Zero-Divisor Graphs of $\mathbb{Z}_n$, their products and $D_n$

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Abstract. This paper is an endeavor to discuss some properties of zero-divisor graphs of the ring \( \mathbb{Z}_n \), the ring of integers modulo \( n \). The zero divisor graph of a commutative ring \( R \), is an undirected graph whose vertices are the nonzero zero-divisors of \( R \), where two distinct vertices are adjacent if their product is zero. The zero-divisor graph of \( R \) is denoted by \( \Gamma(R) \). We discussed \( \Gamma(\mathbb{Z}_n) \)'s by the attributes of completeness, \( k \)-partite structure, complete \( k \)-partite structure, regularity, chordality, \( \gamma-\beta \) perfectness, simplicial vertices. The clique number for arbitrary \( \Gamma(\mathbb{Z}_n) \) was also found. This work also explores related attributes of finite products \( \Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \), seeking to extend certain results to the product rings. We find all \( \Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \) that are perfect. Likewise, a lower bound of clique number of \( \Gamma(\mathbb{Z}_m \times \mathbb{Z}_n) \) was found. Later, in this paper, we discuss some properties of the zero divisor graph of the poset \( D_n \), the set of positive divisors of a positive integer \( n \) partially ordered by divisibility.

1. Introduction

Zero-divisor graphs were first discussed by Beck [1] as a way to color commutative rings. They were further discussed by Livingston and Anderson in [4] and [5]. A zero-divisor graph of a ring \( R \), denoted by \( \Gamma(R) \), is a graph whose vertices are all the zero-divisors of \( R \). Two distinct vertices \( u \) and \( v \) are adjacent if \( uv = 0 \). Beck [1] considered every element of \( R \) a vertex, with 0 sharing an edge with all other vertices. Since then, others have chosen to omit 0 from zero-divisor graphs [2, 3, 4, 5]. For our purposes, we omit 0 so that the vertex set of \( \Gamma(\mathbb{Z}_n) \) denoted by \( ZD(\mathbb{Z}_n) \) will only be the non-zero zero-divisors.

In the first section, we explore a concept explored by Smith [3] called type graphs. In [3], type graphs were used to find all perfect \( \Gamma(\mathbb{Z}_n) \). We extended the notion of type graphs for \( \Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \) to find all perfect zero-divisor graphs of such products, where \( n_1, n_2, \ldots, n_k \)

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are positive integers and $\Gamma(Z_{n_1} \times \cdots \times Z_{n_k})$ is the direct product of $Z_{n_i}', 1 \leq i \leq k$. We then move on to various properties of $\Gamma(Z_n)$ and $\Gamma(Z_{n_1} \times \cdots \times Z_{n_k})$. In the last section, we explore zero divisor graphs of the poset $D_n$, the set of positive divisors of a positive integer $n$ partially ordered by divisibility and we catalog them in a similar way. Zero divisor graph of poset is studied in [8], [9], [10].

2. Type Graphs

When we consider zero-divisor graphs of $\Gamma(Z_n)$, it is useful to consider the type graphs of these rings. A type graph has vertices of $T_a$ where $a$ is a factor of $n$ that is neither 1 nor 0. The set of all such $T_a$ forms a partition of the vertex set of $\Gamma(Z_n)$ where $T_a = \{x \in ZD(Z_n) | gcf(x, n) = a\}$. This concept was shown by Smith [3], where the type graph was used to find all perfect $\Gamma(Z_n)$. Smith used the notation $\Gamma^T(Z_n)$ to denote the type graph. In that paper, four key observations were shown to be true regarding the type graphs on $Z_n$. In this section, we modify the definition of type graph to fit the zero divisor graph of the finite direct product $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$, $n_1, n_2, \ldots, n_k$ being $k$ many positive integers. Additionally, we show these observations to be true over this type graph as well. We then use analogues of some theorems from [3] to characterize the perfectness of $\Gamma(Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k})$.

