

## Research Article

# Machine-Checked Bell-Violating Correlations with Deterministic Outcomes and Local Measurement Dynamics: A Lean 4 Formalisation

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We present a machine-checked Bell-consistency analysis for the finite-dimensional singlet sector of Constraint-Surface Dynamics (CSD), in which realised outcomes are deterministic almost everywhere and reproduce Bell-violating singlet correlations, while the finite measurement dynamics remain tensor-factorised and the observable marginals satisfy no-signalling. Given a singlet preparation sector and measurement independence, deterministic context-indexed outcome maps on a shared probability space reproduce

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4},$$

and attain the maximal quantum CHSH value  $2\sqrt{2}$ . We formally prove that no compatible setting-local four-response assignment  $A_1, A_2, B_1, B_2$  can reproduce the singlet correlations at the four specified CHSH contexts, namely that the realised maps cannot all satisfy

$$F_{ij}(x) = (A_i(x), B_j(x)).$$

The Bell obstruction therefore excludes a compatible setting-local four-response representation of the singlet correlations at the four specified CHSH contexts, not determinism of the realised outcome.

The formal development also studies the structure of the finite measurement interaction in the same singlet sector, without assuming that the contextual outcome family is derived from that finite measurement dynamics. Local setting transformations  $U_A(a) \otimes U_B(b)$  and local detector couplings  $V_A \otimes V_B$  compose as

$$(V_A \otimes V_B)(U_A(a) \otimes U_B(b))^\dagger = (V_A U_A(a)^\dagger) \otimes (V_B U_B(b)^\dagger),$$

so the complete finite measurement chain remains tensor-factorised. Pointer-volume probabilities reproduce the singlet law for all detector settings, including the perfect-correlation and perfect-anticorrelation limits. Operational no-signalling is verified at both the singlet-kernel and pointer-volume levels, with each local marginal equal to  $1/2$  and invariant under the remote setting.

The singlet preparation sector and measurement independence are explicit assumptions, and no general no-signalling theorem for arbitrary non-factorising ontic dynamics is asserted. The contribution is not a new Bell inequality or a new general result that deterministic models can reproduce Bell-violating correlations. Rather, it is a CSD-specific consistency construction and a machine-checked separation of contextual determinism, Bell-local response factorisation, finite measurement factorisation, and operational no-signalling. Within this stated scope, the Lean 4 development provides a formally verified example in which deterministic realised outcomes almost everywhere, Bell-violating singlet correlations, tensor-factorised finite measurement dynamics, Bell-nonfactorisable joint outcomes, and operational no-signalling coexist, while any compatible Bell-local four-response assignment reproducing the singlet correlations at the four specified CHSH contexts is mathematically excluded.

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## 1. Introduction

### 1.1. Fine's Theorem and the Structure of Bell Constraints

In the bipartite Bell–CHSH scenario, two separated parties choose between measurement settings  $a_1, a_2$  and  $b_1, b_2$ , with each measurement yielding a binary outcome. A Bell-local deterministic representation assigns four response variables

$$A_1, A_2, B_1, B_2 \in \{\pm 1\}$$

on a common probability space, with the observed outcome pair for context  $(a_i, b_j)$  obtained from the corresponding pair  $(A_i, B_j)$ .

For binary outcomes with mutually consistent one-party marginals, Fine's theorem establishes the equivalence between the existence of a joint probability distribution

$$P(A_1, A_2, B_1, B_2)$$

whose pairwise marginals reproduce the four experimentally accessible contexts and satisfaction of the Bell–CHSH inequalities (Fine 1982). In particular, any such setting-local four-response representation obeys the CHSH inequality

$$|S_{\text{CHSH}}| \leq 2$$

(Clauser et al. 1969), whereas quantum mechanics permits values up to the Cirel'son bound

$$|S_{\text{CHSH}}| \leq 2\sqrt{2}$$

(Cirel'son 1980), with the upper value attained by suitable measurements on the two-qubit singlet state.

The distinction relevant here is more precise than the statement that “all outcomes cannot coexist”. Context-indexed outcome maps may be defined on a common state space without contradiction. What Bell–CHSH excludes is a particular compatibility structure: the outcome assigned to Alice's setting  $a_i$  must be represented by the same response function  $A_i$  irrespective of Bob's setting, and conversely for  $B_j$ . If the four realised context maps are denoted

$$F_{ij} : \Sigma \rightarrow \{\pm 1\} \times \{\pm 1\},$$

the prohibited Bell-local gluing is therefore

$$F_{ij}(x) = (A_i(x), B_j(x))$$

for all four CHSH contexts on the same underlying probability space.

This formulation separates two logically distinct notions. Determinism requires a definite outcome for the measurement context that is realised. Bell locality requires, in addition, a compatible collection of setting-local responses that can be reused across the unrealised alternative contexts. Bell violation excludes the latter structure; it does not, by itself, exclude deterministic realised outcomes.

The formal development used in this paper implements this distinction directly. Rather than taking Fine's theorem as an additional formal axiom, the central Lean 4 obstruction is obtained from the machine-checked CHSH bound for measurable setting-local responses and the verified singlet violation. Fine's theorem provides the broader probabilistic interpretation of that obstruction.

### *Relation to prior work and novelty*

The logical ingredients of the present Bell obstruction are established results. Deterministic completions of quantum mechanics have longstanding precedents, most notably Bohm's hidden-variable theory <sup>[1][2]</sup>. Bell's subsequent analysis of hidden-variable assumptions and the Kochen–Specker theorem established the central role of context dependence in hidden-variable descriptions <sup>[3][4]</sup>. Modern ontological-model and contextuality frameworks sharpen these distinctions <sup>[5][6][7]</sup>, while the different notions of locality entering Bell arguments have been analysed extensively in the subsequent literature <sup>[8][9][10]</sup>. Accordingly, the present paper does not claim novelty for the Bell–CHSH obstruction, for deterministic contextual models as such, or for the general distinction between operational no-signalling and Bell-local hidden responses.

Formal verification of Bell results is likewise not new. The CHSH inequality and Cirel'son bound have been formalised in Isabelle/HOL [\[11\]](#), and CHSH rigidity has been formalised in Lean 4 [\[12\]](#). The novelty claimed here is narrower: the CSD-specific singlet consistency architecture and its machine-checked theorem package, including full singlet-table reproduction, the shared-domain Bell obstruction, model-level no-signalling, factorised finite measurement dynamics, explicit non-vacuity checks, and repository-level axiom auditing.

## 1.2. Determinism and the Risk of Global Outcome Maps

Constraint-Surface Dynamics (CSD) describes a finite physical system in terms of an ontic state  $x$  on a constraint surface  $\Sigma$ , equipped with a measure  $\mu_L$ , deterministic dynamics  $\Phi$ , and a many-to-one projection

$$\pi : \Sigma \rightarrow \mathbb{CP}^{N-1}$$

to the effective projective state space. Quantum probabilities are represented operationally by measures of context-dependent outcome sectors, while a realised measurement produces a definite record for the actual ontic state.

For a local qubit, the effective pure-state space is

$$\mathbb{CP}^1 \simeq S^2.$$

A two-qubit pure state, however, belongs to the joint effective space

$$\mathbb{CP}^3.$$

The Bell experiment considered here is therefore a two-qubit composite problem with local  $\mathbb{CP}^1$  sectors, not a single  $\mathbb{CP}^1$  Bell state space.

The deterministic character of CSD raises an immediate Bell-facing question. If the underlying state  $x$  is definite and the realised measurement context has a definite outcome almost everywhere with respect to the preparation measure, must there also exist four measurable response functions

$$A_1(x), A_2(x), B_1(x), B_2(x)$$

such that every context-indexed outcome map can be recovered as

$$F_{ij}(x) = (A_i(x), B_j(x))?$$

If such a representation existed for a fixed preparation measure, the four responses would define precisely the joint structure to which the CHSH bound applies.

It is important that the issue is not whether the four experimental contexts can be described on a shared probability space. In the construction below they are. A single preparation measure is used across the four setting choices, making measurement independence an explicit assumption. Nor is the issue whether one may mathematically collect the four context-indexed maps into a larger tuple. One may do so. The substantive question is whether those maps admit the stronger setting-local compatibility condition required to identify Alice's response at  $a_i$  across Bob's alternative settings, and Bob's response at  $b_j$  across Alice's alternatives.

For the singlet correlations, the answer is negative. The same shared-domain construction that supplies deterministic outcomes for each realised context reproduces a CHSH value of  $2\sqrt{2}$ . Consequently, no compatible setting-local four-response assignment can reproduce the singlet correlations at those four CHSH contexts. The resulting determinism is therefore contextual in a precise Bell sense: the actual outcome is definite, but the family of realised-context maps cannot be reduced to a Bell-local table of pre-existing setting responses.

### 1.3. Scope and Strategy

We analyse the minimal two-qubit Bell-CHSH setting in which this distinction can be tested explicitly. The singlet preparation is used as the worked Bell-violating sector because its joint probability kernel takes the simple form

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4}, s, t \in \{\pm 1\},$$

giving

$$E(a, b) = -a \cdot b$$

and the canonical quantum value

$$|S_{\text{CHSH}}| = 2\sqrt{2}.$$

The singlet is therefore the explicit witness used to resolve the Bell-consistency question; it is not assumed to exhaust the scope of contextual deterministic measurement in CSD.

