Research Article

Emergent Quantum Mechanics: How the Classical Laws Can Mimic Non-Local ("Spooky") Interactions

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In Feynman's "Lectures on Physics" it is observed that close to an oscillating dipole the electromagnetic field is almost quasi-static, as if retardation is absent and the speed of transmission thus would approach infinity. Within the framework of Emergent Quantum Mechanics (Em.QM) non-singular electron models are being revisited, showing similar phenomena near a droplet of charge. In the literature, the matter has been called an as yet unsolved dilemma because – like the "spooky" non-local effects of Quantum Mechanics (QM) – it seems to be in conflict with relativity theory. In the present paper, an asymptotic expansion of the electromagnetic field around a moving and deforming droplet of charge is used to investigate the matter in detail. It is concluded that Feynman's formulae are correct, be it *after* initial conditions of the boundary value problem have been established by a field of normal travelling waves. When such a reset has been completed, the field of travelling waves can be written as a superposition of groups of standing waves. In each standing wave system, the variables belonging to it display synchronicity, which causes the illusion of non-retarded interactions. But in fact, there are no causal relations between the singularities (such as charges) and the other field variables (such as electromagnetic forces), even though the illusion may be given that superluminal signalling takes place. The conclusions are important for the theory of Em.QM, since they show that the classical laws are able to mimic non-local phenomena.

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1. Introduction and overview

In his *Lectures on Physics*, Feynman points out a remarkable and somewhat puzzling fact about the electromagnetic forces close to an accelerating charge or an oscillating dipole [1]. To quote Feynman: "Eq...... gives [force] fields very much like the instantaneous theory – much closer than the instantaneous theory with a delay; .... The static formulas are very accurate, much more accurate than you might think. Of course, the compensation only works for points close in". In Feynman's formulae, the frequency of the oscillating dipole is unrestricted; the quasi-static phenomenon is therefore not associated with a low-frequency approximation.

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The physical character of this near-field region, in antenna theory called the "reactive near field", is clearly seen when the flows of energy are analysed (see e.g. the calculation of the Poynting vector around a moving and deforming collection of charges in [2]). At large distances, one finds a smooth field, where a superposition of running waves carries energy away to infinity, in agreement with Larmor's law. Closer in, orders of magnitude more intense flows of energy "slosh around" without contributing to the radiation at infinity. A visualisation of this kind of radiation field is the fierce turmoil in a pond beneath a waterfall, from which at some distance a smooth wave pattern emanates. It is in this highly disturbed near field that, according to Feynman, an unexpectedly small retardation of electromagnetic interactions occurs, much smaller than the ratio "distance of a field point from the point of emission divided by speed of light" would imply. It would seem *as if* the speed of interactions is superluminal (i.e. the kind of interactions Einstein called "spooky"). After mathematically analysing this curious near-field phenomenon, Feynman alas did not philosophise further about possible physical interpretations, at least not in the "Lectures".

Later, Walker [3] performed experiments in the near field of a dipole antenna and discussed a few possible interpretations of the measured quasi-static phenomenon. If one were to accept that superluminal propagation is indeed possible, there is clearly a conflict with relativity theory. Relativity theory predicts that if an information signal is propagated superluminally, then it can be reflected by a moving frame and arrive at the source before the information was transmitted, thereby violating causality. Consequently, Einstein in 1907 stated that superluminal signal velocities are incompatible with relativity theory. Another possible interpretation mentioned by Walker is that there is a superluminal *illusion* caused by the fact that the near field could be a type of standing wave system, as suggested by the fact that high-intensity flows of energy leave the source region and—since they for the most part do not reach infinity—must subsequently return. The system would thus largely consist of a field which grows and collapses synchronised with the oscillation of the electric dipole. This hypothesis was rejected by Walker on the grounds that the analysis in [3] and [4] does not show such a standing wave system. Walker in [4] therefore states: "at present it is unclear how to resolve this dilemma".

It is one of the goals of the present paper to revisit the latter "illusion interpretation". In the paper, the near field is derived in slightly more detail compared with Feynman's and Walker's field of a variable point-dipole. A point singularity is a mathematical idealisation of a physically more complex situation. Therefore, going back to the physical origin of the dipole model can give additional insight.

Recently, another case of non-retarded interactions has been found in the so-called "droplet theory" of Emergent Quantum Mechanics (hereafter abbreviated as Em.QM). It is this theory that is able to provide more details of the electromagnetic field than can be found in Feynman's book. The present paper therefore first of all introduces and summarises the droplet theory of Em.QM.

The terminology "Em.QM" is here used for the theory describing cases where the classical laws can replicate the results of usual Quantum Mechanics (see [5][6][7]). The aim of Em.QM is certainly not to compete with QM as far as practical calculations are concerned. However, Em.QM could become a complementary theory to QM, helping to obtain interpretations of quantum phenomena that are less “magical” than current ones, and—more importantly—to get causality back in the atomic world.

The droplet theory of Em.QM is based on the idea that a suitable theoretical model of electrons should include the inherent “zitter” motion. Time-averaging then leads to a non-singular electron model consisting of a deformable, distributed charge distribution, contained by surface-tension-like forces. The latter are non-physical “apparent” forces, belonging only to the time-averaged model world, and necessary as modelling additions to cancel the unrealistic “explosion forces” that have been artificially introduced by the time averaging. The background of this so-called “droplet model” is further described in [5][6][7]. Using the droplet model, the basic QM-laws $E = h\nu$ and $p = h/\lambda$ can be reproduced by the classical laws, whilst a practically correct value of Planck’s constant h is found. Other results of the droplet theory include a full replication of the quantum harmonic oscillator ([7]), and (in [6]) a possible solution to the long-standing “enigma” of the muon and tau-particle (why do three “generations” of leptons exist, and how can the corresponding mass differences between them be explained?).

A non-uniformly moving and deforming charge distribution as considered in the droplet theory is always subjected to so-called electromagnetic self-forces, arising from internal retardation effects [8]. On a non-deforming charge distribution, as originally investigated by Lorentz [9] and more recently by Yaghjian [10], two self-forces act: an inertia force associated with so-called “electromagnetic mass”, and radiation resistance. The extra feature of droplet theory is that it requires the determination of self-forces on a lump of charges which changes its shape, apart from its translational motion. The tool used in this case for determining the associated electromagnetic field is the “matched asymptotic expansion” technique (hereafter abbreviated as the “max” method). The method uses local “coordinate-stretching” in the vicinity of singularities and enables one to determine the field inside and outside charge distributions in detail, as if seen through a (mathematical) microscope. Here again, the same seemingly superluminal phenomena as mentioned by Feynman are found, be it that more details can be discovered.

The present paper has a twofold aim. The first is to investigate more deeply the phenomenon of non-retarded interactions: how they can be consequences of Maxwell’s laws, and how they might be reconciled with relativity theory.

Secondly, the subject of the present paper has direct and important consequences for Em.QM in general. In spite of past successes, it is often said that Em.QM will always be limited in its scope because “it is believed that the classical laws never will be able to replicate non-local effects (i.e. events causing a remote effect without retardation) ”. Or in jargon terminology: any theory containing hidden variables must encompass "actio-in-distans", which—it is thought—excludes the classical laws from being candidates to offer a valid interpretation of quantum theory. This opinion was in the past already contradicted in a paper by Anderson and Brady ([11]), where the authors showed that a certain type of classical system can satisfy the so-called Bell test, which test is able to prove the existence of non-local effects. If the occurrence of non-retarded interactions as found by Feynman and Walker as well as in the ”droplet theory” can be ascertained beyond doubt and physically explained, this would show that Em.QM is not fundamentally barred from covering the whole of QM.

Until now the results of the droplet theory have been obtained by using the simplest possible model of a deformable lump of charges. The limited aim so far has been to reconsider non-singular electron models and perform a first exploration to find the consequences of such models. For this reason a “minimum complexity” model has been adopted (fig.1), consisting of a one-dimensional potential well in which a slender “droplet” of charge moves back and forth with so-called “pulsation” freedom (i.e. lengthwise stretching and contracting). In sec. 2 the boundary value problem is formulated which has to be satisfied by the scalar and vector potentials associated with this kind of model.

The next sec. 3 describes the principles of the “matched asymptotic expansion” technique, or “max”-method. The expansion parameter of the asymptotic series is the ratio $\frac{a}{\lambda}$, where a denotes a typical dimension of the potential well and the near field, and λ a typical wavelength in the far field. For small $\frac{a}{\lambda}$ the potentials in the near field can be written as quasi-static fields augmented by an asymptotic series. On the other hand, in the far field the limiting process $\frac{a}{\lambda} \to 0$ implies that the field may be approximated by a superposition of multipoles in the origin. Both the near- and far-field approximations are not unambiguous because of missing boundary conditions either at infinity or near the origin respectively. This causes the near field to contain fields that do not vanish at infinity, and these additional fields even prove essential when determining the forces in the near field. “Matching” through the mediation of a so-called “common field” finally achieves an unambiguous “composite” solution which satisfies all the boundary conditions and where any singularities have been cancelled by the matching.

How the in sec. 3 outlined “max” procedure can be applied in actual cases is illustrated in sec. 4, where the potential fields of a moving point charge are calculated, as well as the force field around it. In the theory of a moving and deforming droplet the contribution by infinitesimal elements of the droplet is integrated, using the point charge results (sec. 5).

In this way an inertial self-force (usually expressed in terms of an “electromagnetic mass”) is found as well as the so-called radiation resistance, in agreement with Lorentz’s classical results. Both these self-forces are related without retardation to corresponding properties of the motion: the electromagnetic inertia is associated with the instantaneous acceleration, and the radiation resistance is proportional to the instantaneous jerk. It is only thanks to the non-retardation that the concept of electromagnetic mass can be introduced at all. Were it not so, then the value of electromagnetic mass would depend on the motion history of the droplet. The same is true for the radiation resistance self-force.

Also in sec. 5 the non-retarded forces between separate droplets of charge are considered. A lump of charge is taken as an example, breaking up into two separate droplets. The pair of two fragments may be viewed as an extreme form of deformation. As long as the dimensions of the total constellation remain small compared with the wavelength of disturbances in the far field, both fragments share the same near field and interactions between them appear to be non-retarded.

A convergence investigation in sec. 6 shows that the asymptotic series solution is entirely equivalent to a superposition of travelling waves if certain conditions are fulfilled. These requirements entail that the boundary conditions determining the field are smoothly evolving in time, without discontinuities in any of the time derivatives. A simple example would be a system of two droplets of charge, sharing a common near field, and sinusoidally closing in on and moving away from each other under the influence of an external potential. If the requirement of continuity is not satisfied (for instance during the starting-up of a harmonic motion), during a short interval of time the asymptotic series do not converge to the correct value, and the only way to describe the field is by a superposition of travelling waves (with full retardation effects). During such a transient, both new boundary conditions are being established as well as new initial conditions. After the "reset", the further development of the near field can be described again by an asymptotic series (including the phenomena of non-retarded electromagnetic forces).

