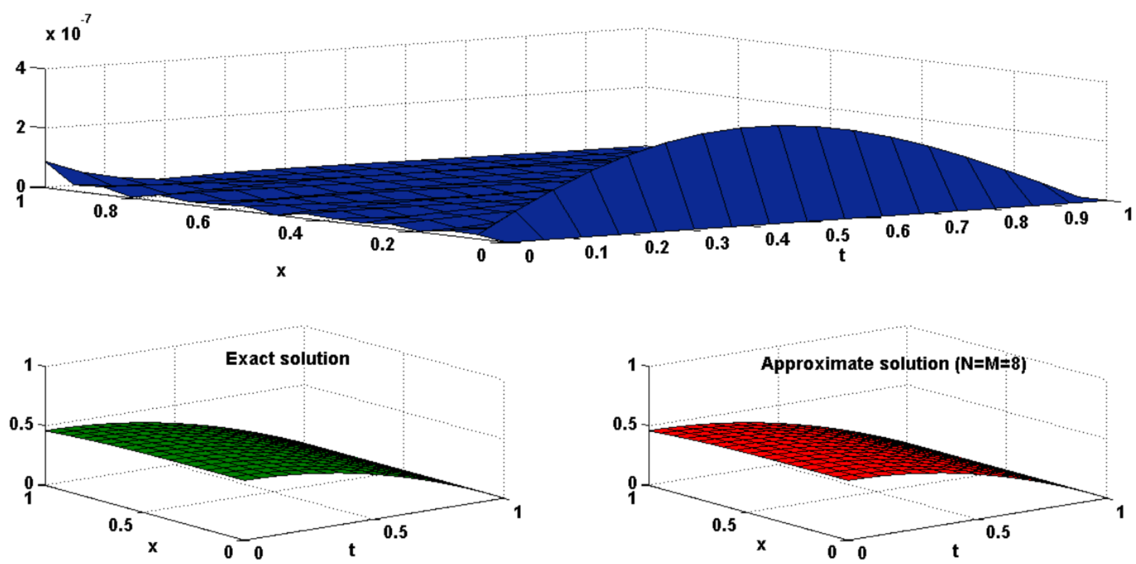

Figure 4. Errors, Exact (left) and approximate (right) solutions for example 5 for (N = M = 8).

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Legendre multiwavelets Method [27]				Present Method
t	$N = 8 \times 8$	$N = 16 \times 16$	$N = 64 \times 64$	Present method N=8
0.1	1.8049 E-6	9.2110 E-7	1.1342 E-7	2.2094 E-7
0.2	1.3464 E-5	2.3295 E-6	4.3291 E-8	4.5662 E-8
0.3	7.2956 E-5	1.9806 E-6	1.8598 E-7	4.2211 E-8
0.4	4.4007 E-5	1.7830 E-5	5.6575 E-7	9.7235 E-9
0.5	3.7671 E-4	7.7404 E-5	4.0517 E-6	1.7920 E-8
0.6	4.5192 E-5	9.5238 E-6	1.0850 E-6	1.7605 E-8
0.7	2.3648 E-5	2.0197 E-5	1.1987 E-6	7.2310 E-9
0.8	8.3275 E-5	2.3185 E-5	1.5333 E-6	2.0556 E-8
0.9	1.0790 E-4	1.1975 E-5	7.5777 E-7	1.8646 E-8
1.0	3.4302 E-4	8.6424 E-5	4.5429 E-6	2.5563 E-8

Table 5. Errors using the OBBP method for test (5).

with $u(x,0) = x$, $u(0,t) = 0$, $x \in [0,1]$, $u(1,t) = e^{-t}$, $t \in [0,1]$, where $g(x,t)$ is determined such that $u(x,t) = x\exp(-xt)$ is the analytical solution. We remark that when N increases, the error decreases. The errors obtained by our method for $N = 10$ are presented in 5 and give better results than those given by the Hermite-Taylor matrix method for $N = 12$ [9] and radial basis functions $N = 40$ [14].