The analysis proceeds in three steps while retaining the structure of the original C1 argument. First, we formulate deterministic context-indexed outcome maps on one shared probability space and define the stronger compatibility condition required for a Bell-local four-response assignment. For the four CHSH contexts, the Lean 4 development proves that no such compatible assignment can reproduce the singlet correlations. This is a direct no-go result, rather than an inference from the mere type-level distinction between context maps and global assignments.

Second, we examine the measurement implementation itself. The finite construction uses local setting transformations

$$U_A(a) \otimes U_B(b)$$

and a local detector coupling

$$V_A \otimes V_B.$$

Their composition is formally verified to satisfy

$$(V_A \otimes V_B)(U_A(a) \otimes U_B(b))^\dagger = (V_A U_A(a)^\dagger) \otimes (V_B U_B(b)^\dagger).$$

Thus the complete finite measurement chain is tensor-factorised. This result concerns the dynamical structure of the explicit finite measurement construction. It does not restore Bell-local outcome factorisation, which is excluded by the preceding CHSH argument.

Third, we verify operational no-signalling within the same singlet framework, both for the contextual outcome model and at the level of the finite pointer-volume construction. At the singlet-kernel level,

$$\sum_{t=\pm 1} P_{st}(a, b) = \frac{1}{2}, \quad \sum_{s=\pm 1} P_{st}(a, b) = \frac{1}{2},$$

so each local marginal is independent of the remote setting. The Lean 4 development also establishes the corresponding equality of marginal pointer volumes in the explicit measurement construction, and the pointer-volume realisation reproduces the singlet probabilities for all detector settings, including the perfect-correlation and perfect-anticorrelation endpoints.

The resulting claim is deliberately narrower than a claim of Bell locality. CSD does not recover setting-local hidden responses, and the tensor factorisation established here concerns the explicit finite measurement dynamics rather than arbitrary ontic dynamics on a general non-factorising  $\Sigma$ . Likewise, the singlet preparation sector and measurement independence are inputs to the analysis, and a general derivation of no-signalling from arbitrary CSD primitives is not asserted.

Within these boundaries, the paper closes a CSD-specific Bell-consistency obligation for the exhibited singlet sector. It does not introduce a new Bell theorem. Rather, it shows within one formally verified framework that deterministic contextual outcomes, singlet correlations, tensor-factorised finite measurement dynamics, and operational no-signalling are mutually consistent, while the Bell-local four-response structure excluded by CHSH cannot be imposed. The novelty claimed here lies in the CSD-specific architecture and its machine-checked theorem package, not in the Bell-CHSH obstruction itself. The principal theorem chain is formalised

in Lean 4 in the public `csd-lean4` repository, with the Bell-consistency, locality, probability-reproduction, and no-signalling results separately axiom-audited.

## 2. Formal Structure of Measurement in the Two-Qubit Bell Scenario

The Bell analysis requires two complementary levels of description. At the probabilistic level, all four CHSH contexts are defined on one shared measurable state space with one fixed preparation measure. At the finite measurement level, the detector settings and pointer coupling are represented explicitly and shown to factorise between the two wings. Keeping these levels distinct is essential: factorisation of the measurement implementation is not the Bell-local factorisation of outcomes that will be excluded in Section 3.

### 2.1. Ontic Manifold and Preparation

Let

$$(\Sigma, \mathcal{F}, \mu)$$

denote the shared probability space underlying the Bell analysis. The state  $x \in \Sigma$  is sampled from a single preparation measure  $\mu$ , and the same measure is used for every subsequent choice of detector settings. This setting independence of the preparation measure is the measurement-independence assumption used throughout the CHSH argument.

In the conventional notation of Bell-type hidden-variable models, this shared state space would ordinarily be denoted by  $\Lambda$ . We use  $\Sigma$  here to avoid collision with notation used elsewhere in the CSD programme. At this stage  $\Sigma$  is simply an arbitrary measurable space carrying a probability measure; no CSD-specific geometric structure is assumed. This generality is important: the Bell obstruction proved below therefore applies to any theory admitting the same shared-probability-space representation. The literature-facing quantum-information module `CsdLean4/Empirical/QM/Crypto/E91.lean` retains Bell's conventional  $\Lambda$  notation deliberately.

Within the concrete Constraint-Surface Dynamics (CSD) singlet witness, the shared state space is realised as

$$\Sigma_{\text{Bell}} = \mathbb{CP}^3 \times T^2,$$

where  $\mathbb{CP}^3$  is the projective state space of the two-qubit system and  $T^2$  is a compact internal fibre. The projection to the effective quantum state is

$$\pi : \Sigma_{\text{Bell}} \rightarrow \mathbb{CP}^3, \pi(p, \theta) = p.$$

The associated reference measure is the product measure

$$\mu_L = \mu_{\text{FS}} \otimes \text{vol}_{T^2},$$

where  $\mu_{\text{FS}}$  is the Fubini–Study probability measure on  $\mathbb{CP}^3$ .

For the singlet preparation used in this paper, the relevant probability law is

$$\mu_{\psi^-} = \delta_{[\psi^-]} \otimes \text{vol}_{T^2}.$$

Consequently,

$$\pi_* \mu_{\psi^-} = \delta_{[\psi^-]},$$

so the projective component is fixed at the singlet ray while the internal fibre remains distributed according to its normalised volume measure. The singlet preparation sector is therefore an explicit input to the construction, not a result derived in the present paper.

This formulation avoids any need to express the preparation as a density with respect to  $\mu_L$ . The Bell argument requires only a common probability measure, and the concrete Lean development uses  $\mu_{\psi^-}$  directly.

Each local qubit retains the usual effective pure-state space

$$\mathbb{CP}^1 \simeq S^2,$$

whereas the composite two-qubit pure state belongs to

$$\mathbb{CP}^3.$$

The distinction is important because the Bell experiment is a composite-system problem rather than a single- $\mathbb{CP}^1$  construction.

#### *Lean correspondence:*

The concrete product space, reference measure, projection, and measure bridge are defined in `CsdLean4/LF4/KahlerInstance.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF4/KahlerInstance.lean>

The relevant definitions and theorem are `KSigma`, `kMuL`, `kSectorData`, and `k_measure_bridge`.

The singlet ray and singlet preparation measure are constructed in `CsdLean4/LF4/SingletKahler.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF4/SingletKahler.lean>

Here `singletRay`, `kMuPsi`, and `kMuPsi_push` implement the preparation used above, with `kMuPsi_push` machine-checking

$$\pi_* \mu_{\psi^-} = \delta_{[\psi^-]}.$$

The Lean development verifies the product types, measures, projection, and measure identities used here. The identification of the factors with their standard compact Kähler geometries is the corresponding differential-geometric interpretation rather than a separately formalised Kähler-manifold structure in Lean.

## 2.2. Measurement Settings as Local Finite Dynamics

A detector setting is represented by a unit vector

$$a \in S^2$$

on Alice's wing and

$$b \in S^2$$

on Bob's wing. For each setting, the eigenvectors of the corresponding local spin observable define a unitary change of basis. We denote the local transformations by

$$U_A(a), U_B(b).$$

Their joint action is the tensor product

$$U_{AB}(a, b) = U_A(a) \otimes U_B(b).$$

Thus the choice of measurement axes is implemented by separate transformations on the two local factors.

For the singlet

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

the state expressed in the selected local measurement bases is

$$|\psi_{a,b}^-\rangle = U_{AB}(a, b)^\dagger |\psi^-\rangle.$$

For outcomes  $s, t \in \{\pm 1\}$ , the squared modulus of the corresponding coordinate is

$$|\langle s_a, t_b | \psi^- | s_a, t_b \rangle|^2 = \frac{1 - s_a \cdot b}{4}.$$

This identity is established for the product-unitary construction for every pair of detector settings, without excluding the parallel or antiparallel limits.

The detector interaction is represented by a local dilation on each wing. Let

$$V_A : \mathcal{H}_A \rightarrow \mathcal{H}_A \otimes \mathcal{H}_{P_A}, V_B : \mathcal{H}_B \rightarrow \mathcal{H}_B \otimes \mathcal{H}_{P_B}$$

denote the two local system–pointer isometries. The joint detector map is constructed as

$$V_{\text{loc}} = V_A \otimes V_B,$$

up to the canonical reindexing that groups the two system factors and the two pointer factors.

The complete setting-dependent finite measurement map therefore satisfies

$$\begin{aligned} \mathcal{M}_{a,b} &= (V_A \otimes V_B)(U_A(a) \otimes U_B(b))^\dagger \\ &= \left( V_A U_A(a)^\dagger \right) \otimes \left( V_B U_B(b)^\dagger \right). \end{aligned}$$

Hence the complete finite measurement chain is tensor-factorised between the two wings. No cross-wing interaction is required by this finite implementation.