How these findings can be interpreted physically, and how they can be reconciled with relativity theory is discussed in sec. 7. It is shown that the complete asymptotic series, equivalent to the field of travelling waves, can be separated into a superposition of standing waves. In case the boundary conditions require a harmonically variable field, the variables belonging to one particular standing wave system are in phase throughout the field. This synchronous behaviour pertains to the entire standing wave system including any embedded singularities, but does not pertain to the variables belonging to another standing wave. The synchronous variation of the quantities belonging to a particular standing wave system does not imply an infinite speed of signal transmission from a singularity to a field point (as if a singularity could exist individually, and as if there would be a causal relation between the singularity and the field, such as incorrectly suggested by the common parlance about a "field induced by a charge"). The field, including its embedded singularities and force fields, develops as a whole, simultaneously in all points. This gives rise to the illusion of non-retarded interactions.

In the conclusions (sec. 8) a further discussion about the significance of the findings is included. The dilemma "superluminal effects versus relativity" raised in the publications by Feynman [1] and Walker [4] would seem to be solved. Furthermore, one may safely state that Em.QM indeed covers phenomena that may be called holistic, non-local or - in Einstein's words - "spooky". However, prior to a future further development of the droplet theory to include "spin" it is not possible to compare these conclusions with the non-local phenomena known in the usual QM.

2. The boundary value problem of the droplet of charge

The droplet model of electrons, depicted in fig.1, has its origin in the principle to model electrons including their "zitter" ("shudder" or "tremble") motion, in such a way that the real physics becomes amenable to a reasonably simple, although approximate, analysis. Another point of view, which will be taken here for the sake of brevity is, that one may propose any kind of model – however unrealistic it might seem at first sight – as long as the consequences agree with experiments. In this respect the droplet model has proven its value (see [5][6][7]). The model configuration analysed in these references consisted of a charge distribution contained by "surface-tension-like" apparent forces, moving back and forth along the Z-axis inside a two-dimensional potential well. The distribution has an elongated shape, with rotational symmetry around the Z-axis and furthermore symmetry w.r.t. an “equator-plane”. All of these model assumptions were introduced in order to simplify the mathematics and obtain quick and simple answers in an exploratory study whether non-singular electron models could make sense. The two degrees of freedom of this model were translation $z_m(t)$ of the midpoint (which is the centre of charge as well as the centre of mass), and the variable length $s(t)$.

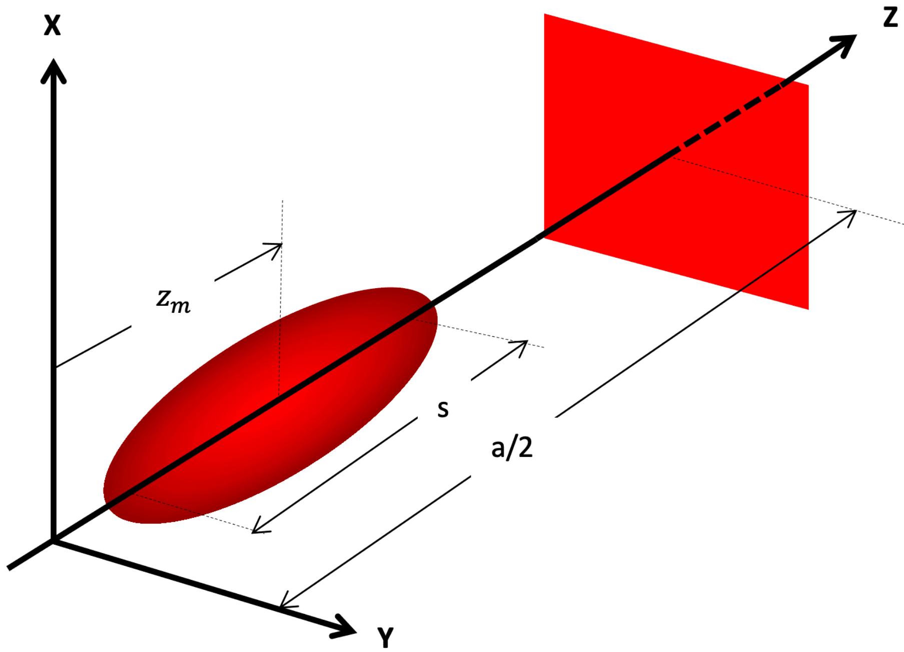

Figure 1. *Notations droplet in potential well*

The (electric) scalar potential $\Phi(x,\ y,\ z,\ t)$ should satisfy the field equation

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\sigma}{\varepsilon_0} \tag{1}$$

where $\sigma(x,\ y,\ z,\ t)$ denotes the charge density. The (magnetic) vector potential $\underline{A}(x,\ y,\ z,\ t)$ must satisfy a similar field equation where the current density $\underline{j}$ is on the r.h.s.:

$$\nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} = -\frac{1}{c^2}\frac{\underline{j}}{\varepsilon_0} \tag{2}$$

In the free space outside the charge- and current distributions the scalar potential and the three Cartesian components $(A_x\ , A_y\ , A_z)$ of the vector potential satisfy wave equations such as $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = 0$ and are required to vanish at infinity.

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3. Matched Asymptotic Expansion ("max") analysis of the non-stationary electromagnetic field

3.1. Near-field approximation

When a "lump" of electrical charge is moving back and forth with some regularity along a stretch of the Z-axis, one can assign a characteristic time scale T to the motion. T can for instance denote the typical average duration to complete a cycle of the -not necessarily harmonic- motion. If the motion covers a variable length of the Z-axis characterised by a scale a, a characteristic velocity of the motion may be defined as $v = a/T$.

The region of space close to the stretch of the Z-axis where the motion takes place will be called the "near field". It is a volume of space with global dimensions of the same order of magnitude a. Within this near field, the spatial variations of the electrical potential Φ will have a scale of order a, or smaller. This strictly applies to the near field only and will not be true in the whole of space stretching out to infinity. A typical length scale far away from the charge will be the wavelength $\lambda = cT$ of electromagnetic waves associated with the moving charge, with c being the speed of light. In the near field (outside the charges), the wave equation may be written in terms of the characteristic coordinates $\frac{x}{a},\ \frac{y}{a},\ \frac{z}{a},\ \frac{t}{T}$:

$$\frac{\partial^2\Phi_{near}}{\partial\left(\frac{x}{a}\right)^2} + \frac{\partial^2\Phi_{near}}{\partial\left(\frac{y}{a}\right)^2} + \frac{\partial^2\Phi_{near}}{\partial\left(\frac{z}{a}\right)^2} = \left(\frac{a}{\lambda}\right)^2 \frac{\partial^2\Phi_{near}}{\partial\left(\frac{t}{T}\right)^2} \tag{3}$$

where all the partial derivatives in eq. (3) have the same order of magnitude, on account of the physical assumptions about the space and time scales in the near field.

The basic assumption of the matched asymptotic expansion procedure is that the ratio a/λ is small, so that it may be used as an expansion parameter in an asymptotic series developed for $a/\lambda \to 0$. An equivalent expansion parameter is the ratio $\frac{v}{c = \frac{1}{c}\frac{a}{T} = \frac{a}{\lambda}}$.

According to eq. (3), the field $\Phi_{near}(x,\ y,\ z,\ t)$ in the limit $a/\lambda \to 0$ will approach a quasi-static field $\Phi_{qu.static}$ satisfying the Laplace equation $\nabla^2\Phi_{qu.static} = 0$ outside the distribution of charges. The terminology "quasi-static" is used to indicate that the field at any instant of time is spatially identical to the field of a static distribution of charges, even if the shape and strength of the charge distribution are functions of time. Changing boundary conditions thus lead to a time sequence of static-like fields.

The quasi-static field is used as the first term in an asymptotic expansion, whilst the other terms of the series describe how the field is influenced by retardation effects:

$$\Phi_{near} = \Phi_{qu.static} + \frac{a}{\lambda}\Phi_1 + \left(\frac{a}{\lambda}\right)^2\Phi_2 + \left(\frac{a}{\lambda}\right)^3\Phi_3 + \ldots \quad \left(\frac{a}{\lambda} \to 0\right) \tag{4}$$

To determine the equations that have to be satisfied by the additional fields $\Phi_1,\ \Phi_2,\ \ldots$, expression (4) is substituted into the wave equation (3):

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$$\left(\frac{\partial^2}{\partial\left(\frac{x}{a}\right)^2} + \ldots\right)\left[\Phi_{qu.static} + \frac{a}{\lambda}\Phi_1 + \ldots\right] = \left(\frac{a}{\lambda}\right)^2 \frac{\partial^2}{\partial\left(\frac{t}{T}\right)^2}\left[\Phi_{qu.static} + \frac{a}{\lambda}\Phi_1 + \ldots\right] \tag{5}$$

As a next step, $\nabla^2\Phi_{qu.static} = 0$ is subtracted from eq. (5) and the equation is divided by $\frac{a}{\lambda}$. Taking the limit $a/\lambda \to 0$ again gives an equation which has to be satisfied by Φ_1,

etc. The same results may be obtained by equating terms on the l.h.s. to terms of the same asymptotic order on the r.h.s. The procedure shows that up to and including the order $O\left(\frac{a}{\lambda}\right)^3$, the near field satisfies the following Poisson equation:

$$\nabla^2\Phi_{near} = \frac{1}{c^2}\frac{\partial^2\Phi_{qu.static}}{\partial t^2} + O\left(\frac{a}{\lambda}\right)^4 \tag{6}$$

The solution of this Poisson equation again has the character of a spatially static-like field, now "induced" by a virtual distribution of sources and sinks, as specified by the r.h.s. of eq. (6). This virtual charge distribution covers the entire space, in contrast to the physical charges inducing $\Phi_{qu.static}$.

Higher-order approximations accurate up to and including the order $O\left(\frac{a}{\lambda}\right)^n$, with $n \geq 4$, will similarly obey the recurrence relations

$$\left(\frac{\partial^2}{\partial\left(\frac{x}{a}\right)^2} + \ldots\right)\Phi_n = \frac{\partial^2}{\partial\left(\frac{t}{T}\right)^2}\Phi_{n-2} \qquad (n \geq 4) \tag{7}$$

from which it follows

$$\nabla^2(\Phi_{near})_n = \frac{1}{c^2}\frac{\partial^2(\Phi_{near})_{n-2}}{\partial t^2} + O\left(\frac{a}{\lambda}\right)^{n+1} \qquad (n \geq 4) \tag{8}$$

where the notation $(\Phi_{near})_n$ stands for the partial sum

$$(\Phi_{near})_n = \Phi_{qu.static} + \frac{a}{\lambda}\Phi_1 + \left(\frac{a}{\lambda}\right)^2\Phi_2 + \ldots + \left(\frac{a}{\lambda}\right)^n\Phi_n \tag{9}$$

When n is increased step by step, the "stack" of fields $(\Phi_{near})_n$, determined by eqs. (6) and (8), may be viewed as a sequence of successive approximations, calculated for a given time. If this sequence of approximations converges, i.e. if $\Phi_n \to 0$ and $(\Phi_{near})_{n-2} \to (\Phi_{near})_n \; for \; n \to \infty$, then according to eq. (8) the converged field $(\Phi_{near})_{n\to\infty}$ exactly satisfies the original wave equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(\Phi_{near})_n = 0 \quad for \; n \to \infty \tag{10}$$

In sec. 6, it is investigated under which physically realisable conditions convergence may be expected.