The following are two important theorems from [3].

**Theorem 2.1** (Smith’s Main Theorem). [3] A graph $\Gamma(Z_n)$ is perfect iff $n$ is of one of the following forms:

1. $n = p^a$ for prime $p$ and positive integer $a$.
2. $n = p^aq^b$ for distinct primes $p,q$ and positive integers $a,b$.
3. $n = p^aq^br$ for distinct primes $p,q,r$ and positive integer $a$.
4. $n = pqrs$ for distinct primes $p,q,r,s$.

**Theorem 2.2** (Smith’s Theorem 4.1). [3] $\Gamma(Z_n)$ is perfect iff its type graph $\Gamma^T(Z_n)$ is perfect.

**Definition 2.3** (Type graph of $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$). The type graph of $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$ denoted by $\Gamma^T(Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k})$ has a vertex set of the type classes $T(x_1, x_2, \ldots, x_k)$ where $(x_1, x_2, \ldots, x_k) \neq (0, 0, \ldots, 0)$ nor $(1, 1, \cdots, 1)$, and $x_i$ is a divisor of $n_i$, 1, or 0. $T(x_1, x_2, \cdots, x_k) = \{(a_1, a_2, \cdots, a_k) | a_i \in Z_{n_i}/0$ and $gcf(a_i, n_i) = x_i$ or $a_i = 0$ if $x_i = 0 \}$. Arbitrary $T(x_1, x_2, \cdots, x_k)$ shares an edge with arbitrary $T(y_1, y_2, \cdots, y_k)$ iff $x_iy_i = 0$ for all $i$. 
Smith [3] gave the following four observations for the type graph of \( \Gamma(\mathbb{Z}_n) \).

**Theorem 2.4.** Each vertex of \( \Gamma(\mathbb{Z}_n) \) is in exactly one type class.

**Theorem 2.5.** Arbitrary distinct vertices \( T_x \) and \( T_y \) share an edge in \( \Gamma^T(\mathbb{Z}_n) \) iff each \( a \in T_x \) shares an edge with each \( b \in T_y \) in \( \Gamma(\mathbb{Z}_n) \).

**Theorem 2.6.** Arbitrary distinct vertices \( T_x \) and \( T_y \) don’t share an edge in \( \Gamma^T(\mathbb{Z}_n) \) iff each \( a \in T_x \) doesn’t share an edge with each \( b \in T_y \) in \( \Gamma(\mathbb{Z}_n) \).

**Theorem 2.7.** In \( \Gamma(\mathbb{Z}_n) \) consider arbitrary \( a \) and \( b \) in the same type class. An arbitrary vertex \( c \) in \( \Gamma(\mathbb{Z}_n) \) shares an edge with \( b \) iff it shares an edge with \( a \) also.

Following are the four analogues to the above results for \( \Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \).

**Theorem 2.8.** Each vertex of \( \Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \) is in exactly one type class.

**Theorem 2.9.** Arbitrary distinct vertices \( T_x = T(x_1, x_2, \cdots, x_k) \) and \( T_y = T(y_1, y_2, \cdots, y_k) \) share an edge in \( \Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \) iff each \( a \in T_x \) shares an edge with each \( b \in T_y \) in \( \Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \).

**Theorem 2.10.** Arbitrary distinct vertices \( T_x = T(x_1, x_2, \cdots, x_k) \) and \( T_y = T(y_1, y_2, \cdots, y_k) \) don’t share an edge in \( \Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \) iff each \( a \in T_x \) doesn’t share an edge with each \( b \in T_y \) in \( \Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \).

**Theorem 2.11.** In \( \Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \) consider arbitrary \( a = (a_1, a_2, \cdots, a_k) \) and \( b = (b_1, b_2, \cdots, b_k) \) in the same type class \( T(t_1, t_2, \cdots, t_k) \). An arbitrary vertex \( c = (c_1, c_2, \cdots, c_k) \) shares an edge with \( b \) iff it shares an edge with \( a \) also.