This is the locality statement used in the present paper. It concerns the structure of the explicit finite measurement dynamics. It does not assert Bell-local factorisation of the resulting outcome functions, nor does it establish a general locality theorem for arbitrary ontic dynamics on an unrestricted constraint surface.

The result is stated at the level established by the formal construction. The setting transformations are proved unitary and the detector maps are proved local isometric dilations. No stronger claim that these particular finite maps are here derived from specified continuous local Hamiltonian generators is required for the Bell-consistency result.

### *Lean correspondence:*

The detector-axis spinors and local basis unitaries are constructed in `CsdLean4/LF3/Spinor.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/Spinor.lean>

The relevant results include `spinor_normSq`, `spinProj_eq_outer`, and `wingBasisUnitary_mem_unitaryGroup`.

The product setting transformation and the corrected local representation of the singlet in the selected detector bases are defined in `CsdLean4/LF6/NudgeLocality.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/NudgeLocality.lean>

Here `wingPairUnitary` is the product  $U_A(a) \otimes U_B(b)$ ; `wingPairUnitary_mem_unitary` and `wingPairUnitary_mem_unitaryGroup` prove that this product is unitary; `localNudge` implements

$$(U_A(a) \otimes U_B(b))^\dagger |\psi^-\rangle;$$

and `localNudge_born` proves the corresponding singlet probability identity.

The local detector dilation is constructed in `CsdLean4/LF6/LocalDeisolationFlow.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/LocalDeisolationFlow.lean>

The theorem `localDeisolation_factorises` proves that the local de-isolation map reduces, under the explicit reindexing, to the tensor product of the two wing dilations.

The complete-chain identity is proved in `CsdLean4/LF6/NudgeLocality.lean` by `localMeasurementChain_factorises`.

### 2.3. Local Readout and Context-Indexed Outcome Maps

A measurement context is the pair

$$C = (a, b)$$

of detector settings. The Bell-facing outcome structure is defined on the same underlying state space for every context. Thus, for each  $C$ , we consider a measurable deterministic map

$$F_C : \Sigma \rightarrow \{\pm 1\} \times \{\pm 1\},$$

with

$$F_C(x) = (F_A(C, x), F_B(C, x)).$$

As a context-indexed measurable partition,  $F_C$  is a total mathematical map on  $\Sigma$ . For its physical interpretation we retain the conservative almost-everywhere qualification appropriate to CSD's dynamical record layer: the realised physical record is taken to be definite for  $\mu$ -almost every  $x \in \Sigma$ .

The theorem `CSD.RecordLayer.no_everywhere_correlation` continues to constrain continuous record-forming witnesses with connected ready regions and disjoint open record regions. Subsequent CSD record-layer work now also provides explicit deterministic measurement propagators with Born-weighted readout, persistence and externalisation.

Those later dynamical results are not premises of the Bell theorem proved here. The Bell argument requires only the shared-domain context-indexed maps, their singlet probabilities and the stated measurement-independence premise. The physical record interpretation is therefore consistent with, but logically downstream of, the CHSH obstruction.

The family

$$\{F_C\}_C$$

is therefore context-indexed but defined on one common domain. This shared domain is essential. Bell consistency cannot be tested by assigning unrelated state spaces to different contexts, because there would then be no common state variable on which to formulate compatibility with a four-response CHSH assignment.

The local labels  $F_A$  and  $F_B$  identify the two components of the joint readout. No pointwise remote-setting independence is imposed. In particular, we do not assume

$$F_A(a, b, x) = F_A(a, b', x)$$

or

$$F_B(a, b, x) = F_B(a', b, x)$$

for every  $x$ . Imposing both conditions would already provide setting-local response functions

$$A(a, x), B(b, x),$$

and would therefore build in the Bell-local structure whose compatibility with the singlet is precisely the question to be decided.

This distinction is central. Locality of the finite measurement map established in Section 2.2 concerns the tensor structure of the setting transformations and detector couplings. It does not imply that the resulting outcome components are independent setting-local hidden responses.

In the explicit finite pointer construction, each realised context has four pointer sectors labelled by

$$(s, t) \in \{\pm 1\}^2.$$

These sectors provide the operational readout associated with the joint outcome labels. Their measures are related to the singlet probability kernel in Section 2.4.

*Lean correspondence:*

The dynamical limitation motivating the conservative almost-everywhere physical interpretation is formalised by `CsdLean4/RecordLayer.no_everywhere_correlation` in `CsdLean4/RecordLayer/MeasurementConstraints.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/RecordLayer/MeasurementConstraints.lean>

Measurement contexts and the distinction between context-indexed maps and global CHSH assignments are defined in `CsdLean4/LF3/ContextMap.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/ContextMap.lean>

The file defines `MeasurementContext`, `ContextIndexedOutcomeMaps`, and `GlobalCHSHAssignment`. The latter two structures are deliberately not treated as mathematically incompatible merely because they are different Lean types.

The corrected shared-domain object required for the Bell analysis is defined in `CsdLean4/LF3/SharedContextMap.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/SharedContextMap.lean>

The structure `SharedContextOutcomeMaps` contains a single state space  $\Sigma$  and a context-indexed map

$$F : C \mapsto (\Sigma \rightarrow \{\pm 1\}^2).$$

The predicate `MeasurableSharedContextOutcomeMaps` requires each joint context map to be measurable. The component functions `wingA` and `wingB` and their measurability properties are derived in the same module.

The compatibility condition with a setting-local four-response assignment is defined separately in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean>

Its predicate `CompatibleWithGlobalCHSH` formalises the stronger condition

$$F_{ij}(x) = (A_i(x), B_j(x)),$$

which will be tested in Section 3 rather than assumed here.

## 2.4. Probabilities and Correlations

For a measurable context map  $F_C$  and preparation measure  $\mu$ , the operational joint probability is

$$P(s, t \mid a, b) = \mu \left( F_{(a,b)}^{-1} \{(s, t)\} \right).$$

Equivalently,

$$P(s, t \mid a, b) = \int_{\Sigma} \mathbf{1}_{\{F_{(a,b)}(x) = (s,t)\}}(x) d\mu(x).$$

The correlation associated with the context is

$$E(a, b) = \sum_{s,t=\pm 1} st P(s, t \mid a, b),$$

or equivalently

$$E(a, b) = \int_{\Sigma} F_A(a, b, x) F_B(a, b, x) d\mu(x),$$

where the sign-valued outcomes are identified with their numerical values  $\pm 1$ .

For the singlet sector, the formally verified kernel is

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4}.$$

The four probabilities are non-negative and normalised,

$$\sum_{s,t=\pm 1} P_{st}(a, b) = 1,$$

and their correlation is

$$\begin{aligned} E(a, b) &= \sum_{s,t=\pm 1} st P_{st}(a, b) \\ &= -a \cdot b. \end{aligned}$$

Thus the singlet correlation used by the later Bell argument is an explicit theorem of the formal development rather than a conditional placeholder.

The finite measurement construction also realises these probabilities through pointer-sector volumes. For each outcome pair  $(s, t)$ , the total pointer-block Fubini–Study volume after the local dilation satisfies

$$\mu_{\text{ptr}}(s, t \mid a, b) = P_{st}(a, b).$$

The corrected construction establishes this identity for every detector pair, including

$$a \cdot b = \pm 1.$$

At these endpoints two outcome sectors have zero probability and the remaining two have probability  $1/2$ , giving the expected perfect correlation or perfect anticorrelation. The earlier requirement that every  $P_{st}$  be strictly positive is therefore not needed by the local construction used in this paper.

The single-wing marginals follow directly:

$$\sum_{t=\pm 1} P_{st}(a, b) = \frac{1}{2}, \sum_{s=\pm 1} P_{st}(a, b) = \frac{1}{2}.$$

These identities will provide the kernel-level basis for the operational no-signalling analysis in Section 4.

At this stage no Bell-local compatibility condition has been imposed. We have one shared preparation space, deterministic measurable outcome maps for realised contexts, an explicit tensor-factorised finite measurement implementation, and the singlet probability law. Section 3 asks whether the four CHSH context maps can additionally be represented by one compatible setting-local four-response assignment.

### *Lean correspondence:*

The singlet probability kernel is defined as `P_st` in `CsdLean4/LF3/Singlet/Kernel.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/Singlet/Kernel.lean>

This module proves `P_st_nonneg`, the squared-amplitude identity `cst_squared_eq`, the correlation theorem `correlation_eq_neg_dot`, the marginal identities `marginal_a_eq_half` and `marginal_b_eq_half`, and the corresponding kernel-level no-signalling corollaries.

The genuine local-basis Born identity is proved in `CsdLean4/LF6/NudgeLocality.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/NudgeLocality.lean>

The theorem `localNudge_born` proves

$$|\langle s_a, t_b | \psi^- | s_a, t_b | \psi^- \rangle|^2 = P_{st}(a, b),$$

and `localDeisolation_pointer_volume_local` proves the corresponding pointer-volume identity for all detector settings, without the earlier genericity hypothesis.