The conclusion is that, under conditions guaranteeing convergence, the solution of the wave equation can be formulated in two different but equivalent forms: either as a superposition of travelling waves or alternatively as a series expansion like eq. (4). The travelling wave solution shows the retardations explicitly, whereas the series expansion is a cross-section for a given time, where time enters as a parameter only and retardation effects are not explicit. Of course, retardation effects are physical realities, but in the representation by an asymptotic series they are implicit: the effects of retardation are mimicked thanks to the presence of the r.h.s. of the Poisson equations (6) and (8).

To return to eq. (6), valid up to and including the order $O\left(\frac{a}{\lambda}\right)^3$, sec. 4 will deal with its solution. The complete solution will consist of a particular solution of the Poisson equation, summed with appropriate solutions of the Laplace equation $\nabla^2\Phi_{near} = 0$. The latter solutions of the homogeneous equation must anyway comprise the quasi-static field $\Phi_{qu.static}$, whose singularities represent the physical charge distribution. However, the near-field approximation misses boundary conditions at infinity (because the derivation of the Poisson equation is valid in the near field only), so another contribution to the near field $\Delta\Phi$ is possible (and actually does occur), satisfying the Laplace equation $\nabla^2(\Delta\Phi) = 0$ and not necessarily vanishing at infinity. This additional field $\Delta\Phi$ cannot yet be determined because of the incompleteness of the boundary value problem. The ambiguity of the solution will be removed by so-called "matching" with a far-field approximation.

3.2. Far-field approximation

In the far field, the characteristic scale length is λ, implying that from a distant point of view, the near-field scale a reduces to zero in the limit $\frac{a}{\lambda} \to 0$. The far field will thus consist of a series of multipoles at the origin. In this case, however, the "inner boundary conditions" are absent, so the number and type of the poles must be determined by the matching process.

3.3. The matching condition

The complete solution Φ, which is valid everywhere in space (up to a certain—chosen—asymptotic accuracy), is the so-called "composite field" $\Phi_{composite}$, built up as follows:

$$\Phi_{composite} = \Phi_{near} + \Phi_{far} - \Phi_{common} \tag{11}$$

where the so-called "common field" Φ_{common} is subtracted to avoid double counting. It is a field satisfying the following conditions:

$$\Phi_{common} \sim \left(behaviour\ of\ \ \Phi_{near}\ far\ away,\ \ whilst\ \frac{a}{\lambda} \to 0\right) \tag{12}$$

and also

$$\Phi_{common} \sim \left(behaviour\ of\ \ \Phi_{far}\ nearby,\ \ whilst\ \ \frac{v}{c} \to 0\right) \tag{13}$$

where the symbol $\sim$ stands for "equality to the required asymptotic order". In the region close to the physical charges, the singular behaviour of the far field Φ_{far} will be cancelled by the common part according to eq. (13), and the composite field there reduces to the near field. Likewise, eq. (12) sees to it that the composite field becomes equal to the far field at large distances so that, as required, the fields vanish at infinity. The two relations (12) and (13) can be summarised in the form of the so-called "matching condition":

$$[\ \Phi_{near}]_{r\to O(\lambda)}\ \sim\ [\ \Phi_{far}]_{\rho\to O(a)}\quad for\ \ \frac{a}{\lambda} \to 0 \tag{14}$$

where r denotes the distance between an element of the droplet and a field point, and ρ is the distance of a field point from the origin. The symbol $[\Phi_{near}]_{r\to O(\lambda)}$ is often called "the outer expansion of the near field" in the text and, likewise, the terminology "inner expansion of the far field" is used for $[\Phi_{far}]_{\rho\to O(a)}$.

All the ambiguities and singularities remaining in both the near and far fields separately are removed by applying the matching condition.

4. The near-field potentials of a moving point charge

The complete field of a moving and deforming droplet of charge as described in sec. 2 will be built up by integrating the contributions from infinitesimal elements. We will therefore first construct the potentials of a moving point charge dq whose position is given by $(x_0,\ y_0,\ z_0(t))$, having a velocity in the Z-direction $v(t) = \dot{z}_0(t)$, an acceleration $\dot{v}(t)$, and a jerk $\ddot{v}(t)$. The notations are shown in Fig. 2. The value of the potentials at a field point will be denoted by the lower-case symbols $\phi(x,y,z,t)$ and $a_z(x,y,z,t)$ instead of $\Phi(x,y,z,t)$ and $A_z(x,y,z,t)$ to emphasise that they are associated with a small element dq of the complete droplet with charge q. The quasi-static scalar potential of the point charge is

$$\phi_{qu.static} = \phi_{ref}\,\frac{a}{r} \qquad\qquad \phi_{ref} = \frac{dq}{4\pi\varepsilon_0}\,\frac{1}{a} \tag{15}$$

where

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \tag{16}$$

In the present paper, the solution of Φ_{near} is pursued up to and including $O\left(\frac{a}{\lambda}\right)^3$, for which the following Poisson equation is applicable:

$$\nabla^2\phi_{near} = \frac{1}{c^2}\,\frac{\partial^2\phi_{qu.static}}{\partial t^2} \tag{17}$$

Because the point charge is moving in the Z-direction, the vector potential $\underline{a}$ has only one component, viz. $a_z(x,y,z,t)$. The quasi-static field is given by

$$\frac{c.\,(a_z)_{qu.static}}{\phi_{ref}} = \frac{v}{c}\,\frac{a}{r} \tag{18}$$

which must be substituted into

$$\nabla^2 a_{z\,near} = \frac{1}{c^2}\,\frac{\partial^2 (a_z)_{qu.static}}{\partial t^2} \tag{19}$$

According to Appendices A and B, the near fields, up to the order $O\left(\frac{a}{\lambda}\right)^3$ and including the terms inserted by the matching, are given by

$$\frac{(\phi_{near})}{\phi_{ref}} = \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)\frac{a}{r} - \frac{z-z_0}{2}\left[\frac{\dot{v}}{c^2} - \frac{v^2}{c^2}\frac{\partial}{\partial z}\right]\left(\frac{a}{r}\right) \mp a\,\frac{v\dot{v}}{c^3} \pm \frac{1}{3}\,a\,(z-z_0)\,\frac{\ddot{v}}{c^3} \tag{20}$$

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$$\frac{c.\,a_{z\,near}}{\phi_{ref}} = \left[\frac{v}{c} + \frac{\ddot{v}}{c^3}\frac{r^2}{2} - 3\frac{v\dot{v}}{c^3}\frac{z-z_0}{2} + \frac{v^3}{c^3}\frac{z-z_0}{2}\frac{\partial}{\partial z}\right]\left(\frac{a}{r}\right) \mp \frac{\dot{v}}{c^2}a \tag{21}$$

In the case of outgoing waves in the far field, the – sign of the symbol $\mp$ is applicable, and the + sign of the symbol $\pm$. Similarly, the lower signs in these symbols are associated with incoming waves. Note the presence of terms that do not vanish at infinity. These particular terms will be cancelled when the complete composite field is formed. They influence the electromagnetic forces acting on charges in the near field because of the presence of the derivatives $\frac{\partial \Phi_{near}}{\partial z}$ and $\frac{\partial a_{z\,near}}{\partial t}$ in Lorentz's force equation (sec. 5).

5. Forces within the near-field

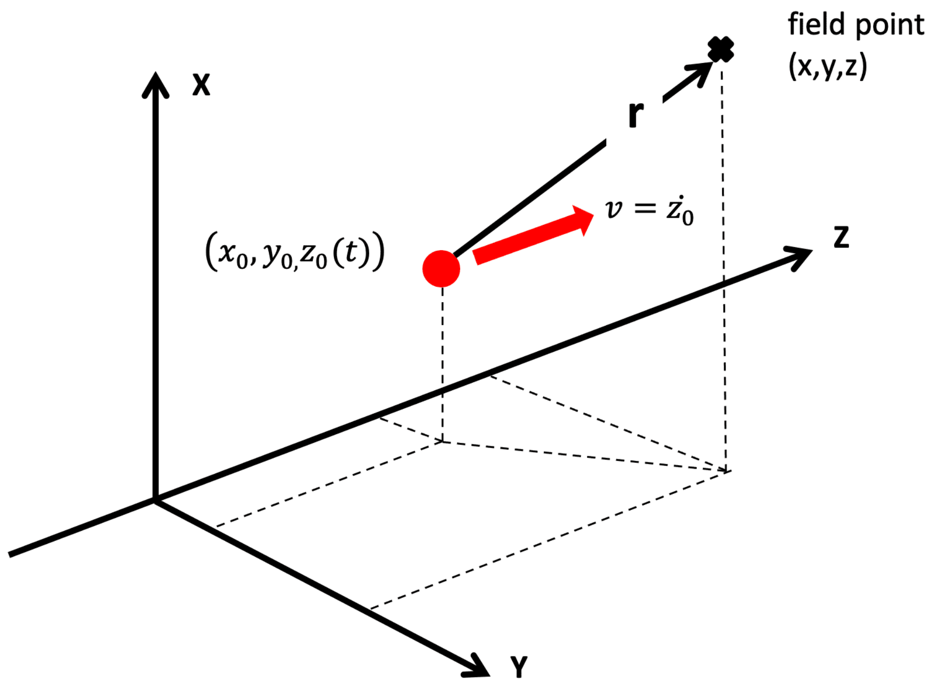

Figure 2. Notations point charge

5.1. Force on a unit "test" charge exerted by an element dq of the droplet

Lorentz's force equation reads:

$$\underline{F} = Q\left(\underline{E} + \underline{V} \times \underline{B}\right) \tag{22}$$

where

$$\underline{E} = -grad\Phi - \frac{\partial \underline{A}}{\partial t} \qquad \underline{B} = rot\ \underline{A} \tag{23}$$

$$\underline{V} = velocity\ of\ charge\ Q\ \ on\ which\ \underline{F}\ \ acts$$

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As a preliminary to determining the self-forces on the entire droplet of charge, a unit “test” charge in the near field will be considered with a velocity V in the Z-direction. The velocity $\underline{v}$ of a charge element dq and $\underline{V}$ of the unit “test” charge are both in the Z-direction. Consequently, the contribution by $\underline{V} \times \underline{B}$ is zero, and $\underline{F}$ does not depend on the velocity of the test charge. According to eq. (22), the Z-force associated with a droplet element dq exerted on a unit test charge in the near field is

$$(f_z)_{near} = -\frac{\partial \phi_{near}}{\partial z} - \frac{\partial a_{z\,near}}{\partial t} \tag{24}$$

The expressions for the near field of the scalar and vector potential (20) and (21) are substituted into (24). Omitting terms of the order $O\left(\frac{a}{\lambda}\right)^4$ leads to the result

$$\frac{(f_z)_{near}}{\phi_{ref}} = -\frac{1}{2}\frac{\dot{v}}{c^2}\frac{a}{r}\left(1+\cos^2\theta\right) - \left[1+\frac{1}{2}\frac{v^2}{c^2}\left(1-3\cos^2\theta\right)\right]\frac{\partial}{\partial z}\left(\frac{a}{r}\right) \pm \frac{2}{3}a\frac{\ddot{v}}{c^3} \tag{25}$$

where

$$cos\theta = \frac{z-z_0}{r} \quad and \quad \frac{\partial}{\partial z}\left(\frac{a}{r}\right) = -a\frac{cos\theta}{r^2} \tag{26}$$

An approximation giving better insight can be introduced when the connecting line between the “test charge” and the droplet element includes a small angle with the Z-axis, such that one may approximate $\cos\theta = sgn(z-z_0)$ and $cos^2\theta = 1$:

$$\frac{(f_z)_{near}}{\phi_{ref}} = -\frac{\dot{v}}{c^2}\frac{a}{r} + \left(1-\frac{v^2}{c^2}\right)\frac{a}{r^2}sgn\,(z-z_0) \pm \frac{2}{3}a\frac{\ddot{v}}{c^3} \tag{27}$$

5.2. Self-forces

The self-force in the translational direction on a droplet is given by the double volume integral

$$F_{z\,self} = \iiint_{droplet} \sigma\, dx\, dy\, dz \iiint_{droplet} \frac{(f_z)_{near}}{\phi_{ref}}\,\frac{1}{a}\,\frac{dq}{4\pi\varepsilon_0} \tag{28}$$

Despite the singularity of the integrand, this type of integral can be defined exactly (see [12]). For now, a rigid (or relativistically rigid) droplet is assumed, so that the results of the “max” procedure can be verified by a comparison with the classical results. The second term in eqs. (25) and (27), expressing a Coulomb force including a relativistic correction, would clearly lead to electrostatic attraction or repulsion on *another* droplet but does not contribute to a translational *self*-force.