**Proof.** Follows from Theorem 2.5 and 2.6. \( \Box \)

Next, we have the following theorem:

**Theorem 2.12.** \( \Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \) is perfect iff its type graph \( \Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}) \) is perfect.

To show this, we will use the following three theorems, whose proofs are analogous to the corresponding proofs in [3].
Theorem 2.13. Given arbitrary hole or antihole $H$ of length greater than 4 in $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$, every vertex in $H$ belongs to a different type class.

Theorem 2.14. Let there be a hole or antihole $H$ length $l > 4$ in $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$. Then the type graph $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ must also contain a hole or antihole length $l$.

Theorem 2.15. Let there be a hole or antihole $H$ length $l > 4$ in the type class $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$. Then the graph $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ must also contain a hole or antihole length $l$.

Using these theorems, now we can establish the following proof of Theorem 2.12.

Proof. The proof is analogous to the proof in [3].

Now that we know perfectness in the type graph implies perfectness in the zero-divisor graph, it is possible to find all such perfect $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$. As it turns out, for both $\Gamma^T(\mathbb{Z}_n)$ and $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$, we can exchange the primes of each $n_i$, and as long as the form of the primes (the amount of distinct primes and the power of each prime) stays the same, the type graph will be isomorphic. To illustrate this, consider $\Gamma^T(\mathbb{Z}_p \times \mathbb{Z}_q)$ where $p, q$ are prime. This type graph is isomorphic to $\Gamma^T(\mathbb{Z}_{pq})$ where $r, s$ are prime, even if the value of the primes are different. We will use this to find all perfect $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$.

Theorem 2.16. Consider some $\Gamma^T(\mathbb{Z}_n)$ and $\Gamma^T(\mathbb{Z}_m)$ such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$. Then $\Gamma^T(\mathbb{Z}_n) \cong \Gamma^T(\mathbb{Z}_m)$.

Proof. Consider arbitrary vertex $u$ in $\Gamma^T(\mathbb{Z}_n)$. $u$ is a factor of $n$, so we can write $u = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$. Note that $0 \leq x_i \leq \alpha_i$, $\forall i$. Define a function $f : \Gamma^T(\mathbb{Z}_n) \rightarrow \Gamma^T(\mathbb{Z}_m)$ as $f(u) = f(p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}) = q_1^{x_1} q_2^{x_2} \cdots q_k^{x_k}$. Since $n$ and $m$ both have the same amount of prime factors, and each corresponding prime has the same power $\alpha_i$, the result follows.

Theorem 2.17. Consider $\Gamma^T(\mathbb{Z}_n \times \cdots \times \mathbb{Z}_k)$ and $\Gamma^T(\mathbb{Z}_m \times \cdots \times \mathbb{Z}_k)$ where the prime factorization of $n_i$ has the same form as $m_i$ for each $i$. That is, $n_i$ and $m_i$ have the same amount of prime factors and the same power for each prime. Then $\Gamma^T(\mathbb{Z}_n \times \cdots \times \mathbb{Z}_k) \cong \Gamma^T(\mathbb{Z}_m \times \cdots \times \mathbb{Z}_k)$.

Proof. Take arbitrary $n_i$.

Denote the prime factorization of $n_i = p_{i,1}^{\alpha_{i,1}} \cdots p_{i,j_i}^{\alpha_{i,j_i}}$ where $j_i$ is the
amount of prime of $n_i$. Likewise, $m_i = q_i^{a_{i1}} \cdots q_i^{a_{i\ell_i}}$. Note that the only difference between these factorizations is the values of the primes that are used. The powers and the number of distinct primes in the respective factorizations are the same. Consider arbitrary $(u_1, \cdots, u_k) \in \Gamma^T(Z_{n_1} \times \cdots \times Z_{n_k})$. Each $u_i$ is a factor of $n_i$ or 0. We can write $u_i = p_i^{x_{i1}} \cdots p_i^{x_{i\ell_i}}$ where $0 \leq x_{i\ell_i} \leq \alpha_{i\ell_i}$. Note that if $u_i$ is 1, each $x_{i\ell_i}$ is 0 and if $u_i$ is 0, $x_{i\ell_i} = \alpha_{i\ell_i}$ for every $l$.