At the shared-context outcome-map level, the positive half of the C1 construction is proved in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean>

The theorem `singletContextualModel_table` proves that the exhibited measurable context-indexed outcome family reproduces the full singlet table

$$\mu(\{x : F_C(x) = (s, t)\}) = P_{st}(a, b)$$

for every measurement context and every outcome pair. The theorem `singletContextualModel_reproduces` therefore establishes the corresponding singlet correlations at the four CHSH contexts. Finally, `c1_singlet_contextual_capstone` packages the positive and negative halves into one statement: such a measurable singlet-reproducing shared-context family exists, while no compatible global CHSH assignment exists for it.

The concrete singlet preparation and Bell witness are exercised on the actual CSD arena in `CsdLean4/Tests/Witnesses/SingletBell.lean`:

<https://github.com/zblorc/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Tests/Witnesses/SingletBell.lean>

In particular, `P_st_setting_dependent` verifies that the kernel is genuinely setting dependent rather than a constant artefact, while `kMuPsi_no_global_chsh_assignment` instantiates the later Bell obstruction on the same concrete  $\mathbb{CP}^3 \times T^2$  arena.

### 3. Fine-Bell Obstruction to Compatible Setting-Local Outcome Assignment

Section 2 established the ingredients required for the Bell-consistency question: one shared preparation space, deterministic measurable outcome maps for each realised context, and the singlet correlations

$$E(a, b) = -a \cdot b.$$

We now ask whether the four context maps entering a CHSH experiment can additionally be represented by one setting-local assignment of responses to the two alternative settings on each wing.

The distinction is important. Nothing prevents four context-indexed maps from being defined on the same state space, nor from being collected mathematically into a single context-labelled object. The Bell obstruction concerns the stronger requirement that Alice's response to a given local setting be the same function of the underlying state for both of Bob's alternative settings, and conversely for Bob. It is this setting-local compatibility condition that is incompatible with the singlet correlations.

### 3.1. Global Joint Assignments

For the C1 obstruction we use the four detector settings fixed in the formalisation:

$$a_1 = \text{chshA} = (1, 0, 0), a_2 = \text{chshA}' = (0, 1, 0),$$

and

$$b_1 = \text{chshB} = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), b_2 = \text{chshB}' = \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right).$$

With

$$E(a, b) = -a \cdot b$$

and the convention

$$S = E(a_1, b_1) - E(a_1, b_2) + E(a_2, b_1) + E(a_2, b_2),$$

these settings give

$$S = -2\sqrt{2}, |S| = 2\sqrt{2}.$$

Let

$$F_{ij} : \Sigma \rightarrow \{\pm 1\} \times \{\pm 1\}, i, j \in \{1, 2\},$$

denote the deterministic outcome map associated with the measurement context

$$C_{ij} = (a_i, b_j).$$

A setting-local global CHSH assignment consists of four response functions

$$A_1, A_2, B_1, B_2 : \Sigma \rightarrow \{\pm 1\}.$$

Equivalently, these may be packaged as a map

$$G : \Sigma \rightarrow \{\pm 1\}^4,$$

with

$$G(x) = (A_1(x), A_2(x), B_1(x), B_2(x)).$$

The existence of such a map alone is not prohibited. The relevant additional condition is compatibility with the realised context maps:

$$F_{ij}(x) = (A_i(x), B_j(x)) \quad \text{for all } i, j \in \{1, 2\} \text{ and } x \in \Sigma.$$

This condition requires the same  $A_i$  to be reused across both choices of Bob's setting and the same  $B_j$  to be reused across both choices of Alice's setting.

If this compatibility condition holds, the four response variables

$$A_1, A_2, B_1, B_2$$

are defined on the same probability space

$$(\Sigma, \mathcal{F}, \mu).$$

Their joint pushforward therefore defines a probability distribution

$$P(A_1, A_2, B_1, B_2).$$

For binary outcomes, Fine's theorem relates the existence of such a compatible joint distribution to satisfaction of the Bell-CHSH inequalities, subject to the usual consistency of the observed pairwise marginals (Fine 1982).

The shared domain is essential. A collection of outcome maps whose domains themselves depend on the context would not provide a common state  $x$  on which the compatibility equations above could even be formulated. For this reason, the formal Bell analysis uses one common state space  $\Sigma$  and allows only the outcome map  $F_{ij}$  to vary with context.

### *Lean correspondence:*

The underlying measurement-context and global-response structures are defined in `CsdLean4/LF3/ContextMap.lean`:

<https://github.com/zblore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/ContextMap.lean>

The structure `GlobalCHSHAssignment` contains the four functions  $A_1, A_2, B_1$ , and  $B_2$ .

The common-domain context family required for the actual obstruction is defined in `CsdLean4/LF3/SharedContextMap.lean`:

<https://github.com/zblore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/SharedContextMap.lean>

Its structure `SharedContextOutcomeMaps` contains

$$F : \text{MeasurementContext} \rightarrow \Sigma \rightarrow \text{Sign} \times \text{Sign},$$

with a single state type  $\Sigma$  for every context. The predicate `MeasurableSharedContextOutcomeMaps` requires measurability only of these joint context maps.

The setting-local compatibility condition is defined in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean>

There `CompatibleWithGlobalCHSH` states precisely that, at each of the four CHSH contexts,

$$F_{ij}(x) = (A_i(x), B_j(x)).$$

Thus the formal obstruction below is a theorem about compatibility on a shared domain, not a consequence of representing context maps and global assignments by different Lean types.

### 3.2. Fine–Bell Obstruction

We now specialise to the four CHSH contexts defined by the detector settings specified in Section 3.1. Define

$$E_{ij} = \int_{\Sigma} A_i(x) B_j(x) d\mu(x),$$

and use the CHSH convention

$$S = E_{11} - E_{12} + E_{21} + E_{22}.$$

For any probability space  $(\Sigma, \mu)$  and measurable response functions

$$A_i, B_j \in \{\pm 1\},$$

the pointwise CHSH combination satisfies

$$A_1 B_1 - A_1 B_2 + A_2 B_1 + A_2 B_2 \in \{-2, +2\}.$$

Integration therefore gives

$$|S| \leq 2.$$

This is the Bell–CHSH bound for deterministic setting-local responses (Bell 1964; Clauser et al. 1969).

For the singlet kernel established in Section 2.4,

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4},$$

the corresponding correlation is

$$E(a, b) = -a \cdot b.$$

At the CHSH detector settings this gives

$$|S_{\psi^-}| = 2\sqrt{2}.$$

Since

$$2\sqrt{2} > 2,$$

the singlet correlations cannot be reproduced by a compatible setting-local four-response assignment.

The result can be stated directly as follows.

*Theorem 3.1 (Shared-domain CHSH obstruction):*

Let

$$(\Sigma, \mathcal{F}, \mu)$$

be a probability space, with the same measure  $\mu$  used for all four CHSH contexts. Let

$$\{F_{ij}\}_{i,j=1}^2$$

be measurable deterministic outcome maps

$$F_{ij} : \Sigma \rightarrow \{\pm 1\}^2$$

whose correlations reproduce the singlet correlations at the four detector settings specified in Section 3.1, for which

$$|S_{\psi^-}| = 2\sqrt{2}.$$

Then there do not exist response functions

$$A_1, A_2, B_1, B_2 : \Sigma \rightarrow \{\pm 1\}$$

such that

$$F_{ij}(x) = (A_i(x), B_j(x))$$

for every  $i, j$  and every  $x \in \Sigma$ .

*Proof:*

Assume that such a compatible assignment exists. Because each joint map  $F_{ij}$  is measurable, its two component maps are measurable. Compatibility identifies these components with the corresponding  $A_i$  and

$B_j$ , so the setting-local responses required by the CHSH inequality are measurable consequences of the assumed context maps. Separate measurability of the global assignment need not be postulated.

The Bell–CHSH theorem then gives

$$|S| \leq 2.$$

By hypothesis, however, the four context correlations equal the singlet correlations at the CHSH settings, for which

$$|S| = 2\sqrt{2}.$$

Since

$$2\sqrt{2} > 2,$$

the two statements are incompatible. Hence no such compatible setting-local four-response assignment exists.  $\square$

The theorem uses one fixed preparation measure  $\mu$  across the four contexts. This is the measurement-independence premise identified in Sections 2.1 and 2.3. No superdeterministic or setting-dependent preparation measure is introduced.

The result also requires singlet reproduction only at the four CHSH settings. It does not assume reproduction of the singlet correlation for every possible pair  $(a, b)$ . The obstruction is therefore already present at the minimal four-context level required to violate CHSH.

Fine’s theorem provides the corresponding joint-distribution interpretation. If a compatible assignment

$$(A_1, A_2, B_1, B_2)$$

existed, its joint distribution would supply the Fine joint distribution whose relevant pairwise marginals reproduce the four contexts. The CHSH violation excludes that possibility. The machine-checked proof itself does not import Fine’s theorem as an axiom. It proves the obstruction directly from the measure-theoretic CHSH bound and the formally verified singlet value.

### *Lean correspondence:*

The Bell–CHSH inequality for an arbitrary hidden-variable probability space is machine-checked as `lhvCHSH_abs_le_two` in `CsdLean4/Empirical/QM/Crypto/E91.lean`:

<https://github.com/zbllore/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Empirical/QM/Crypto/E91.lean](https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Empirical/QM/Crypto/E91.lean)

The theorem quantifies over an arbitrary probability space and arbitrary measurable  $\pm 1$ -valued local response functions. It contains no CSD-specific premise.