Assuming a slender shape, an approximation of the self-force may be obtained by substituting eq. (27) into (28). The first term on the r.h.s. of eq. (27) expresses a force field opposite to the acceleration of the droplet elements and leads after integration to an inertia force:

$$F_{z\,inert} = -\frac{\dot{v}}{c^2}\iiint_{droplet} \sigma\, dx\, dy\, dz \iiint_{droplet} \frac{1}{r}\,\frac{dq}{4\pi\varepsilon_0} =$$

$$= -\frac{\dot{v}}{c^2}\iiint_{droplet} \sigma\, \Phi_{static}\;\, dx\, dy\, dz = -\frac{\dot{v}}{c^2}2U_{es} \tag{29}$$

where U_{es} denotes the so-called "electrostatic energy", representing the formation energy expended by bringing all the elements of the charge together against their mutual repulsive forces. In Lorentz's time, this result gave rise to the concept of "electromagnetic mass" m_{em}, see e.g. [8]. Caution should be exercised when using m_{em} in dynamic equations since it must be corrected for the binding energy to get agreement with relativity theory, see e.g. [6] or [13]. Also, analogous to the second term in eq. (27), a relativistic correction would be expected:

$$\frac{(f_z)_{near}}{\phi_{ref}} = -\frac{\dot{v}}{c^2}\frac{a}{r}\sqrt{1-\frac{v^2}{c^2}} + \ \ldots . \tag{30}$$

which would make the electromagnetic mass dependent on velocity in the same way as mechanical mass under the influence of relativity effects. However, this refinement would have an asymptotic order $O\left(\frac{v}{c}\right)^4$ and will be neglected in the following.

The last term on the r.h.s. of eq. (27) leads to Lorentz's "radiation resistance" (sometimes "radiation reaction"):

$$F_{z\,rad} = \pm\frac{2}{3}\frac{\ddot{v}}{c^3}\iiint_{droplet}\sigma\ dx\ dy\ dz\ \iiint_{droplet}\frac{dq}{4\pi\varepsilon_0} = \pm\frac{2}{3}\frac{\ddot{v}}{c^3}\frac{q^2}{4\pi\varepsilon_0} \tag{31}$$

The name "radiation resistance" is derived from the fact that, in the case of an outgoing wave (+ sign applicable), an external force overcoming this self-force would on average supply as much energy to the system as escapes at infinity due to radiation. An interesting situation occurs when the radiating charge is surrounded by a wave barrier, so that the system is described by outgoing and incoming waves of equal strength. In such a case there can be no energy loss from the system by radiation to infinity, and according to eq. (31) the radiation resistance vanishes. In contrast, the electromagnetic mass effects do not depend on the direction of the radiation, and have the same value for open and shrouded systems.

The results obtained by the "max" approach are fully in agreement with the classical results obtained by Lorentz and Abraham for a rigid charge configuration. A deformable droplet has more translational self-forces, amongst them forces coupling the translation and the pulsation. These are derived and discussed in [5][6][7].

5.3. Non-retarded forces

When the equations (25) or (27) in sec. 5.1 are used to calculate the self-forces on a rigid or deforming object, it is assumed that both charges (i.e. the unit "test" charge and the droplet element dq) are part of the same—coherent—charge distribution. However, eqs. (25) and (27) are not restricted to coherent configurations, and may be applied to pairs of charges situated in separate lumps of charge with different velocities. An example is shown in fig. 3, where one coherent distribution splits into two separate parts. The resulting change of configuration may be viewed as an "extreme form of deformation". Eqs. (25) and (27) may be applied as long as both the slender droplets share the same near field, i.e. as long as the characteristic dimensions of the total constellation remain small compared with typical wavelengths in the far field.

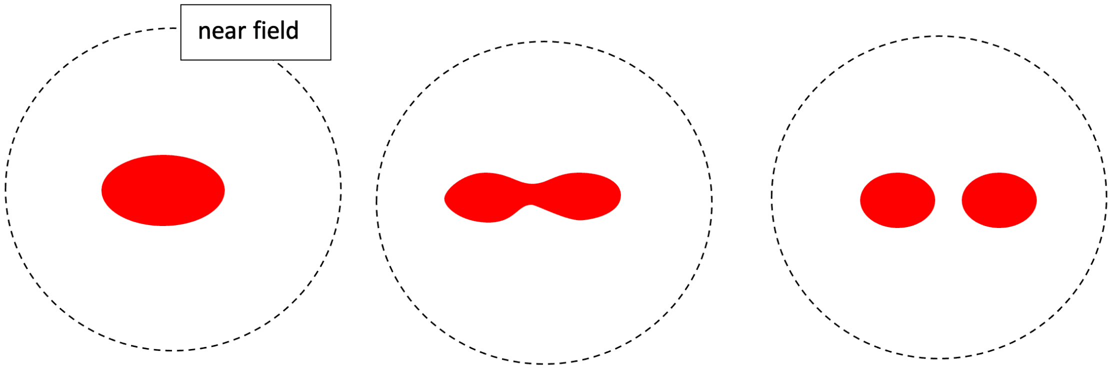

Figure 3. *The concept of "two droplets sharing the same near field", viewed as an extreme type of deformation.*

Eqs. (25) or (27) seem to imply that the position, the acceleration and jerk of one of the fragments influence the forces on the other fragment *without time retardations.* Note that this is true in spite of the fact that the retardation effects implied by the wave equation were duly accounted for by the presence of the r.h.s. in the Poisson equations (17) and (19). A discussion of this paradox will be given in sec. 7, after a closer consideration of the convergence of the asymptotic series which in the above theory replaced the wave-like solution of the field equations.

6. Convergence of the asymptotic series

In the present section it is investigated how and under which conditions the metamorphosis of the retardations from explicit to implicit takes place in the "max" theory. This transition takes place during the determination of the inner expansion of the far field. Consider as a "toy model" first a monopole with variable strength, of a type as occurs in the far field of $a_{z\,far}$:

$$\phi_{monop.} = \frac{v\left(t \mp \frac{\rho}{c}\right)}{c} \frac{1}{\rho} \tag{32}$$

The minus-sign, denoting a retardation, is associated with outgoing waves. The plus-sign is associated with an incoming wave. The field (32) satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{v\left(t \mp \frac{\rho}{c}\right)}{c}\frac{1}{\rho} = 0 \tag{33}$$

The function $v\left(t \mp \frac{\rho}{c}\right)$ is expanded in a Taylor series (as was done in app. A and B to determine the inner expansion of $a_{z\,far}$). Using the Taylor formula with a Lagrange remainder:

$$\frac{v\left(t \mp \frac{\rho}{c}\right)}{c}\frac{1}{\rho} = \frac{v(t) \mp \left(\frac{\rho}{c}\right)\dot{v}(t) + \frac{1}{2}\left(\frac{\rho}{c}\right)^2 \ddot{v}(t) \mp \ldots + \frac{(\mp 1)^n}{n!}\left(\frac{\rho}{c}\right)^n v^{(n)}\left(t \mp \vartheta\frac{\rho}{c}\right)}{c\,\rho} \tag{34}$$

with $0 < \vartheta < 1$. On the l.h.s. the retardation is still explicit; on the r.h.s. the role of time has been reduced to just a parameter and the retardations are implicit. The mathematical condition for the validity of the Taylor formula is that

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there must exist a certain time interval in which the function $v(t)$ can be differentiated n times. The endpoints t and $\left(t \mp \frac{\rho}{c}\right)$ should be points of this interval.

The general term as well as the remainder R_n can be written in terms of non-dimensional quantities:

$$R_n = c\frac{(\mp 1)^n}{n!} \left(\frac{\rho}{\lambda}\right)^n \left. \frac{d^n\left(\frac{v}{c}\right)}{d\left(\frac{t}{T}\right)^n}\right|_{t \mp \vartheta \frac{\rho}{c}} \tag{35}$$

If we let $n \to \infty$ (implying that all the derivatives of $v(t)$ are differentiable within the chosen interval) the remainder vanishes ($R_n \to 0$), due to

$$\lim_{n\to\infty} \frac{\left(\frac{\rho}{\lambda}\right)^n}{n!} = 0 \tag{36}$$

The numerator on the r.h.s. of eq. (34) becomes an infinite Taylor series for $n \to \infty$ converging towards $v\left(t \mp \frac{\rho}{c}\right)$. The convergence is assured no matter how far the field point is removed from the singularity in the origin (i.e. $\frac{\rho}{\lambda}$ may be arbitrarily large in eq. (36)). However, at large distances in the far field the speed of convergence will be very poor. It is only when very many terms are included that some amount of convergence of successive terms will be noticed. Often such a poor convergence is physically associated with the fact that the chosen type of series does not “naturally fit” the physics. In this case, it is an indication that at far distances from a variable multipole the description by travelling waves is the most natural and fits the physical appearance of the field best. In contrast, it is seen that at small distances where $\frac{\rho}{\lambda} \to O\left(\frac{a}{\lambda}\right)$ and $\frac{a}{\lambda} \to 0$ the series will converge fast, which may be a sign that in the inner regions of the multipole field the series description is relatively “natural”.

In Fig. 4 a possible—but arbitrary—function $v(t)$ of the monopole strength is sketched. What is meant is the pole strength without retardation, as would be observed at the origin $\rho = 0$. It is assumed that the field is actually observed at a distance ρ_1 from the origin, at the time t_1. The pole strength “experienced” at the observation point is retarded (assuming an outgoing wave), i.e. the relevant pole strength is $v(t_1 - \frac{\rho_1}{c})$.