Defining a function $f : \Gamma^T(Z_{n_1} \times \cdots \times Z_{n_k}) \to \Gamma^T(Z_{m_1} \times \cdots \times Z_{m_k})$ in a natural way component-wise, by using the bijective function in the proof of the last theorem we get the desired bijection. □

Theorem 2.18. $\Gamma^T(Z_{n_1} \times \cdots \times Z_{n_k})$ is isomorphic to $\Gamma^T(Z_{n_1 \cdots n_k})$ if all $n_i$'s are mutually co-prime.

Proof. The proof follows by Chinese Remainder theorem. □

The next theorem will show how we can characterize the perfectness of $\Gamma(Z_{n_1} \times \cdots \times Z_{n_k})$. Because now by the above three theorems, without loss of generality, we can simply choose primes that will make the $n_i$'s mutually co-prime. Then we know the type graph will be isomorphic to $\Gamma(Z_n)$ where $n$ is the product of all such co-prime $n_i$. So, we have the following theorem.

Theorem 2.19. $\Gamma(Z_{n_1} \times Z_{n_2} \cdots Z_{n_k})$ is perfect iff it is possible to find mutually coprime positive integers $m_1, m_2, \cdots m_k$, so that each $m_i$ has same amount of prime factors with same exponent in its prime factorization as that in $n_i$ and $\Gamma(Z_{m_1 \cdots m_k})$ is perfect.

Example 2.20. For example, $\Gamma(Z_{p^2q} \times Z_p)$ is perfect because $\Gamma(Z_{u^2bc})$ is perfect as shown by [3]. Also note, no product with a dimension greater than four can be perfect. $\Gamma(Z_{p_1} \times \cdots \times Z_{p_5})$ is not perfect since no $\Gamma(Z_{p_1 \cdots p_5})$ is perfect as shown by [3].

3. Some properties of $\Gamma(Z_n)$

In this section, we characterize $\Gamma(Z_n)$ by various qualities such as completeness, cordiality and clique number. A helpful construction used is the strong type graph. We define the strong type graph as the type graph with self-loops. We normally do not consider self-loops, in zero-divisor graphs and type graphs, but in the strong type graph, a vertex has a loop at it if it annihilates itself. We denote the strong type graph of $\Gamma(Z_n)$ as $\Gamma^S(Z_n)$.

Another construction used commonly in this section is $n^*$. Consider some $\Gamma(Z_n)$. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, $n^* = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ where $\beta_i = \lceil \frac{\alpha_i}{2} \rceil$. 
Lemma 3.1. Two arbitrary vertices $u$ and $v$ in $\Gamma(\mathbb{Z}_n)$ that are both in the same type class $T_i$ share an edge iff $T_i$ has a self-loop in the strong type graph.

Proof. Let $T_i$ have a self-loop. Then $i^2 = 0$. Since every $u, v \in T_i$ are multiples of $i$, $u$ and $v$ will share an edge.

Conversely, let $T_i$ does not have a self-loop. Take arbitrary $u$ and $v$ in $T_i$. According to the definition of type class, $u$ and $v$ are some multiple of $i$ where $\text{gcf}(u,n) = i$ and likewise for $v$. We can write $u = ai$ and $v = bi$ where $\text{gcf}(a,n/i) = 1$ and $\text{gcf}(b,n/i) = 1$. Assume $u$ and $v$ share an edge. Then $uv = cn$, $ab_i^2 = cn$ where $c$ is a natural number. So $\frac{ab_i^2}{n} = c$. Since $T_i$ does not have a self-loop, $i^2 \neq 0$ which means $n$ has a factor that is not a factor of $i^2$. Let this factor be called $d$. Let $\frac{i^2}{d}$ represent the simplified form of the fraction $\frac{i^2}{n}$ where $d$ is not 1. By substitution, $\frac{ab_i^2}{d} = c$. This is a contradiction since $a$, $b$ and $g$ do not share a factor with $n/i$, so cannot cancel the $d$ out of the denominator. Therefore, the expression cannot be equal to $c$, a natural number. So, $u$ and $v$ do not share an edge. \hfill \Box