The singlet CHSH value at the detector settings specified above is proved in `CsdLean4/Empirical/QM/Bell.lean`:

<https://github.com/zblorc/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Empirical/QM/Bell.lean>

The relevant theorem is `chsh_singlet_at_optimal_angles`, built from the formally verified singlet correlation

$$E(a, b) = -a \cdot b.$$

The C1 theorem itself is `no_compatible_global_chsh_assignment_realises_singlet` in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zblorc/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean>

The formal statement assumes:

1. one shared probability measure  $\mu$ ;
2. a measurable shared-context outcome family;
3. compatibility with a four-response global CHSH assignment; and
4. reproduction of the singlet correlations at the four CHSH contexts.

It then derives `False` by applying `lhvCHSH_abs_le_two` and the machine-checked singlet CHSH violation.

Importantly, measurability of the four functions in `GlobalCHSHAssignment` is not independently assumed.

The proof derives it from measurability of the context maps together with compatibility.

The same file proves `compatibleGlobalCHSH_nonvacuous`. Compatible measurable context families therefore exist. For example, the constant all- $+1$  assignment is compatible and yields correlation 1. The obstruction arises specifically from requiring such a compatible family to reproduce the singlet correlations; it is not an artefact of an inconsistent compatibility predicate.

The more general every-setting product-response obstruction is proved separately as `no_product_partition_realises_singlet` in `CsdLean4/LF6/ForcedContextuality.lean`:

<https://github.com/zblorc/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/ForcedContextuality.lean>

That theorem requires singlet reproduction at every detector-setting pair. The C1 theorem used here is deliberately narrower and requires only the four CHSH contexts.

Finally, the obstruction is instantiated on the same concrete CSD singlet arena used in Section 2.1. In `CsdLean4/Tests/Witnesses/SingletBell.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Tests/Witnesses/SingletBell.lean>

the theorem `kMuPsi_no_global_chsh_assignment` instantiates the shared-domain obstruction on

$$(\mathbb{CP}^3 \times T^2, \mu_{\psi^-}),$$

while `kMuPsi_chsh_obstruction_nonvacuous` establishes non-vacuity on that same concrete space.

### 3.3. Interpretation

Theorem 3.1 fixes the precise relationship between determinism and Bell violation in the present construction.

For every realised measurement context

$$C = (a, b),$$

the contextual map

$$F_C : \Sigma \rightarrow \{\pm 1\}^2$$

is a total deterministic measurable map. For its interpretation as a physical record we retain the conservative almost-everywhere qualification discussed in Section 2.3. Nothing in the Bell argument requires that physical record to be intrinsically stochastic.

What fails is the stronger identification

$$F_{ij}(x) = (A_i(x), B_j(x))$$

with the same  $A_i$  reusable across both remote settings and the same  $B_j$  reusable across both settings on Alice's wing. That structure would convert contextual determinism into a Bell-local hidden-response model, and the formal CHSH theorem proves that it cannot reproduce the singlet.

It is therefore not correct to say that the four context maps cannot coexist as measurable functions, or that no single mathematical object may contain them. The family

$$(F_{11}, F_{12}, F_{21}, F_{22})$$

can be defined on a common state space. The impossibility lies in gluing their components into the setting-local form

$$(A_1, A_2, B_1, B_2)$$

while preserving the four singlet correlations.

This distinction also prevents the locality result of Section 2.2 from being misinterpreted. The finite measurement chain satisfies

$$(V_A \otimes V_B)(U_A \otimes U_B)^\dagger = (V_A U_A^\dagger) \otimes (V_B U_B^\dagger),$$

so the implemented setting transformations and detector couplings factorise between the two wings. Theorem 3.1 simultaneously proves that the resulting deterministic joint outcomes cannot factorise into Bell-local hidden responses. These are different mathematical statements:

$$\text{tensor-factorised finite measurement dynamics} \Rightarrow \neg \text{Bell-local outcome factorisation}.$$

The Bell obstruction is therefore located in the attempted setting-local gluing of counterfactual responses, rather than in determinism of the realised outcome itself. This does not evade or weaken Bell's theorem. It identifies explicitly which Bell-local structure the deterministic construction cannot possess.

The result also does not establish a new experimental departure from quantum mechanics. Within the singlet sector the construction reproduces the standard quantum probabilities and CHSH value. Its significance is structural and formal: deterministic realised outcomes almost everywhere, Bell-violating correlations, and factorised finite measurement dynamics are established within the same formally verified singlet framework, and the precise failure of Bell-local outcome assignment is machine-checked rather than left as an interpretive assertion.

The formal support and assumption ledger for these C1 claims is recorded in:

<https://github.com/zblorc/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/docs/C1-FORMAL-SUPPORT.md>

The repository records the shared-domain obstruction as the load-bearing C1 theorem, separately identifies measurement independence and the singlet preparation sector as assumptions, and distinguishes the finite dynamical-locality result from Bell-local response factorisation.

Section 4 next examines the remaining operational condition. It verifies that the same singlet construction has remote-setting-independent local marginals, without inferring that result from Bell-local outcome factorisation.

## 4. No-Signalling and Marginal Stability

The Bell obstruction established in Section 3 rules out a compatible setting-local assignment of hidden responses, but it does not by itself determine whether the resulting contextual outcome structure permits operational signalling. These are logically separate questions.

Operational no-signalling requires only that the probability distribution observed at one wing be invariant under changes of the measurement setting at the remote wing. It does not require the microscopic outcome regions, or the deterministic outcomes associated with individual ontic states, to remain unchanged.

This distinction is essential in the present construction. Pointwise conditions such as

$$F_A(a, b, x) = F_A(a, b', x)$$

for every  $x$  would be substantially stronger than operational no-signalling. Together with the corresponding condition on Bob's wing, they would reconstruct the setting-local response functions excluded by Theorem 3.1. The appropriate condition is therefore equality of marginal probabilities, not equality of context-dependent microscopic outcomes.

We verify this condition at two formal levels. First, it follows directly from the machine-checked singlet probability kernel. Second, the same marginal invariance is established for the explicit pointer-volume construction introduced in Section 2.2. In both cases, one fixed preparation measure is used across the alternative measurement contexts, so measurement independence remains an explicit premise.

### 4.1. Marginal Probabilities

For a context

$$C = (a, b),$$

the joint probability of outcomes

$$(s, t) \in \{\pm 1\}^2$$

is

$$P(s, t \mid a, b) = \mu \left( F_{(a,b)}^{-1} \{(s, t)\} \right).$$

The local marginal probabilities are therefore

$$P_A(s \mid a, b) = \sum_{t=\pm 1} P(s, t \mid a, b)$$

and

$$P_B(t \mid a, b) = \sum_{s=\pm 1} P(s, t \mid a, b).$$

Operational no-signalling requires

$$P_A(s \mid a, b) = P_A(s \mid a, b')$$

for every  $a, b, b'$  and  $s$ , together with

$$P_B(t \mid a, b) = P_B(t \mid a', b)$$

for every  $a, a', b$  and  $t$ .

For the singlet kernel derived in Section 2.4,

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4}.$$

Alice's marginal is

$$\begin{aligned} P_A(s \mid a, b) &= \sum_{t=\pm 1} P_{st}(a, b) \\ &= \sum_{t=\pm 1} \frac{1 - sta \cdot b}{4} \\ &= \frac{1}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} P_B(t \mid a, b) &= \sum_{s=\pm 1} P_{st}(a, b) \\ &= \frac{1}{2}. \end{aligned}$$

Thus

$$P_A(s \mid a, b) = P_A(s \mid a, b') = \frac{1}{2}$$

for arbitrary  $b, b'$ , and

$$P_B(t \mid a, b) = P_B(t \mid a', b) = \frac{1}{2}$$

for arbitrary  $a, a'$ .

The local singlet marginals are therefore maximally mixed and contain no information about the remote detector setting, despite the explicit dependence of the joint distribution

$$P_{st}(a, b)$$

on both  $a$  and  $b$ .

The relevant distinction can be expressed compactly as

$$P_{st}(a, b) \neq P_{st}(a, b') \quad \text{in general,}$$

while nevertheless

$$\sum_t P_{st}(a, b) = \sum_t P_{st}(a, b').$$

Context dependence of the joint outcomes is therefore compatible with remote-setting invariance of the observable marginals.

*Lean correspondence:*

The singlet kernel and marginal identities are machine-checked in `CsdLean4/LF3/Singlet/Kernel.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/Singlet/Kernel.lean>

The relevant theorems are `marginal_a_eq_half` and `marginal_b_eq_half`. The same module proves the corresponding remote-setting identities `no_signalling_strong_readout_a` and `no_signalling_strong_readout_b`.

## 4.2. No-Signalling Theorem

The marginal calculation above already establishes operational no-signalling at the level of the singlet probability kernel. The explicit finite measurement construction allows a stronger consistency check: the same property can be verified directly for the pointer-sector volumes that realise those probabilities.