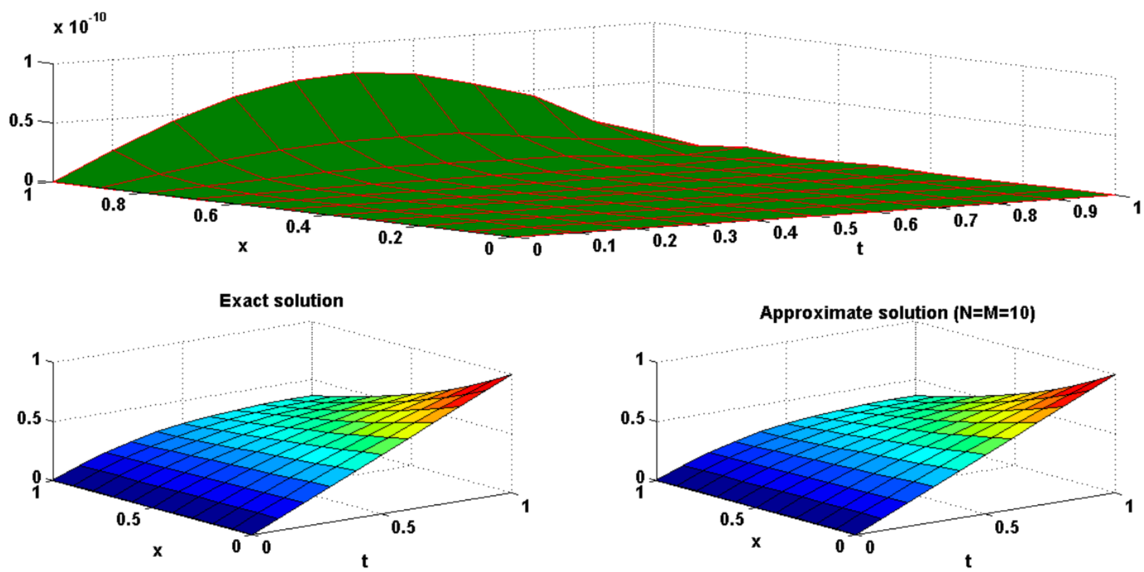

Figure 5. Errors, Exact (left) and approximate (right) solutions for example 6 for (N = M = 10).

6. Conclusion

In this article, a new numerical approach was proposed. This approach was utilized to solve partial integro-differential equations with Volterra and Fredholm types. The matrices of orthonormal Bernoulli polynomials were derived and used to obtain the approximate solution of PIDEs. After, we take Gauss-Legendre nodes in the intervals $[0, b]$ and $[0, T]$ as collocation points. The approach was applied to obtain numerical solutions of some test problems. The numerical results show the high accuracy of the scheduled algorithm. The presented method is easily implementable and simple and can be used for different types of PIDEs and also for differential equations. Many test problems were inserted and compared with other algorithms to appreciate the good efficiency of the proposed methodology. The proposed algorithm can be employed in more dimensions.

Other References

- C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, Spectral Methods: Fundamentals in Single Domains, Springer, New York, NY, USA, 2006.
- E. Tohidi, A.H. Bhrawy, K. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. Appl. Math. Model. 37(6), (2013) 4283–429.
- F. Toutounian, E. Tohidi, A new Bernoulli matrix method for solving second order linear partial differential equations with the convergence analysis. Appl. Math. Comput. 223, (2013) 298–310.
- F.A. Costabile, F. DellAccio, Expansions over a rectangle of real functions in Bernoulli polynomials and applications. BIT 41(3), (2001) 451–464.
- P. Natalini, A. Bernaridini, A generalization of the Bernoulli polynomials, J. Appl. Math.3 (2003) 155-163.
- S. Mashayekhi, Y. Ordokhani, M. Razzaghi, Hybrid functions approach for nonlinear constrained optimal control problems. Commun. Nonlinear Sci. Numer. Simul. 17, (2012) 1831–1843.