As a result, we have

Theorem 3.2. $\Gamma(\mathbb{Z}_{p^2})$ is complete where $p$ is prime.

Theorem 3.3. $\Gamma(\mathbb{Z}_{p^x})$ where $p$ is prime and $x \geq 3$ is not complete.

Proof. Let $x \geq 3$.

Case 1: $p = 2$: $p$ and $3p$ are distinct non-zero zero-divisors that are not connected.

Case 2: $p \neq 2$: $p$ and $2p$ are distinct non-zero zero-divisors that are not connected.

\hfill \Box

Theorem 3.4. $\Gamma(\mathbb{Z}_n)$, where $n \geq 2$ is complete iff $n = p^2$.

Proof. Let $\Gamma(\mathbb{Z}_n)$ be complete. Assume $n$ has two or more distinct prime factors. Label the smallest such factor by $p$. Now choose another one as $q$. $p$ is a zero divisor and shares an edge with $n/p$. Since $p$ and $q$ are both prime factors of $n$, $pq \leq n$. Also, since $p < q$, $p^2 < pq$. So $p^2 < pq \leq n$ which means $p^2$ is non-zero and distinct from $p$. $p^2$ shares an edge with $n/p$ so $p^2$ is a distinct zero-divisor that does not share an edge with $p$, making $\Gamma(\mathbb{Z}_n)$ not complete. The converse follows by the above two Theorems. \hfill \Box

Theorem 3.5. $\Gamma(\mathbb{Z}_n)$ is $k$-partite if $\Gamma^S(\mathbb{Z}_n)$ is $k$-partite.
Proof. Let $\Gamma^S(Z_n)$ be $k$-partite. Then $\Gamma^S(Z_n)$ can be partitioned into $k$ disjoint subsets $S_1, S_2, \ldots, S_k$ such that no vertex in the same set share an edge. Partition $\Gamma(Z_n)$ into a similar grouping $Q_1, Q_2, \ldots, Q_k$ where $u \in Q_i$ iff $u \in T_u \in S_i$. Consider arbitrary $u$ and $v$, vertices of $\Gamma(Z_n)$ that are in the same partitioned set $Q_i$.

Case 1: $u$ and $v$ are in different type classes. 
Call such classes $T_u$ and $T_v$. Then since $u$ and $v$ are both in $Q_i$, $T_u$ and $T_v$ are both in $S_i$ which means $T_u$ does not share an edge with $T_v$. So, by [3] $u$ and $v$ do not share an edge.

Case 2: $u$ and $v$ are in the same type class. 
Call this class $T_u$. Then since $\Gamma^S(Z_n)$ is $k$-partite, $T_u$ does not form a loop with itself. Hence, by Lemma 3.1, $u$ and $v$ do not share an edge.

$\Box$

Theorem 3.6. $\Gamma(Z_n)$ is complete k-partite if $\Gamma^S(Z_n)$ is complete k-partite.