Let

$$\mathcal{V}_{st}(a, b)$$

denote the total Fubini–Study volume of the pointer blocks corresponding to joint outcome  $(s, t)$  after the local finite measurement chain associated with settings  $(a, b)$ . The construction of Section 2.2 establishes

$$\mathcal{V}_{st}(a, b) = P_{st}(a, b) = \frac{1 - sta \cdot b}{4}$$

for every pair of detector settings, including the perfect-correlation and perfect-anticorrelation cases.

The corresponding Alice marginal pointer volume is

$$\mathcal{V}_{A,s}(a, b) = \sum_{t=\pm 1} \mathcal{V}_{st}(a, b),$$

and hence

$$\mathcal{V}_{A,s}(a, b) = \sum_{t=\pm 1} P_{st}(a, b) = \frac{1}{2}.$$

Similarly,

$$\mathcal{V}_{B,t}(a, b) = \sum_{s=\pm 1} \mathcal{V}_{st}(a, b) = \frac{1}{2}.$$

It follows immediately that

$$\mathcal{V}_{A,s}(a, b) = \mathcal{V}_{A,s}(a, b')$$

and

$$\mathcal{V}_{B,t}(a, b) = \mathcal{V}_{B,t}(a', b).$$

We may therefore state the result as follows.

*Theorem 4.1 (Operational no-signalling in the constructed singlet sector):*

Assume a single preparation distribution is used independently of the subsequently selected detector settings. For the finite local measurement construction described in Section 2.2, the joint pointer-sector volumes reproduce

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4}$$

for all detector settings. Consequently,

$$\mathcal{V}_{A,s}(a, b) = \frac{1}{2}$$

for every  $a, b, s$ , and

$$\mathcal{V}_{B,t}(a, b) = \frac{1}{2}$$

for every  $a, b, t$ . Therefore,

$$\mathcal{V}_{A,s}(a, b) = \mathcal{V}_{A,s}(a, b')$$

and

$$\mathcal{V}_{B,t}(a, b) = \mathcal{V}_{B,t}(a', b),$$

so the explicit finite construction is operationally no-signalling.

*Proof:*

The all-settings pointer-volume theorem gives

$$\mathcal{V}_{st}(a, b) = P_{st}(a, b).$$

Summing over Bob's outcome gives

$$\mathcal{V}_{A,s}(a, b) = \sum_t P_{st}(a, b) = \frac{1}{2},$$

independently of  $b$ . Summing over Alice's outcome gives

$$\mathcal{V}_{B,t}(a, b) = \sum_s P_{st}(a, b) = \frac{1}{2},$$

independently of  $a$ . The remote-setting invariance follows directly.  $\square$

The theorem concerns equality of marginal *volumes*. It does not assert equality of the underlying microscopic pointer regions. In general, changing the remote setting changes the context and may therefore change the detailed partition of the state space. Operational no-signalling requires only

$$\mu(\Omega_{A,s}^{a,b}) = \mu(\Omega_{A,s}^{a,b'})$$

and not

$$\Omega_{A,s}^{a,b} = \Omega_{A,s}^{a,b'}.$$

This measure-level distinction is required by the contextual structure established in Section 3.

The use of one preparation measure across all contexts is also substantive. It is precisely the measurement-independence assumption:

$$\mu_{a,b} = \mu$$

for all detector choices. The no-signalling verification presented here is conditional on that premise, as is the Bell analysis of Section 3.

*Lean correspondence:*

The corresponding measure-level statement for the exhibited shared-context outcome family is proved as `CSD.LF6.singletContextualModel_no_signalling` in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zbllore/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean](https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean)

This theorem inhabits the `OperationalNoSignalling` predicate for the outcome family and measure, whereas the kernel-level results verify the same marginal identities directly from  $P_{st}$ .

The measure-level definition of operational no-signalling is formalised in `CsdLean4/LF3/OperationalNoSignalling.lean`:

<https://github.com/zbl0re/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/OperationalNoSignalling.lean](https://github.com/zbl0re/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/OperationalNoSignalling.lean)

The predicates `RemoteMarginalInvariantA` and `RemoteMarginalInvariantB` state equality of marginal measures under changes of the remote setting. Their conjunction is `OperationalNoSignalling`.

At the closed-form kernel level, the equal-marginal identities are machine-checked as `singlet_marginals_eq_half` and `singlet_operational_no_signalling`. These theorems establish the corresponding identities directly from the singlet kernel  $P_{st}$ .

The outcome-map statement used here is stronger in type. The theorem `singletContextualModel_no_signalling` inhabits the measure-level predicate `OperationalNoSignalling` for the exhibited shared-context outcome family itself. Thus the equalities written above as measures of outcome-map fibres are now verified at the same mathematical level at which they are stated in this section.

This establishes

$$P_A(s \mid a, b) = P_B(t \mid a, b) = \frac{1}{2}$$

at every context.

The pointer-volume version is machine-checked in `CsdLean4/LF6/NudgeLocality.lean`:

<https://github.com/zbl0re/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/NudgeLocality.lean](https://github.com/zbl0re/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/NudgeLocality.lean)

The theorem `localDeisolation_pointer_volume_local` first establishes

$$\mathcal{V}_{st}(a, b) = P_{st}(a, b)$$

for every setting pair.

The marginal-volume identities are then proved as `localDeisolation_A_marginal_volume_eq_half` and `localDeisolation_B_marginal_volume_eq_half`. Finally, `localDeisolation_no_signalling_A` and `localDeisolation_no_signalling_B` machine-check equality of the marginal pointer volumes when the remote setting is changed.

Importantly, the two sides of these Lean equalities are constructed from different setting-dependent states and therefore need not correspond to identical microscopic regions. What the theorems establish is equality of their measures.

### 4.3. Structural Consequences

Combining the results of Sections 2, 3, and 4 gives four distinct structural statements.

First, the contextual outcome map is deterministic as a total measurable partition. For every actual context

$$C = (a, b),$$

the map

$$F_C : \Sigma \rightarrow \{\pm 1\}^2$$

assigns a definite outcome pair mathematically for every  $x \in \Sigma$ . For its physical interpretation we retain the conservative almost-everywhere qualification discussed in Section 2.3. The present paper does not derive this physical record map from continuous ontic dynamics.

Second, the finite measurement implementation is tensor-factorised:

$$(V_A \otimes V_B)(U_A(a) \otimes U_B(b))^\dagger = (V_A U_A(a)^\dagger) \otimes (V_B U_B(b)^\dagger).$$

Thus the finite setting and detector operations used in the construction separate into wing-local maps.

Third, Bell-local outcome factorisation is impossible. The four CHSH context maps cannot simultaneously satisfy

$$F_{ij}(x) = (A_i(x), B_j(x))$$

for any compatible setting-local four-response assignment reproducing the singlet correlations.

Fourth, this failure of Bell-local outcome factorisation does not enable operational signalling. The observable marginals satisfy

$$P_A(s \mid a, b) = P_A(s \mid a, b') = \frac{1}{2}$$

and

$$P_B(t \mid a, b) = P_B(t \mid a', b) = \frac{1}{2},$$

and the explicit pointer-volume construction satisfies the corresponding marginal-volume identities.

The resulting structure is therefore

deterministic realised pointer records almost everywhere, tensor-factorised finite measurement dynamics,  
 Bell-nonfactorisable joint outcomes,  
 operational no-signalling.

There is no contradiction among these properties because they constrain different mathematical objects. Tensor factorisation concerns the finite measurement maps. Bell factorisation concerns whether the joint deterministic outcomes can be decomposed into setting-local hidden response functions. No-signalling concerns only the setting dependence of observable marginal measures.

In particular,

Bell-nonfactorisable outcomes  $\Rightarrow$   $\neg$ operational signalling,

just as

tensor-factorised measurement dynamics  $\Rightarrow$   $\neg$ Bell-local hidden outcomes.

The scope of the no-signalling result should also be stated precisely. The present paper verifies no-signalling in the explicit finite singlet construction. It does not derive a general no-signalling theorem from arbitrary CSD primitives on an unrestricted, potentially non-factorising ontic space  $\Sigma$ . Establishing sufficient primitive dynamical conditions that force remote marginal invariance in that general setting is a separate problem.

Accordingly, no-signalling enters the present result as a property *proved of the constructed sector*, not as an assumed consequence of locality and not as a universal theorem about all possible CSD ontic dynamics.

This boundary is recorded explicitly in the repository's C1 support map:

<https://github.com/zblorc/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/docs/C1-FORMAL-SUPPORT.md>

The support ledger distinguishes the proved kernel-level and pointer-volume no-signalling results from the still-open problem of deriving remote marginal invariance for arbitrary non-factorising  $\Sigma$ .

The combination of Theorems 3.1 and 4.1 therefore gives the central Bell-facing result required for the subsequent discussion. Within the same formally verified singlet framework, the contextual outcome model violates the Bell-local CHSH bound and admits no compatible setting-local hidden outcome assignment, while the finite measurement chain remains tensor-factorised and the corresponding operational statistics preserve no-signalling.

## 5. Discussion and Scope

The remaining interpretive question is therefore not whether these properties are mutually consistent. Within the finite construction considered here, their consistency is machine-checked. The relevant question is what this combination of results does, and does not, establish about determinism, locality, and contextuality.