References

1. ^*A.V. Manzhirov, On a method of solving two-dimensional integral equations of axisymmetric contact problems for bodies with complex rheology, J. Appl. Math. Mech. 49 (6) (1985) 777–782.*
2. ^*G.P. Galdi, K. Pileckas, A.L. Silvestre, On the unsteady Poiseuille flow in a pipe, Z. Angew. Math. Phys. 58 (6) (2007) 994–1007.*
3. ^*H.J. Ding, H.M. Wang, W.Q. Chen, Analytical solution for the electroelastic dynamics of a nonhomogeneous spherically isotropic piezoelectric hollow spher e, Arch. Appl. Mech. 73 (1–2) (2003) 49–62.*
4. ^*M.A. Bartoshevich, A heat-conduction problem, J. Eng. Phys. 28 (2) (1975) 240–244.*
5. ^*M.V. Mirkin, A.J. Bard, Multidimensional integral equations. Part 1. A new approach to solving microelectrode diffusion problems, J. Electroad. Chem. 323 (1 –2) (1992) 1–27.*
6. ^*P.C. Hansen, T.K. Jensen, Large-Scale Methods in Image Deblurring, Lect. Notes Comput. Sci., vol. 4699, 2007.*
7. ^*P. Baratella, A Nystrom interpolant for some weakly singular linear Volterra integral equations, Comput. Appl. Math. 231 (2) (2009) 725–734.*
8. ^*A.H. Bhrawy, E. Tohidi, F. Soleymani, A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals. Appl. Math. Comput. 219(2), (2012) 482–497.*
9. a, b, c*E. Yalc¸ın, Ö.K. Kürkxcxxux, M. Sezer, Amatched Hermite-Taylor matrixmethod to solve the combined¨ partial integro-differential equations having no nlinearity and delay terms. Comput. Appl. Math. 39, 280 (2020).*
10. ^*E. Tohidi, F. Toutounian, Convergence analysis of Bernoulli matrix approach for one-dimensional matrix hyperbolic equations of the first order. Comput. Math. Appl. 68(1) (2014) 1–12.*
11. ^*P.K. Sahu, S.S. Ray, A new Bernoulli wavelet method for accurate solutions of nonlinear fuzzy Hammerstein–Volterra delay integral equations. Fuzzy. Sets Syst. 309, (2017) 131–144.*
12. a, b*R. Zeghdane, Numerical solution of stochastic integral equations by using Bernoulli operational matrix. Mathematics and Computers in Simulation 165, (2019) 238-254.*
13. ^*S. Bazm, Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations. J. Comput. Appl. Math. 275 (2015) 44 –60.*
14. a, b, c*Z. Avazzadeh, Z.B. Rizi, F.M. Maalek Ghaini, G.B. Loghmani, A numerical solution of nonlinear parabolictype Volterra partial integro-differential equat ions using radial basis functions. Eng. Anal. Bound. Elem. 36 (2012) 881–893.*
15. a, b, c*F. Fakhar-Izadi and M. Dehghan, The spectral methods for parabolic Volterra integro-differential equations, Journal of Computational and Applied M athematics, vol. 235, no. 14 (2011) 4032–4046.*

16. ^*H. Han, L. Zhua, H. Brunner, J. Ma, The numerical solution of parabolic Volterra integro-differential equations on unbounded spatial domains. Appl. Numer. Math. 55 (2005) 83–99.*

17. ^*B.G. Pachpatte, On a nonlinear diffusion system arising in reactor dynamics. J. Math. Anal. Appl. 94 (1983) 501–508.*

18. ^*C.V. Pao, Solution of a nonlinear integrodifferential system arising in nuclear reactor dynamics. J. Math. Anal. Appl. 48 (1974) 470–492.*

19. ^*F. Abeergel, R. Tachet, A nonlinear partial integro-differential equation from mathematical finance. AIMS, 27 (2010) 907–917.*

20. ^*Zogheib, B., Tohidi, E.: A new matrix method for solving two-dimensional time-dependent diffusion equations with Dirichlet boundary conditions. Appl. Math. Comput. 291, (2016) 1–13.*

21. ^*H. Engler, On some parabolic integro-differential equations—Existence and asymptotics of solutions. Lect. Notes Math. 1017 (1983) 161–167.*

22. ^*H.M. Srivastava, J.Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic, Dordreeht, 2001.*

23. ^*E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons Press, New York, 1978.*

24. ^*M. H. Heydari, Z. Avazzadeh, New formulation of the orthonormal Bernoulli polynomials for solving the variable-order time fractional coupled Boussinesq–Burger's, Engineering with Computers, Springer-Verlag London Ltd., part of Springer Nature 2020. https://doi.org/10.1007/s00366-020-01007-w.*

25. ^*Mirzaee Farshid, Nasrin Samadyar, Explicit representation of orthonormal Bernoulli polynomials and its application for solving Volterra–Fredholm-Hammerstein integral equations, SeMA Journal, 77 (2020) 81-96.*

26. a, b, c*C. Canuto and A. Quarteroni, Approximation Results for Orthogonal Polynomials in Sobolev Spaces. MAathematics of Computation Volume 38, Number 157 (1982).*

27. a, b, c, d, e*Aziz Imran and Imran Khan, Numerical Solution of Partial Integrodifferential Equations of Diffusion Type. Hindawi Mathematical Problems in Engineering Volume (2017), Article ID 2853679, 11 pages https://doi.org/10.1155/2017/2853679.*

28. ^*Fairouz Tchier, Ioannis Dassios, Ferdous Tawfiq and Lakhdar Ragoub, On the Approximate Solution of Partial Integro-Differential Equations Using the Pseudospectral Method Based on Chebyshev Cardinal Functions. Mathematics, 9, 286 (2021). https://doi.org/10.3390/math9030286.*

Declarations

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