Proof. Let $\Gamma^S(Z_n)$ be complete k-partite. Then by the above theorem $\Gamma(Z_n)$ is k-partite. Using the partition used in the above Theorem, if we let $\Gamma^S(Z_n)$ be partitioned into $k$ disjoint subsets $S_1, S_2, \ldots, S_k$, then $\Gamma(Z_n)$ can be partitioned into $k$ disjoint subsets $Q_1, Q_2, \ldots, Q_k$, where arbitrary vertex of $\Gamma(Z_n)$ is in $Q_i$ if its type class is in $S_i$. Consider arbitrary vertices in $\Gamma(Z_n)$, $u$ and $v$, that are not in the same $Q_i$. Then $u$ and $v$ must be in different type classes in two different $S_i$'s. Call these classes $T_u$ and $T_v$. Since $\Gamma^S(Z_n)$ is complete k-partite, $T_u$ and $T_v$ share an edge. Then $u$ and $v$ share an edge by [3]. $\Box$

Remark 3.7. The converse of Theorem 3.5 and 3.6 is not always true. If the zero-divisor graph is k-partite, but has a self-annihilating vertex, the strong type graph will have a self-loop, which prevents it from being k-partite. For example, $\Gamma(Z_9)$ is complete bi-partite, whereas $\Gamma^S(Z_9)$ is not.

Theorem 3.8. If $n$ is square free, $\Gamma(Z_n)$ is k-partite, where $k$ is the number of distinct prime factors of $n$.

Proof. Consider the strong type graph $\Gamma^S(Z_n)$. Let, $n = p_1 p_2 \cdots p_k$. Partition the graph into $k$ sets $S_1, S_2, \ldots, S_k$. A vertex $T_a$ in the strong type graph is in $S_i$ if $gcd(a, p_i) = 1$ and $gcd(a, p_h) > 1$ for all $h < i$.
We now claim that $S_1, S_2, \ldots, S_k$ covers all the vertices of $\Gamma^S(Z_n)$.
Assume there is a $T_a$ that is not in any $S_i$. Since $T_a$ is a vertex, $a$ must
be a factor of \( n \) that is also less than \( n \). So \( a \) must omit at least one \( p_i \). So \( \gcd(a, p_i) = 1 \). Since \( T_a \) is not in any \( S_i \), there must exist some \( h < i \) such that \( \gcd(a, p_h) = 1 \). Choose the smallest index \( h \) of such \( p_h \). Then \( T_a \) must be in \( S_h \) which is a contradiction.

Our next claim is any two vertices \( u \) and \( v \) in the same partition do not share an edge.

Consider arbitrary \( u \) and \( v \) in \( S_i \). Both \( u \) and \( v \) do not contain \( p_i \) so they do not share an edge. So the strong type graph is \( k \)-partite.

By Theorem 3.5, \( \Gamma(Z_{p_1 p_2 \cdots p_k}) \) is \( k \)-partite. □

**Lemma 3.9.** Arbitrary type class \( T_a \) in \( \Gamma^T(Z_n) \) contains only one element iff \( a = \frac{n}{2} \).

**Proof.** Let \( T_a \in \Gamma(Z_n) \) have a type class that has only one element. Assume \( a \neq \frac{n}{2} \). Since \( a \) is a factor of \( n \), \( \frac{n}{a} = f \) is also a factor of \( n \). Note that \( f \geq 3 \).

Consider the vertex \( a(f-1) \) of \( \Gamma(Z_n) \). The quantity \( (f-1) \) does not share any factors with \( f \). Since \( af = n \), \( \gcd(a(f-1), n) = a \). So \( a(f-1) \in T_a \). Also note that \( a < a(f-1) < n \). So \( a(f-1) \) is a distinct vertex in \( T_a \) which is a contradiction. So \( a = \frac{n}{2} \)

Let \( a = \frac{n}{2} \). Then \( a \) is the only element in \( T_a \) since \( 2a = n \). □

**Corollary 3.10.** Analogous to above, \( T_{n/p} \) in \( \Gamma^T(Z_n) \) contains exactly \( p-1 \) elements if \( p \) is the smallest prime factor of \( n \).