### 5.1. Deterministic Outcomes vs. Joint Outcome Assignments

The central distinction of this paper is between two logically different claims.

The first is contextual determinism: for every implemented context  $C$ ,

$$F_C : \Sigma \rightarrow \{\pm 1\}^2$$

is a total deterministic measurable map. For its interpretation as a physical record we retain the conservative almost-everywhere qualification discussed in Section 2.3. This qualification anticipates the dynamical record layer and is not a restriction on the mathematical totality of  $F_C$ .

The second is Bell-local counterfactual compatibility: there exist four setting-local responses

$$A_1, A_2, B_1, B_2 : \Sigma \rightarrow \{\pm 1\}$$

such that the four CHSH context maps satisfy

$$F_{ij}(x) = (A_i(x), B_j(x))$$

on the same underlying state space.

The first statement does not imply the second.

For the exhibited singlet contextual model, this separation is now itself a machine-checked theorem. Alice's realised response at setting  $a$  cannot be detached from the complete measurement context and represented by one setting-local response  $R_A(a)$  that is reusable across Bob's alternative settings, while simultaneously doing the analogous construction on Bob's wing. The theorem `singletContextualModel_not_product` proves that no pair of setting-local response families  $R_A(a)$  and  $R_B(b)$  can reproduce the correlations realised by the exhibited model.

Theorem 3.1 makes this distinction mathematical rather than semantic. Suppose that the four realised context maps could be glued into

$$(A_1, A_2, B_1, B_2).$$

Then their correlations would take the Bell-local form

$$E_{ij} = \int_{\Sigma} A_i(x) B_j(x) d\mu(x),$$

and therefore satisfy

$$|S| \leq 2.$$

The singlet construction instead gives

$$|S| = 2\sqrt{2}.$$

The compatible setting-local gluing is therefore impossible.

It would be incorrect to strengthen this result into the statement that the four context maps themselves cannot coexist. They can. In particular, the object

$$(F_{11}, F_{12}, F_{21}, F_{22})$$

may be defined on one common measurable state space. The obstruction concerns the additional identities

$$F_{ij}(x) = (A_i(x), B_j(x)),$$

not the existence of the context-labelled maps themselves.

This distinction also clarifies the role of Fine's theorem. Fine's result provides the joint-distribution interpretation of Bell compatibility: a suitable joint distribution for the four setting outcomes exists precisely under the Bell-consistency conditions (Fine 1982). The formal C1 result does not require Fine's theorem as an imported premise. Instead, the Lean development proves the relevant implication directly by deriving the measurable setting-local responses from the assumed compatibility condition, applying the general CHSH theorem

$$|S| \leq 2,$$

and contradicting the formally verified singlet value.

The result should therefore not be described as an evasion or weakening of Bell's theorem. Bell-local outcome factorisation is not restored. It is explicitly proved impossible for the singlet correlations. The result instead identifies which notion of locality is absent and which remains available.

In particular, the finite measurement dynamics satisfy

$$(V_A \otimes V_B)(U_A(a) \otimes U_B(b))^\dagger = (V_A U_A(a)^\dagger) \otimes (V_B U_B(b)^\dagger).$$

Thus the setting transformations and detector couplings of the finite construction are tensor-factorised. At the same time,

$$F_{ij}(x) \neq (A_i(x), B_j(x))$$

as a universal Bell-local representation of the singlet context maps.

The two statements concern different levels:

factorisation of measurement operations  $\neq$  factorisation of hidden outcome responses.

Operational no-signalling supplies a third, distinct level. Although the joint outcome structure depends on the complete context, the local marginal probabilities obey

$$P_A(s \mid a, b) = P_A(s \mid a, b')$$

and

$$P_B(t \mid a, b) = P_B(t \mid a', b).$$

Hence contextual dependence of the joint deterministic outcome structure does not imply an operational signalling channel.

The formal construction therefore separates three notions that can otherwise be conflated:

determinism of the realised outcome,  
 tensor factorisation of the finite measurement dynamics,  
 Bell-local factorisation of counterfactual outcome responses.

The first two hold in the construction. The third is excluded by the CHSH theorem.

### *Lean correspondence:*

The shared-domain distinction is encoded in `CsdLean4/LF3/SharedContextMap.lean`:

<https://github.com/zblore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/SharedContextMap.lean>

The compatible four-response structure and its obstruction are machine-checked in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zblore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean>

The central theorem is `no_compatible_global_chsh_assignment_realises_singlet`. Its companion theorem `compatibleGlobalCHSH_nonvacuous` shows that compatible measurable families exist in general,

so the obstruction is specifically to singlet reproduction rather than to the compatibility definition itself.

The stronger every-setting result `no_product_partition_realises_singlet` is proved independently in `CsdLean4/LF6/ForcedContextuality.lean`:

<https://github.com/zblore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/ForcedContextuality.lean>

The factorisation of the complete finite measurement chain is proved as `localMeasurementChain_factorises` in `CsdLean4/LF6/NudgeLocality.lean`:

<https://github.com/zblore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/NudgeLocality.lean>

For the exhibited singlet contextual model, the failure of setting-local outcome factorisation is proved directly as `CSD.LF6.singletContextualModel_not_product` in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zblore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean>

This strengthens the conditional Bell obstruction for the purposes of the present discussion: the non-product result is now a theorem about an exhibited outcome family rather than merely a consequence conditional on an unspecified singlet-reproducing family.

## 5.2. Limitations and Future Work

The formal results substantially strengthen the original version of this paper, but their scope remains finite and explicit. Several boundaries are important for interpreting the result correctly.

### *Singlet preparation is an input:*

The construction begins with the singlet preparation sector. It does not derive the physical origin of that entangled sector from more primitive CSD dynamics. The logical dependency is therefore

$$\text{singlet preparation} \implies P_{st}(a, b) \implies |S| = 2\sqrt{2} \implies \text{failure of Bell-local outcome factorisation.}$$

The downstream implications are formally established, but the first element is supplied as preparation data. The present paper therefore does not claim a first-principles derivation of entanglement.

The concrete singlet preparation used by the witness suite is implemented in `CsdLean4/LF4/SingletKahler.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF4/SingletKahler.lean>

*Measurement independence is assumed:*

One fixed preparation measure

$$\mu$$

is used for every measurement context. This is measurement independence, not a consequence of the Bell or no-signalling proofs. The results therefore do not apply to models in which the preparation distribution is allowed to depend on the subsequently selected settings.

This premise is made explicit in `CsdLean4/LF3/OperationalNoSignalling.lean`:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF3/OperationalNoSignalling.lean>

*The locality result is finite and dynamical:*

The theorem

$$(V_A \otimes V_B)(U_A \otimes U_B)^\dagger = (V_A U_A^\dagger) \otimes (V_B U_B^\dagger)$$

establishes tensor factorisation of the explicit finite measurement chain. This is a theorem about the constructed setting transformations and dilated detector maps.

It is not a proof that an arbitrary CSD ontic state space  $\Sigma$  possesses a canonical subsystem tensor decomposition. Nor is it a theorem of relativistic local causality for arbitrary CSD dynamics. The term “local” in this paper therefore refers specifically to the wing-factorised finite measurement implementation unless otherwise stated.

A further dynamical extension would be to connect these finite unitary and isometric maps to explicitly specified continuous ontic Hamiltonian generators, while preserving the factorisation theorem at the appropriate level. Such a construction would deepen the dynamical interpretation, but is not required for the Bell-consistency theorem proved here.

*Relation to the general CSD measurement layer*

The context-indexed singlet outcome family used in this paper is an explicit Bell witness and remains the object consumed by the CHSH proof. The current CSD record layer now separately supplies a dynamical

finite-dimensional measurement closure in which physical readout sectors are generated by a constructed propagator and carry their Born measures.

The present Bell theorem does not need to identify the singlet witness with that more general measurement construction. Its conclusion concerns the compatibility of deterministic realised context maps with a reusable Bell-local four-response assignment. The later measurement closure strengthens the physical interpretation of the outcome maps without altering the Bell logic.

*No-signalling is verified for the construction, not derived universally:*

The singlet kernel and the pointer-volume model both satisfy

$$P_A(s \mid a, b) = P_A(s \mid a, b')$$

and its Bob-wing analogue. These are proved results.

What has not been established is a general theorem of the form

$$\text{CSD primitive assumptions} \implies \text{operational no-signalling}$$

for arbitrary non-factorising ontic dynamics on  $\Sigma$ .

The distinction is substantive. The present work verifies no-signalling in the constructed singlet sector. It does not derive remote marginal invariance solely from the most general CSD primitives.

This open generalisation is recorded in the repository backlog:

<https://github.com/zblorc/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/specs/BACKLOG.md>

*The present Bell result is two-qubit and CHSH-specific:*

The explicit witness uses the two-qubit singlet and the four-setting CHSH scenario. This is sufficient to establish the central incompatibility because a single CHSH violation already excludes the corresponding Bell-local response assignment.