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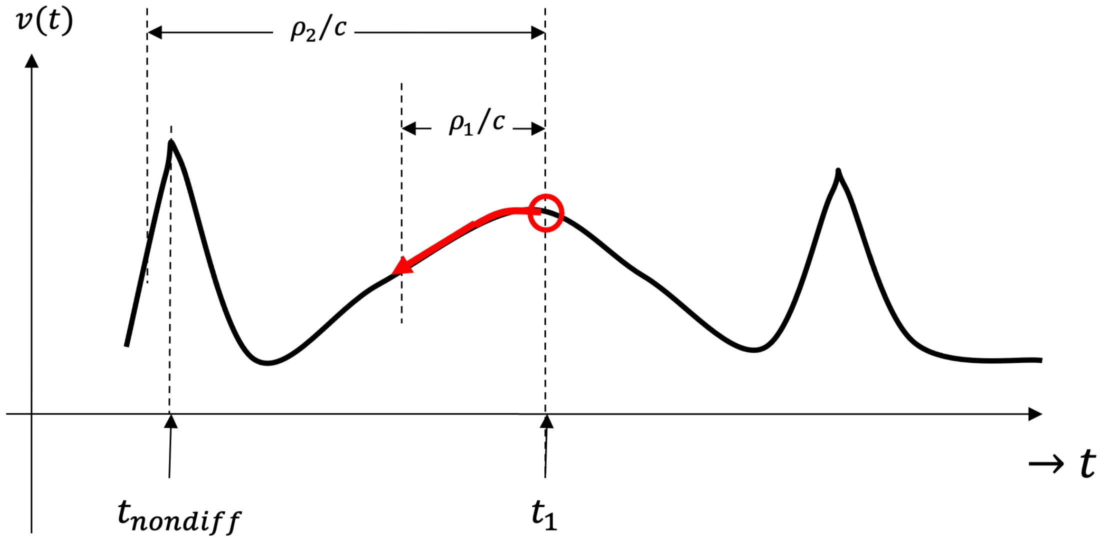

Figure 4. *Pole strength as a function of time*

If the value of $v(t_1 - \frac{\rho_1}{c})$ is determined by a Taylor series, starting from $v(t_1)$ and using the derivatives $\dot{v}(t_1)$, $\ddot{v}\,(t_1)\,,\;\ldots$ (as symbolised by the red arrow in Fig. 4), the requirement is that all the derivatives of the function $v(t)$ are differentiable in the time interval $(t_1 - \frac{\rho_1}{c}, t_1)$. Note that it is not sufficient that all the derivatives $\dot{v}(t_1)$, $\ddot{v}(t_1)$, ..., occurring in the Taylor formula exist. The requirement is more restrictive, viz. that all the derivatives $\dot{v}(t)$, $\ddot{v}(t)$, ... exist in the entire interval $(t_1 - \frac{\rho_1}{c}, t_1)$ and that they are differentiable.

If $t_{nondiff}$ denotes the time where the function $v(t)$ or one of its derivatives cannot be differentiated (in the ordinary analytical meaning; the case of $v(t)$ being defined as a "generalised function" needs special consideration), convergence of the Taylor series towards the value $v(t_1 - \frac{\rho_1}{c})$ does not occur between the time $t_1 = t_{nondiff}$ and $t_1 = t_{nondiff} + \frac{\rho_1}{c}$ (see Fig. 4). The "forbidden" time interval increases proportionally to the distance from the origin; see the case $\rho_2 > \rho_1$ sketched also in Fig. 4. The forbidden regions, in time as well as in space, are summarised in Fig. 5. In this figure, the influence of a later discontinuity is also sketched.

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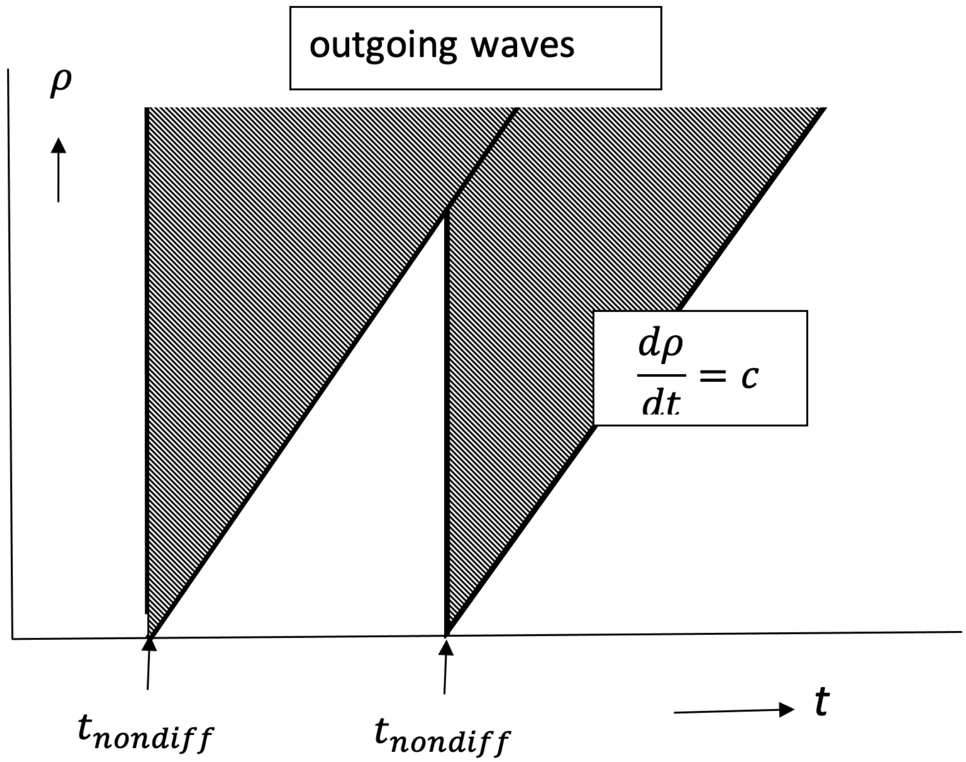

Figure 5. *Regions where asymptotic expansion is not valid, following discontinuity in boundary conditions at* $t_{nondiff}$

So far, the conditions allowing a Taylor expansion of the numerator of the r.h.s. of eq. (34). What also has to be examined is whether the series expansion of the monopole field under all conditions satisfies the wave equation. It thus has to be investigated whether the following equality is true:

$$0 = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left[\frac{v(t)}{c}\frac{1}{\rho} \mp \frac{\dot{v}}{c^2} + \frac{1}{2}\frac{\ddot{v}}{c^3}\rho \mp \frac{1}{6}\frac{\dddot{v}}{c^4}\rho^2 + \frac{1}{24}\frac{v^{(4)}}{c^5}\rho^3 + \dots\right] \tag{37}$$

or:

$$\begin{aligned}
0 = -\frac{\ddot{v}}{c^3}\frac{1}{\rho} + & \qquad\qquad (contribution\ by\ first\ term\ between\ square\ brackets) \\
\pm \frac{\dddot{v}}{c^4} + & \qquad\qquad (by\ 2^{nd}\ term) \\
+\frac{\ddot{v}}{c^3}\frac{1}{\rho} \qquad -\frac{1}{2}\frac{v^{(4)}}{c^5}\rho + & \qquad\qquad (by\ 3^{rd}\ term) \\
\mp \frac{\dddot{v}}{c^4} \qquad \pm\frac{1}{6}\frac{v^{(5)}}{c^6}\rho^2 + & \qquad\qquad (by\ 4^{th}\ term) \\
+\frac{1}{2}\frac{v^{(4)}}{c^5}\rho \qquad -\frac{1}{24}\frac{v^{(6)}}{c^7}\rho^3 + \dots & \qquad\qquad (5^{th}\ term\ and\ remainder)
\end{aligned} \tag{38}$$

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Each term on the upper diagonal is produced by the operation $-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ and is cancelled by a term two rows lower (in the lower diagonal), produced by the operation ∇^2. For complete cancellation (and therefore exact compliance with the wave equation) the series expansion of $v\left(t-\frac{\rho}{c}\right)$ has to be continued indefinitely, i.e. $n \to \infty$. Truncating the Taylor series and adding a Lagrange remainder would always leave "uncancelled" terms on the r.h.s. of expression (38).

The conclusions reached about the convergence of the series describing a monopole field are equally valid for other multipole fields, since any kind of partial derivative of a monopole field will satisfy the wave equation. Any retarded multipole field can thus be expressed as a quasi-static field plus an infinite number of correction terms representing the retardation effects. The series is an exact representation of the retarded wave field, outside certain "forbidden" time intervals as sketched in Fig. 5.

The asymptotic series solution of the near field matches the inner expansion of the far field (i.e. the above-investigated field of a multipole). If the inner expansion of the far field needs relatively few terms, fast convergence of the near-field series obtained from the stack of Poisson equations (8) can also be expected.

In the near field, the "forbidden" time intervals between $t_{nondiff}$ and $t_1 = t_{nondiff} + \frac{\rho_1}{c} = t_{nondiff} + \frac{a}{\lambda}.T$ are a small part of the total travelling time T between reversals of the back-and-forth motion of a droplet on a stretch a.

7. Separation of travelling waves into a superposition of standing waves

If the space around the pole is closed, e.g. by a perfectly reflecting spherical surface, the field consists of a superposition of outgoing and incoming waves of equal strength, leading to standing waves. Eq. (34) shows that the infinite series

$$\phi_1 = \frac{v(t)}{c}\frac{1}{\rho} + \frac{1}{2}\frac{\ddot{v}}{c^3}\rho + \frac{1}{24}\frac{v^{(4)}}{c^5}\rho^3 + \dots \tag{39}$$

thus describes a field of standing waves. This is also shown by eq. (38), where the tabulation proves that the series (39) satisfies the wave equation. If $v(t)$ is harmonic, the potential ϕ_1 is everywhere in the field in phase (or anti-phase) with $v(t)$. This position-independent synchronicity is one of the characteristics of a field of standing waves. The remaining series

$$\phi_2 = \mp\left[\frac{\dot{v}}{c^2} + \frac{1}{6}\frac{\dddot{v}}{c^4}\rho^2 + \frac{1}{120}\frac{v^{(5)}}{c^6}\rho^4 + \dots\right] \tag{40}$$

also satisfies the wave equation according to eq. (38) and displays position-independent synchronicity if $v(t)$ is harmonic. The field ϕ_2 also describes a field of standing waves, although it is not in phase with ϕ_1.

The conclusion is also applicable to the field of any kind of multipole. If it can be represented by an infinite asymptotic series (outside certain "forbidden" time intervals), then the terms can be rearranged into two groups, such that each of the resulting series represents a standing wave system. The complete multipole field of travelling waves can therefore be viewed as a superposition of standing waves.

It should be emphasised that the separate terms of the series, like e.g. $\frac{v(t)}{c}\frac{1}{\rho}$, display position-independent synchronicity of the field variables and the associated singularity but do not satisfy a wave equation. The separation of the field of travelling waves into a superposition of standing waves therefore applies only to the infinite series, unless—in an actual calculation—a certain accuracy threshold is chosen below which all the terms are considered to be negligible. In such a practical case, the separation of a truncated series into two truncated series does represent the separation of a travelling wave system into a superposition of standing wave systems, albeit with a certain error being explicitly accepted. If, for instance, between the square brackets of eq. (37) everything after the third term (which is of order $O\left(\frac{v}{c}\right)^3$) is neglected, it is seen in eq. (38) that the uncancelled terms have an order $O\left(\frac{v}{c}\right)^4$ or smaller. The wave equation is thus satisfied to the same order as the order of accuracy of the truncated expansion of the pole field.