**Lemma 3.11.** There is at most one type class with only one element.

**Proof.** Assume there are two or more distinct type classes that have only one element. Call two of these classes \( T_u \) and \( T_v \). By Lemma 3.9, \( u = v = \frac{n}{2} \) which is a contradiction. □

**Theorem 3.12.** \( \Gamma(Z_n) \) is \( k \)-partite if \( \Gamma^S(Z_n) \) is \( k \)-partite or \( \Gamma^T(Z_n) \) is \( k \)-partite and the only self-connected vertex of \( \Gamma(Z_n) \) is \( T_{\frac{n}{2}} \).

**Proof.** Let \( \Gamma^S(Z_n) \) be \( k \)-partite. By Theorem 3.5, \( \Gamma(Z_n) \) is \( k \)-partite. Let \( \Gamma^T(Z_n) \) be \( k \)-partite and let \( \Gamma^S(Z_n) \) have only one self-connected vertex, \( T_{\frac{n}{2}} \). Consider arbitrary distinct \( u \) and \( v \), zero divisors of \( \Gamma(Z_n) \), that are in the same partition.

Case 1: \( u \) and \( v \) are in the same type class.

By Lemma 3.9, \( T_{\frac{n}{2}} \) has only one element, so if \( u \) and \( v \) are distinct, they cannot be in \( T_{\frac{n}{2}} \). Then, the type class they are in is not self-connected, so \( u \) and \( v \) do not share an edge.
Case 2: $u$ and $v$ are in different type classes.

Since $u$ and $v$ are in the same partition, their type classes are in the same partition and do not share an edge. Thus, $u$ and $v$ do not share an edge.

Lemma 3.13. A vertex in $\Gamma(\mathbb{Z}_n)$ annihilates itself iff it is a multiple of $n^*$.  

Lemma 3.14. Consider two arbitrary vertices in $\Gamma(\mathbb{Z}_n)$, $u$ and $v$ such that $u$ is a factor of $v$. The largest clique containing $v$, $M_v$ has a magnitude greater than or equal to the $M_u$, the largest clique containing $u$.  

Proof. Take arbitrary vertices $u$ and $v$ in $\Gamma(\mathbb{Z}_n)$. Let $u$ be a factor of $v$. Every element $e$ in $M_u \setminus u$ has the property $eu = 0$. Then $\forall e \in M_u, ev = 0$. So a clique $C$ exists with $v$ and each $e$ in $M_u \setminus u$. So, $C$ is a clique containing $v$ magnitude of at least $M_u$.  

Theorem 3.15. $cl(\Gamma(\mathbb{Z}_n)) \geq \frac{n^*}{n} + k - 1$ where $k$ is the number of distinct primes having odd power in the prime factorization of $n$.  

Proof. The multiples of $n^*$ form a clique. Call it $C$. An arbitrary vertex of $C$ will be of the form $an^*$ for $1 < a < \frac{n}{n^*}$. The number of elements in this clique is $\frac{n}{n^*} - 1$, so the clique number of the graph is at least $\frac{n}{n^*} - 1$. Now consider all vertices of the form $n^*/q$ where $q$ is an arbitrary odd-power prime in the prime factorization of $n$. Arbitrary $n^*/q$ shares an edge with each $an^*$ in $C$. Also, each $n^*/q_1$ shares an edge to each other $n^*/q_2$. Since $k$ is the number of distinct odd powered primes in the prime factorization of $n$, $cl(\Gamma(\mathbb{Z}_n)) \geq \frac{n}{n^*} + k - 1$.  

Theorem 3.16. $cl(\Gamma(\mathbb{Z}_n)) \leq \frac{n^*}{n} + k - 1$ where $k$ is the number of odd-power primes in the prime factorization of $n$.  