The paper does not claim that the same formal construction has yet been developed for arbitrary bipartite dimensions, multipartite Bell inequalities, or the complete hierarchy of contextuality scenarios. Such extensions would test how far the same separation between deterministic realised outcomes almost everywhere, finite dynamical factorisation, and Bell-local response factorisation persists beyond the minimal singlet witness.

*No new deviation from quantum mechanics is predicted here:*

Within its stated sector, the construction reproduces

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4}, E(a, b) = -a \cdot b, |S| = 2\sqrt{2},$$

together with the standard no-signalling marginals. The present result therefore does not propose a modified Bell correlation or a new Bell-test signature.

Its contribution is instead structural and formal. It exhibits an explicit single-outcome framework in which realised outcomes are deterministic almost everywhere and in which:

Bell-violating singlet correlations are reproduced,  
the finite measurement chain tensor-factorises,  
operational no-signalling holds,  
and Bell-local hidden outcome factorisation is impossible.

The complete relation among these properties is verified within one Lean 4 development rather than assembled from separate informal arguments.

### *Formal verification boundary:*

The key C1 theorem chain is machine-checked without additional project-specific logical axioms beyond Lean and Mathlib's foundational logical constants recorded by the repository audit. The exact theorem, module, assumption, and axiom status of each C1 claim is maintained in:

<https://github.com/zblorg/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/docs/C1-FORMAL-SUPPORT.md>

The repository-wide axiom ledger is:

<https://github.com/zblorg/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/AXIOMS.md>

### *Formalisation as error detection*

Formalisation did more than certify a finished argument. It exposed two substantive errors in earlier versions of the C1 development.

First, an earlier argument treated the type-level distinction between context-indexed outcome maps and global CHSH assignments as though the different formal types themselves prevented a global Bell assignment. They do not. The corrected construction places the relevant objects on one shared state space and introduces the explicit compatibility condition

$$F_{ij}(x) = (A_i(x), B_j(x)),$$

so that the obstruction follows from CHSH rather than from type separation.

Second, an earlier setting-dependent object, `nudgedSinglet`, was treated as a local-unitary image of the singlet although it did not implement the required product-unitary transformation. The corrected construction replaces it by `localNudge`, defined from

$$(U_A(a) \otimes U_B(b))^{\dagger} |\psi^{-}\rangle,$$

after which the corresponding Born and finite-factorisation statements are proved on the corrected object.

These were load-bearing mathematical corrections rather than typographical changes. The present C1 theorem architecture is therefore partly a result of formalisation rather than merely a transcription of a previously complete informal proof. This error-detection role has precedent in formal quantum information: recent Lean 4 work on CHSH rigidity likewise identified a gap in an existing informal argument [\[12\]](#), while earlier Isabelle/HOL work formalised the CHSH inequality and Cirel'son bound [\[11\]](#).

Within these stated boundaries, the CSD-specific Bell-consistency obligation is closed for the exhibited two-qubit singlet model. This is a model-specific consistency result, not a new Bell theorem and not a derivation of the contextual outcome family from the ontic flow. Deterministic contextual outcomes are retained, Bell-local four-response factorisation is excluded, the finite measurement chain remains tensor-factorised, and the operational marginals remain no-signalling.

## 6. Conclusion

This paper has addressed a model-specific consistency obligation rather than a new Bell theorem. Bell-CHSH and Fine already determine the setting-local response structures incompatible with the singlet correlations. The question here is whether the exhibited finite-dimensional CSD singlet model, with deterministic context-indexed outcomes, instantiates that prohibited structure. Within the explicit assumptions stated throughout the paper, it does not.

The construction begins with one shared probability space, a fixed singlet preparation sector, and measurement independence. For every realised measurement context

$$C = (a, b),$$

the contextual outcome map

$$F_C : \Sigma \rightarrow \{\pm 1\}^2$$

is total and assigns one definite joint outcome mathematically for every state. For its interpretation as a physical record we retain the conservative almost-everywhere qualification used throughout the paper. The resulting probabilities reproduce the singlet kernel

$$P_{st}(a, b) = \frac{1 - sta \cdot b}{4},$$

and hence the correlation

$$E(a, b) = -a \cdot b.$$

At the CHSH settings this gives

$$|S| = 2\sqrt{2},$$

in violation of the Bell-local bound

$$|S| \leq 2.$$

The central Bell-consistency result identifies exactly what must therefore fail. At the four specified CHSH settings, the realised context maps reproducing the singlet correlations cannot be represented by a compatible setting-local assignment

$$A_1, A_2, B_1, B_2 : \Sigma \rightarrow \{\pm 1\}$$

satisfying

$$F_{ij}(x) = (A_i(x), B_j(x))$$

for all four CHSH contexts. The obstruction is thus not to determinism of the realised outcome, nor to the simultaneous mathematical existence of four context-indexed maps. It is to their gluing into the Bell-local counterfactual structure required to derive the CHSH bound.

At the same time, the explicit finite measurement implementation factorises between the two wings. The local setting transformations and detector dilations satisfy

$$(V_A \otimes V_B)(U_A(a) \otimes U_B(b))^\dagger = (V_A U_A(a)^\dagger) \otimes (V_B U_B(b)^\dagger).$$

Thus the finite measurement chain is tensor-factorised even though the resulting joint outcomes cannot be decomposed into Bell-local hidden responses. The construction therefore demonstrates explicitly that

$$\text{factorised finite measurement dynamics} \Rightarrow \neg \text{Bell-local outcome factorisation}.$$

Operational no-signalling is preserved as a distinct operational property. Both the singlet kernel and the pointer-volume construction give

$$P_A(s \mid a, b) = P_A(s \mid a, b') = \frac{1}{2},$$

and

$$P_B(t \mid a, b) = P_B(t \mid a', b) = \frac{1}{2}.$$

Remote settings may therefore change the contextual joint structure without changing the locally observable marginal probabilities. No equality of microscopic outcome regions is required.

The resulting structure can be summarised as

deterministic realised outcomes almost everywhere,  
 Bell-violating singlet correlations,  
 tensor-factorised finite measurement dynamics,  
 operational no-signalling,  
 and no compatible Bell-local four-response assignment  
 reproducing the singlet correlations at the specified CHSH contexts.

These properties constrain different mathematical objects and are mutually consistent within the construction presented here.

The scope of this conclusion is explicit. The singlet preparation sector is supplied as an input rather than derived from more primitive CSD dynamics, and measurement independence is assumed through the use of one fixed preparation measure across measurement contexts. The locality theorem concerns the explicit finite setting transformations and detector dilations rather than arbitrary ontic dynamics on a general constraint surface. Likewise, the no-signalling result is verified for the constructed singlet sector and is not presented as a universal derivation of remote marginal invariance for arbitrary non-factorising ontic dynamics.

Within these boundaries, the principal result is not a modification of Bell's theorem and does not predict a departure from standard quantum-mechanical Bell correlations. Rather, it provides an explicit formally verified separation of notions that are often conflated. Bell violation excludes compatible setting-local hidden responses, but does not by itself exclude deterministic realised outcomes. In the construction studied here, this distinction is realised together with tensor-factorised finite measurement dynamics and operational no-signalling.

The load-bearing results are machine-checked in Lean 4. The shared-domain Bell obstruction is proved as `no_compatible_global_chsh_assignment_realises_singlet` in `CsdLean4/LF6/C1BellConsistency.lean`:

<https://github.com/zbllore/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean](https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/C1BellConsistency.lean)

The general Bell-local CHSH bound is proved as `lhvCHSH_abs_le_two` in `CsdLean4/Empirical/QM/Crypto/E91.lean`:

<https://github.com/zbllore/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Empirical/QM/Crypto/E91.lean](https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Empirical/QM/Crypto/E91.lean)

The singlet CHSH value is proved in `CsdLean4/Empirical/QM/Bell.lean`:

<https://github.com/zbllore/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Empirical/QM/Bell.lean](https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/Empirical/QM/Bell.lean)

The factorisation of the complete finite measurement chain, the all-settings pointer-volume reproduction, and the pointer-volume no-signalling theorems are contained in `CsdLean4/LF6/NudgeLocality.lean`:

<https://github.com/zbllore/csd->

[lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/NudgeLocality.lean](https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/CsdLean4/LF6/NudgeLocality.lean)

The theorem-to-claim and assumption ledger for the paper is maintained at:

<https://github.com/zbllore/csd-lean4/blob/17dcf27ebf055faf7f3c5c6d24adadbe55259629/docs/C1-FORMAL-SUPPORT.md>

The result therefore closes a CSD-specific consistency obligation rather than introducing a new Bell theorem. Bell and Fine determine the prohibited global response structure; the contribution here is to exhibit the CSD contextual singlet model, prove its singlet-table and no-signalling properties, formalise the incompatible Bell-local gluing condition, and machine-check the resulting theorem chain together with its assumption and axiom boundaries. In doing so, formalisation also exposed and removed substantive errors in earlier informal versions of the argument, demonstrating that the proof assistant functioned as part of the scientific method rather than as post hoc verification.

## Notes

[www.constraintsurfacedynamics.com](http://www.constraintsurfacedynamics.com)

<https://github.com/zbllore/csd-lean4>

CSD Companion Paper C1

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