8. Interpretation of non-retarded forces from a physical point of view

If a point charge in $(x_0,\ y_0,\ z_0(t))$ moves in the Z-direction, the force field $f_z(x,y,z,t)$ is determined from the scalar and vector potential by the relation (24). Both ϕ and a_z satisfy the wave equation, so the field $f_z(x,y,z,t)$ also satisfies this equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f_z = 0 \tag{41}$$

An approximation of the near force field satisfying this wave equation to the order $O\left(\frac{v}{c}\right)^3$ was given by expression eq. (25). If a harmonic variation $v = \cos(\omega t)$ is assumed and when the first and third terms of eq. (25) with $\dot{v}$ and $\ddot{v}$ respectively are considered, it is seen that their sum at a given time has a continuously variable phase as a function of the distance r. This is a characteristic of a travelling wave. The truncated series eq. (25) thus describes a travelling wave, either outgoing (+ sign) or incoming (– sign).

If the space around the pole is surrounded by a reflecting surface, the field consists of a superposition of outgoing and incoming waves of equal strength, leading to standing waves. Such a superposition will remove the third term from the field of eq. (25). The standing wave system obtained by the reflection is thus described by the sum of the first and second terms of eq. (25). It has to be noticed that the second term, though adding higher harmonics, does not cause any phase shifts that would depend on the position.

The third part $\pm\frac{2}{3}a\frac{\ddot{v}}{c^3}$ of the force field satisfies the wave equation to the required accuracy and displays synchronicity of all the field points and $v(t)$, no matter the distance. It too can be interpreted as the field of a standing wave, accurate to the order $O\left(\frac{v}{c}\right)^3$.

In a standing wave system, there is synchronicity of the field variables, including the singular points in the field. It is crucial for the argument in this section to emphasise that “synchronicity” does not imply a causal relation between the singularity and the rest of the field. Singularities do not exist in isolation, by definition. The different types of electromagnetic singularities have historically been named “charge”, “current”, “dipole”, etc., as if they were isolated

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physical entities. The wrong impression is thereby given that such singularities would exist in isolation, whilst all these physical quantities are known and measured only by mediation of the corresponding fields. Speaking about “the field of a singularity” wrongly suggests a causal process and should be replaced by “the singularity of a field”, indicating no more than a certain mathematical property of the field. Synchronicity can easily give rise to the *illusion* of a superluminal interaction. As an example in fig. 6, two droplets of charge are drawn, close to the edge of a potential well, the latter symbolised by the red line. Initially, the droplets move away from each other due to electrostatic repulsion.

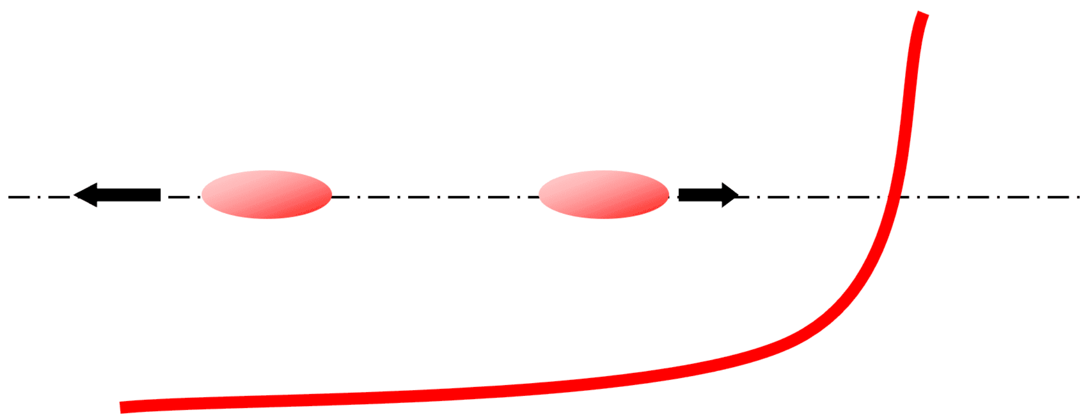

Figure 6. *Two droplets of charge moving near the corner of a potential well*

Nearing the steep edge of the well, the right droplet will decelerate rather suddenly and, depending on its kinetic energy relative to the potential energy inside the well, may even reverse its velocity. According to eq. (27), the inertial forces in the field extend to the droplet on the left-hand side (as long as both droplets still share the same near-field), so the left droplet will immediately slow down too, and in an extreme situation could reverse the direction of its motion. Both droplets simultaneously react to the presence of the potential "wall", which could easily—but incorrectly—be interpreted as superluminal "signalling" between them. The theory derived above has shown that this behaviour is just a consequence of Maxwell's equations and that it is not at all in conflict with relativity theory.

The physical view on the above-sketched situation is that the two droplets sharing the same near field are in essence no different from one deforming charged object. The separation of the two parts can be viewed as just an extreme form of deformation. If a coherent collection of charges possesses electromagnetic mass, this implies that all parts of the collection experience a force opposite to an acceleration. This will be true even if the collection is no longer coherent but has fallen apart into fragments.

Whether electromagnetic mass is indeed a physical reality has been a matter of some discussion. Around a century ago the analyses by Lorentz and Abraham showed that it is one of the consequences of Maxwell's equations. The different mathematical analysis reported in the present paper arrives at the same conclusion. In [8] Feynman discusses electromagnetic mass as a contributor to the total mass of a body, and concludes that its existence is confirmed by experimental data. Finally, based on the premise that all the charged leptons possess electromagnetic mass, in [6] an explanation is proposed for how it is possible that the electron, muon and tau-particle have widely differing masses, even if there are no other known differences between them. This once more adds to the plausibility that electromagnetic mass is a physical reality.

The use of a quantity like "mass of a body" (whatever its origin) is meaningful only if it is not influenced by the acceleration or other kinematics of the body. If in eq.(27) the influence of $\dot{v}$ on f_z had been retarded, then after substitution into eq.(28) the mass would be a function of the time history of $v(t)$. The non-retardation on the scale of the near field is a *conditio sine qua non* for the definition of electromagnetic mass.

The non-retardation in the near force field can also be explained by a mixed physical-mathematical consideration similar to the description of the potential field around a variable pole (sec. 6). The droplet's far field of forces consists of a set of variable multipole fields, and represents travelling waves. As shown in sec. 6 it is possible to divide such a field of travelling waves into two groups of terms, each group apart describing a system of standing waves. Each group thus consists of a multi-harmonic collection of standing waves, where all the phase-angles are position-independent. One of the groups will contain the acceleration $\dot{v}$ of the droplet. All the variables belonging to this particular group will thus show synchronicity with the variable $\dot{v}$, however far removed they are from the origin of the far field multipoles. Thanks to the matching of the near- and far field, the corresponding variables in the near field also show synchronicity with $\dot{v}$, which explains the so-called "non-retardation" of the elemental forces which integrate into the total inertial force on the droplet.

If the boundary conditions display discontinuities, then the standing waves temporarily break down, and new boundary conditions as well as new initial conditions will be established by a process obeying the speed limits of relativity theory. After such a reset the autonomous and holistic development of the field variables is restored.

9. Conclusions

9.1. The existence of so-called "non-retarded interactions at a distance" is not in conflict with Maxwell's equations, and such interactions are not incompatible with relativity theory.

9.2. These consequences of Maxwell's equations are limited to the near field of a moving and deforming charge distribution, where the near field is defined as the region where characteristic dimensions are small w.r.t. typical wavelengths in the far field extending to infinity.

9.3. Another proviso is that the boundary conditions (i.e. a specification of the motion and deformation of the charge distribution, as well as of external potential fields) are smoothly evolving in time, meaning that within the specified margins of accuracy no discontinuities occur of the derivatives w.r.t. time.

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9.4. A consequence is that the mutual interactions between two droplets of charge sharing the same near field are non-retarded, as if the droplets still formed parts of the same coherent lump of charges (the separation is not different from an extreme form of deformation). For instance, a forced deceleration of one of them will immediately result in a simultaneous deceleration of the other.

9.5. The theoretical interpretation is that the common far field of the droplets, which consists of a multitude of travelling waves, can be separated into a superposition of standing waves. Within a standing wave system the variables belonging to this particular system behave synchronously. Synchronous behaviour does not imply a causal relation between the concerned variables, and therefore does not involve any form of signal transfer which could involve a delay. However, the synchronous behaviour of the variables may give rise to the "illusion" of non-delayed signalling.

9.6. An interruption of the field evolution due to discontinuities necessitates the establishment of new boundary conditions and new initial conditions. Such a "reset" will take a finite time and propagate through the field with the speed of light.

9.7. The matched asymptotic expansion technique which in the present paper enabled the detailed analysis of the near field of charges will in future have to be further extended to cover phenomena like "spin", so that a comparison becomes possible with QM-experiments on non-local effects and the EPR-paradox.

9.8. It is often thought that Emergent Quantum Mechanics based on the classical laws of physics will fundamentally be limited in scope because of its supposed incapability to encompass non-local effects. This opinion is disproved by the present analysis, showing that non-local ("spooky") phenomena are compatible with Maxwell's electromagnetic equations and with Einstein's theory of special relativity.

Appendix A. The scalar potential of a moving point charge

A.1. Near field

From eq.(16) the useful relation follows:

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\right) = -v\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \tag{A.1}$$

Using the relation (A.1) the Poisson equation (17) becomes:

$$\nabla^2\left(\frac{\phi_{near}}{\phi_{ref}}\right) = \frac{1}{c^2}\left[-\dot{v}\frac{\partial}{\partial z} + v^2\frac{\partial^2}{\partial z^2}\right]\ \left(\frac{a}{r}\right) \tag{A.2}$$

In order to obtain the particular solution of (A.2) the following general rule is applied:

$$\nabla^2\Phi = \frac{\partial\Psi}{\partial z}\ \ has\ \ the\, particular\ soln.\ \ \Phi = \frac{z}{2}\Psi\ \ \ if\ \ \nabla^2\Psi = 0 \tag{A.3}$$

The complete solution of (A.2) is therefore

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$$\frac{(\phi_{near})}{\phi_{ref}} = \frac{a}{r} - \frac{z}{2}\left[\frac{\dot{v}}{c^2} - \frac{v^2}{c^2}\frac{\partial}{\partial z}\right]\left(\frac{a}{r}\right) + \frac{\Delta\phi}{\phi_{ref}} \tag{A.4}$$

where $\Delta\phi$ should satisfy the Laplace equation

$$\nabla^2(\Delta\phi) = 0 \tag{A.5}$$

whilst $\Delta\phi$ (to be determined by the matching process) may be non-zero or may even have singularities for $r \to \infty$.

A.2. Outer expansion of the near field

At large distances from the origin it is convenient to use the polar coordinates $(\rho,\ \varphi,\ \chi)$ as sketched in fig.7:

$$x = \rho\ sin\varphi\ cos\chi,\ \ y = \rho\ sin\varphi\ sin\chi,\ \ \ z = \rho\ cos\varphi \ \ \ \rho = \sqrt{x^2 + y^2 + z^2} \tag{A.6}$$

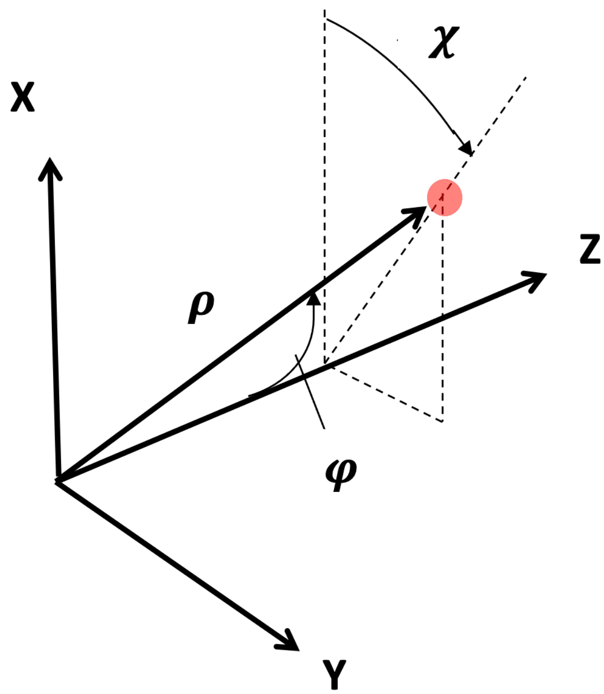

Figure 7. *Notations polar coordinates in far field*

The Cartesian coordinates in (16) are transformed into polar coordinates to read:

$$r = \rho\left[1 + \frac{{x_0}^2 + {y_0}^2 + {z_0}^2}{a^2}\left(\frac{a}{\rho}\right)^2 - 2\frac{x_0}{a}\left(\frac{a}{\rho}\right) sin\varphi\ cos\chi - 2\frac{y_0}{a}\left(\frac{a}{\rho}\right) sin\varphi\ sin\chi - 2\frac{z_0}{a}\left(\frac{a}{\rho}\right) cos\varphi\right]^{\frac{1}{2}} \tag{A.7}$$

so that, using the short notation $[\ldots]$ for the expression between the square brackets in eq. (A.7) and denoting $f\left(\frac{a}{\rho}\right) = [\ldots]^{-\frac{1}{2}}$:

$$\frac{a}{r} = \frac{a}{\rho} f\left(\frac{a}{\rho}\right) = \frac{a}{\rho}\left\{f(0) + \frac{a}{\rho}f'(0) + \frac{1}{2}\left(\frac{a}{\rho}\right)^2 f''(0) + \ldots\right\} \tag{A.8}$$

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In the region of space $r = O(\lambda)$ (implying $\rho = O(\lambda)$), it is seen that an asymptotic accuracy of $O\left(\frac{a}{\lambda}\right)^3$ is achieved when the series expansion in (A.8) is truncated by neglecting the terms symbolised by the dots. Using eq. (A.8) in combination with eq. (A.7), the final result can be written in the form of a series of multipoles in the origin (one monopole, three dipoles, and six quadrupoles):

$$\begin{aligned} \frac{a}{r} = \frac{a}{\rho} - a\frac{\partial}{\partial x}\left(\frac{x_0}{\rho}\right) - a\frac{\partial}{\partial y}\left(\frac{y_0}{\rho}\right) - a\frac{\partial}{\partial z}\left(\frac{z_0}{\rho}\right) + \\ + \frac{a}{2}\frac{\partial^2}{\partial x^2}\left(\frac{{x_0}^2}{\rho}\right) + \frac{a}{2}\frac{\partial^2}{\partial y^2}\left(\frac{{y_0}^2}{\rho}\right) + \frac{a}{2}\frac{\partial^2}{\partial z^2}\left(\frac{{z_0}^2}{\rho}\right) + \\ + a\frac{\partial^2}{\partial x \partial y}\left(\frac{x_0 y_0}{\rho}\right) + a\frac{\partial^2}{\partial x \partial z}\left(\frac{x_0 z_0}{\rho}\right) + a\frac{\partial^2}{\partial y \partial z}\left(\frac{y_0 z_0}{\rho}\right) \end{aligned} \tag{A.9}$$

The multipoles are quasi-static, i.e. their field has the form of poles in an electrostatic field, but their "strength" may depend on time. The complete near field (A.4) possesses an outer expansion given by

$$\begin{aligned} \left[\frac{(\phi_{near})}{\phi_{ref}}\right]_{\rho=O(\lambda)} = \frac{a}{r} - \frac{z}{2}\frac{\dot{v}}{c^2}\left[\frac{a}{\rho} - a\frac{\partial}{\partial x}\left(\frac{x_0}{\rho}\right) - a\frac{\partial}{\partial y}\left(\frac{y_0}{\rho}\right) - a\frac{\partial}{\partial z}\left(\frac{z_0}{\rho}\right)\right] + \\ + \frac{z}{2}\frac{v^2}{c^2}\frac{\partial}{\partial z}\left(\frac{a}{\rho}\right) + \left[\frac{\Delta\phi}{\phi_{ref}}\right]_{\rho=O(\lambda)} \qquad \left(\frac{a}{\lambda} \to 0\right) \end{aligned} \tag{A.10}$$

where the first term $\frac{a}{r}$ on the r.h.s. is written out in eq. (A.9). The truncation of the asymptotic series takes into account that coefficients like $\frac{\dot{v}}{c^2}$ and $\frac{v^2}{c^2}$ are of the order $O\left(\frac{v}{c}\right)^2 = O\left(\frac{a}{\lambda}\right)^2$. It will be noted that the expansion of $\frac{a}{r}$ leads to multipoles, whereas no other pole-fields occur in (A.10), due to the multiplication by z.

A.3. Far-field approximation

From the requirement that the outer expansion of the near field should equal the inner expansion of the far field, it can be concluded that the far field will consist of the same type of multipoles as represented by (A.10). However, in the far field, the retardation effects cannot be neglected, and the far field must satisfy the unabridged wave equation. The wave equation admits both outgoing waves as well as incoming waves as valid solutions. The retardation time therefore may be negative as well as positive, and is for example denoted by $z_0\left(t \mp \frac{\rho}{c}\right)$. Here $z_0(t)$ is the position of the point charge at time t, and $z_0\left(t - \frac{\rho}{c}\right)$ is the earlier, retarded value as "experienced" in a field point at a distance ρ, when an outward moving wave is considered. The lower sign in the symbol $\mp$ is associated with an incoming wave.

Not all the poles have a time-dependent strength, and these static poles are represented by the same expression as in eq. (A.9). The far field, approximated to the asymptotic order $O\left(\frac{a}{\lambda}\right)^3$, is therefore given by:

$$\begin{aligned} \frac{(\phi_{far})}{\phi_{ref}} = \frac{a}{\rho} - a\frac{\partial}{\partial x}\left(\frac{x_0}{\rho}\right) - a\frac{\partial}{\partial y}\left(\frac{y_0}{\rho}\right) - a\frac{\partial}{\partial z}\left(\frac{z_0\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) + \\ + \frac{a}{2}\frac{\partial^2}{\partial x^2}\left(\frac{{x_0}^2}{\rho}\right) + \frac{a}{2}\frac{\partial^2}{\partial y^2}\left(\frac{{y_0}^2}{\rho}\right) + \frac{a}{2}\frac{\partial^2}{\partial z^2}\left(\frac{{z_0}^2\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) + \\ + a\frac{\partial^2}{\partial x \partial y}\left(\frac{x_0 y_0}{\rho}\right) + a\frac{\partial^2}{\partial x \partial z}\left(\frac{x_0 z_0\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) + a\frac{\partial^2}{\partial y \partial z}\left(\frac{y_0 z_0\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) \end{aligned} \tag{A.11}$$

A.4. Inner expansion of the far field

An inner expansion of the far field is based on the fact that for small distances from the origin, i.e. for $\rho \to O(a)$, the retardation is small:

$$\frac{\rho}{c} = \frac{\rho}{\lambda}\frac{\lambda}{c} \to \frac{a}{\lambda}T \quad for\ \rho \to O(a) \tag{A.12}$$

so that in the limit $\frac{a}{\lambda} \to 0$, the behaviour of the multipole fields can be determined by a Taylor expansion. As an example, the Z-dipole in eq. (A.11) may then, on account of $\dot{z}_0 = v$, be approximated by:

$$-a\frac{\partial}{\partial z}\left(\frac{z_0\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) = -a\frac{\partial}{\partial z}\left(\frac{z_0(t) \mp \frac{\rho}{c}v(t) + \frac{1}{2}\left(\frac{\rho}{c}\right)^2\dot{v}(t) \mp \frac{1}{6}\left(\frac{\rho}{c}\right)^3\ddot{v}(t) + \dots}{\rho}\right) =$$

$$= -a\frac{\partial}{\partial z}\left(\frac{z_0}{\rho}\right) - \frac{\dot{v}}{c^2}\frac{z}{2}\left(\frac{a}{\rho}\right) \pm \frac{a}{3}\frac{\ddot{v}}{c^3}z + O\left(\frac{v}{c}\right)^4 \tag{A.13}$$

It is seen that the first term in this inner expansion of the far field is the same quasi-static dipole as found in the near-field's outer expansion (A.10), whereas the second term of (A.13) corresponds with the first term between the square brackets of the near field's outer expansion. The third term in eq. (A.13) apparently represents an effect that must be absorbed by the common field component $\left[\frac{\Delta\phi}{\phi_{ref}}\right]_{\rho=O(\lambda)}$. Similarly, the other poles in the far field eq. (A.11) lead to the following parts of the inner expansion:

$$a\frac{\partial^2}{\partial x \partial z}\left(\frac{x_0 z_0\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) = a\frac{\partial^2}{\partial x \partial z}\left(\frac{x_0 z_0(t)}{\rho}\right) + a\frac{\dot{v}}{c^2}\frac{z}{2}\frac{\partial}{\partial x}\left(\frac{x_0}{\rho}\right) + O\left(\frac{v}{c}\right)^4 \tag{A.14}$$

$$a\frac{\partial^2}{\partial y \partial z}\left(\frac{y_0 z_0\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) = a\frac{\partial^2}{\partial y \partial z}\left(\frac{y_0 z_0(t)}{\rho}\right) + a\frac{\dot{v}}{c^2}\frac{z}{2}\frac{\partial}{\partial y}\left(\frac{y_0}{\rho}\right) + O\left(\frac{v}{c}\right)^4 \tag{A.15}$$

$$\frac{a}{2}\frac{\partial^2}{\partial z^2}\left(\frac{{z_0}^2\left(t \mp \frac{\rho}{c}\right)}{\rho}\right) =$$

$$= \frac{a}{2}\frac{\partial^2}{\partial z^2}\left(\frac{{z_0}^2(t)}{\rho}\right) + a\frac{\dot{v}}{c^2}\frac{z}{2}\frac{\partial}{\partial z}\left(\frac{z_0}{\rho}\right) + \frac{v^2}{c^2}\frac{z}{2}\frac{\partial}{\partial z}\left(\frac{a}{\rho}\right) + \frac{1}{2}\frac{v^2 + z_0\dot{v}}{c^2}\frac{a}{\rho} \mp a\frac{v\dot{v} + \frac{1}{3}z_0\ddot{v}}{c^3} \tag{A.16}$$

A.5. Application of the matching condition

As derived in sec. 3.3, matching requires

$$\left(outer\ expansion\ \ to\ O\left(\frac{a}{\lambda}\right)^3 of\ near\ field\ \right) \sim$$

$$\sim \left(inner\ expansion\ \ to\ O\left(\frac{v}{c}\right)^3 of\ far\ field\right) \quad for\ \frac{a}{\lambda},\ \frac{v}{c}\ \to 0 \tag{A.17}$$

It is thus found that

$$\left[\frac{\Delta\phi}{\phi_{ref}}\right]_{\rho=O(\lambda)} = \frac{1}{2}\frac{v^2 + z_0\dot{v}}{c^2}\frac{a}{\rho} \mp a\frac{v\dot{v} - \frac{1}{3}(z - z_0)\ddot{v}}{c^3} \tag{A.18}$$

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Forming the composite field, the near- and far fields are summed and the common part subtracted. A term such as $\frac{a}{3}\frac{\ddot{v}}{c^3}z$ in eq. (A.18) is thereby cancelled at large distances, so that the composite field satisfies the boundary condition

$$\phi_{composite} \to 0 \quad for \quad \rho \to \infty \tag{A.19}$$

A.6. Completion of the near-field of the scalar potential

The expression in eq. (A.18) is the outer expansion of the near-field component

$$\left[\frac{\Delta\phi}{\phi_{ref}}\right] = \frac{1}{2}\frac{v^2}{c^2}\frac{a}{r} + \frac{z_0}{2}\frac{\dot{v}}{c^2}\frac{a}{r} \mp a\frac{v\dot{v}}{c^3} \pm \frac{1}{3}a\,(z-z_0)\,\frac{\ddot{v}}{c^3} - \frac{z_0}{2}\frac{v^2}{c^2}\frac{\partial}{\partial z}\left(\frac{a}{r}\right) \tag{A.20}$$

where the last term, satisfying a Laplace equation and with an outer expansion of the negligible order $O\left(\frac{a}{\lambda}\right)^4$, has been added to enhance the symmetry of the finally resulting expression for the complete near field:

$$\frac{(\phi_{near})}{\phi_{ref}} = \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\frac{a}{r} - \frac{z-z_0}{2}\left[\frac{\dot{v}}{c^2} - \frac{v^2}{c^2}\frac{\partial}{\partial z}\right]\left(\frac{a}{r}\right) \mp a\frac{v\dot{v}}{c^3} \pm \frac{1}{3}a\,(z-z_0)\,\frac{\ddot{v}}{c^3} \tag{A.21}$$

Appendix B: The (magnetic) vector potential of a moving point charge

B.1. Near field

The component $a_z(x,y,z,t)$ in the Z-direction has to satisfy the Poisson equation

$$\nabla^2 a_{z\,near} = \frac{1}{c^2}\frac{\partial^2 (a_{z\,near})_{qu.static}}{\partial t^2} \tag{B.1}$$

where the quasi-static potential is given by

$$\frac{c.\,(a_{z_{near}})_{qu.static}}{\phi_{ref}} = \frac{v}{c}\frac{a}{r} \tag{B.2}$$

with ϕ_{ref} defined by eq. (15). Using relation (A.1) between the time- and Z-derivative, as well as

$$\frac{\partial r}{\partial t} = -v\frac{\partial r}{\partial z} \tag{B.3}$$

the Poisson equation (B.1) reads

$$\frac{c.\,\nabla^2 a_{z\,near}}{\phi_{ref}} = \left[\frac{\ddot{v}}{c^3} - 3\frac{v\dot{v}}{c^3}\frac{\partial}{\partial z} + \frac{v^3}{c^3}\frac{\partial^2}{\partial z^2}\right]\left(\frac{a}{r}\right) \tag{B.4}$$

In order to determine a particular solution, use can be made again of the general rule (A.3), except in the case of the first term on the right-hand side, which is not a Z-derivative. The complete near-field solution is:

$$\frac{c.\,a_{z\,near}}{\phi_{ref}} = \left[\frac{v}{c} + \frac{\ddot{v}}{c^3}\frac{r^2}{2} - 3\frac{v\dot{v}}{c^3}\frac{z}{2} + \frac{v^3}{c^3}\frac{z}{2}\frac{\partial}{\partial z}\right]\left(\frac{a}{r}\right) + \frac{c.\,\Delta a_{z\,near}}{\phi_{ref}} \tag{B.5}$$

B.2. Outer expansion of near field

To determine the outer expansion of the near field of a_z up to and including $O\left(\frac{a}{\lambda}\right)^3$, eq. (A.9) may be used for the expansion of $\frac{a}{r}$, as well as eq. (A.7) to find the expansion of $\frac{r}{a}$:

$$\frac{r}{a}=\frac{\rho}{a}[\ldots]^{\frac{1}{2}}=\frac{\rho}{a}\left[g(o)+\frac{a}{\rho}g^{'}(0)+\ldots\right] \tag{B.6}$$

where, as earlier, the symbol $[\ldots]$ stands for the expression between the square brackets in eq. (A.7), and the function g is defined as $g\left(\frac{a}{\rho}\right)=[\ldots]^{\frac{1}{2}}$. The second term in (B.5) is thus expanded as

$$\left[\frac{\ddot{v}}{c^3}\frac{r^2}{2}\left(\frac{a}{r}\right)\right]_{r=O(\lambda)}=\frac{\ddot{v}}{c^3}\frac{a^2}{2}\left[\frac{\rho}{a}-\frac{xx_0+yy_0+zz_0}{a\rho}\right]+O\Big(\frac{a}{\lambda}\Big)^4 \qquad \frac{a}{\lambda}\to 0 \tag{B.7}$$

and the complete outer expansion of $a_{z\,near}$ is given by

$$\left[\frac{c.\,a_{z\,near}}{\phi_{ref}}\right]_{r=O(\lambda)}=$$

$$=\frac{v}{c}\left[\frac{a}{\rho}-a\frac{\partial}{\partial x}\left(\frac{x_0}{\rho}\right)-a\frac{\partial}{\partial y}\left(\frac{y_0}{\rho}\right)-a\frac{\partial}{\partial z}\left(\frac{z_0}{\rho}\right)\right]+\frac{\ddot{v}}{c^3}\frac{a^2}{2}\left[\frac{\rho}{a}-\frac{xx_0+yy_0+zz_0}{a\rho}\right]+$$

$$-\frac{3}{2}a\frac{v\dot{v}}{c^3}\frac{z}{\rho}+\left[\frac{c.\,\Delta a_{z\,near}}{\phi_{ref}}\right]_{r=O(\lambda)}+O\Big(\frac{a}{\lambda}\Big)^4 \qquad \frac{a}{\lambda}\to 0 \tag{B.8}$$

B.3. Far field

From eq. (B.8) it is concluded that the singularities of the far field include a monopole and three dipoles, all of them subject to retardation:

$$\frac{c.\,a_{z\,far}}{\phi_{ref}}=\frac{v\left(t\mp\frac{\rho}{c}\right)}{c}\frac{a}{\rho}-a\frac{\partial}{\partial x}\left(\frac{v\left(t\mp\frac{\rho}{c}\right)}{c}\frac{x_0}{\rho}\right)-a\frac{\partial}{\partial y}\left(\frac{v\left(t\mp\frac{\rho}{c}\right)}{c}\frac{y_0}{\rho}\right)+$$

$$-a\frac{\partial}{\partial z}\left(\frac{v(t\mp\frac{\rho}{c})}{c}\frac{z_0(t\mp\frac{\rho}{c})}{\rho}\right) \tag{B.9}$$

The upper sign in the symbol $\mp$ is associated with outgoing waves, the lower one with incoming waves.

B.4. Inner expansion of far field

The inner expansion of the far field reads:

$$\left[\frac{c.\,a_{z\,far}}{\phi_{ref}}\right]_{\rho=O(a)}=\frac{v(t)}{c}\frac{a}{\rho}\mp\frac{\dot{v}}{c^2}a+\frac{a}{2}\frac{\ddot{v}}{c^3}\rho+$$

$$-a\frac{v(t)}{c}\frac{\partial}{\partial x}\left(\frac{x_0}{\rho}\right)-\frac{a}{2}\frac{\ddot{v}}{c^3}\frac{xx_0}{\rho}-a\frac{v(t)}{c}\frac{\partial}{\partial y}\left(\frac{y_0}{\rho}\right)-\frac{a}{2}\frac{\ddot{v}}{c^3}\frac{yy_0}{\rho}+$$

$$-a\frac{v(t)}{c}\frac{\partial}{\partial z}\left(\frac{z_0}{\rho}\right)-\frac{a}{2}\frac{\ddot{v}}{c^3}\frac{zz_0}{\rho}-\frac{3}{2}a\frac{v\dot{v}}{c^3}\frac{z}{\rho}+O\Big(\frac{a}{\lambda}\Big)^4 \qquad \frac{a}{\lambda}\to 0 \tag{B.10}$$

where the asymptotic order of the terms is solely determined by the coefficients $\frac{v}{c=O\left(\frac{a}{\lambda}\right)^1},\ \frac{\dot{v}}{c^2=O\left(\frac{a}{\lambda}\right)^2,\ \frac{\ddot{v}}{c^3}}=O\left(\frac{a}{\lambda}\right)^3$.

B.5. Matching

Comparing eqs. (B.8) and (B.10) it is seen that the near field has to be supplemented by

$$\frac{c.\,\Delta a_{z\,near}}{\phi_{ref}}=\mp\frac{\dot{v}}{c^2}a \tag{B.11}$$

so that the complete near field of the vector potential is given by combining (B.5) and (B.11).

B.6. Complete near field (vector potential)

In the expression below, a few terms have been added to enhance the symmetry. These additional terms satisfy the Laplace equation and add to the outer expansion only terms of an irrelevant order $O\left(\frac{a}{\lambda}\right)^4$:

$$\frac{c.\,a_{z\,near}}{\phi_{ref}} = \left[\frac{v}{c} + \frac{\ddot{v}}{c^3}\frac{r^2}{2} - 3\frac{v\dot{v}}{c^3}\frac{z-z_0}{2} + \frac{v^3}{c^3}\frac{z-z_0}{2}\frac{\partial}{\partial z}\right]\left(\frac{a}{r}\right) \mp \frac{\dot{v}}{c^2}a \tag{B.12}$$

Notations

- $\underline{A}$ (magnetic) vector potential
- a dimension of potential well
- a_z Z-component of vector potential of charge element
- $\underline{B}$ magnetic field strength
- c velocity of light
- $\underline{F}$ electromagnetic force on charge Q
- f_z Z-component of force due to charge element
- $\underline{j}$ current density
- q charge of element
- Q total charge
- r distance between field point and charge element
- s length of droplet
- T typical cycle time
- t time
- $\underline{V}$ velocity vector
- v velocity of charge element
- x, y, z Cartesian coordinates of field point
- x_0, y_0, z_0 Cartesian coordinates of charge element
- z_m Z-coordinate of midpoint
- $\underline{E}$ electric field strength
- ε_0 vacuum permittivity
- θ angle defined in eq. (26)
- λ typical wavelength in far field
- $\rho,\ \varphi,\ \chi$ polar coordinates
- σ charge density
- Φ (electric) scalar potential

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- ϕ scalar potential of charge element

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