Proof. Consider arbitrary clique $C$. Partition $C$ into sets $L$ and $N$ where $L$ is the set of vertices of $C$ that are not multiples of $n^*$ and $N$ is the set of vertices of $C$ that are multiples of $n^*$. Consider arbitrary vertex $l_1$ in $L$. Since $l_1$ is not a multiple of $n^*$, there must be some prime factor $p_1$ of $n$ whose power in $l_1$ is less than half of its power in $n$. Every other $l_i$ in $L$ must have its $p_1$ factor with a power greater than or equal to half its power in $n$ for it to share an edge with $l_1$. Consider another vertex $l_2$ in $L$. $l_2$ must also have a prime factor whose power is less than half its power in $n$, but it cannot be $p_1$. Call it $p_2$. So each $l_i$ in $L$ must have a distinct prime factor $p_i$ that has a power less than
or equal to half its power in $n$. Let $m$ be the number of distinct prime factors of $n$. Then there can be a maximum of $m$ many $l_i$ in $L$. $N$ has a maximum size of $\frac{n}{n^*} - 1$, so the clique number is at most $\frac{n}{n^*} + m - 1$. Consider some $e_1$, a vertex in $L$ whose corresponding $p_1$, has an even power in $n$. $e_1$ does not share an edge with $n^*$. This means the clique number is one less if $n$ has an even-powered prime. Consider another $e_2$ that has an even power $p_2$ whose power is less than half. Then $e_2$ does not share an edge with distinct vertices $p_1p_2\cdots p_{i-1}n^*$. So the size of $C$ is reduced by the number of even powered primes of $n$. This value can be represented by $m - k$ where $k$ is the number of odd-powered primes of $n$. Hence, since $C$ is arbitrary, $cl(\Gamma(\mathbb{Z}_n)) \leq \frac{n}{n^*} + k - 1$. □

Theorem 3.17. $cl(\Gamma(\mathbb{Z}_n)) = \frac{n}{n^*} + k - 1$.

Proof. The proof follows by Theorem 3.15 and Theorem 3.16. □

Theorem 3.18. There are no non-empty, non-complete, regular $\Gamma(\mathbb{Z}_n)$.

Proof. Consider all $\Gamma(\mathbb{Z}_n)$ that are non-empty and not complete. Assume $\exists$ some regular graph among these graphs.

Case 1: $n = p^x$ where $p$ is prime
If $x = 1$, the graph is empty, and if $x = 2$, the graph is complete, so $x \geq 3$. Then $p$ is a vertex that shares an edge with $p - 1$ many other vertices, and $p^2$ is a vertex that shares an edge with $p^2 - 1$ many other vertices. Since the graph is regular, $p - 1 = p^2 - 1$, thus $p = p^2$, which means $p = 1$ i.e., a contradiction.

Case 2: $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, $m \geq 2$ and $p_i$ are all prime
Vertex $p_1$ shares an edge with $p_1 - 1$ many other vertices, and the vertex $p_2$ shares an edge with $p_2 - 1$ many other vertices. Since the graph is regular, $p_1 - 1 = p_2 - 1$, thus $p_1 = p_2$ which is a contradiction since $p_1$ and $p_2$ are distinct.
So the only non-empty regular graphs are complete. □

Theorem 3.19. $\Gamma(\mathbb{Z}_n)$ is chordal iff $n = p^x, 2p$ or $2p^2$, where $p$ is prime and $x$ is a positive integer.

Proof. Let $n = p^x$. Assume that $\Gamma(\mathbb{Z}_{p^x})$ is not chordal. Then $\exists$ a cycle $C$ of length $> 3$, that has no chord. Let $y$ be a vertex of $C$ that is not a multiple of $n^*$. Then, since the power of $p$ in $y$ has a power strictly
less than \( \frac{\alpha}{2} \), each neighbor must be a multiple of \( n^* \). Then the two neighbors of \( y \) in \( C \) share an edge which is a chord. So all vertices in \( C \) must be a multiple of \( n^* \) which also causes a chord. So \( \Gamma(\mathbb{Z}_{\mu^x}) \) is chordal.

Let \( n = 2p \). Then, \( \Gamma(\mathbb{Z}_{2p}) \) is chordal.

Let \( n = 2p^2 \). Assume \( \Gamma(\mathbb{Z}_{2p^2}) \) is non-chordal. Then \( \exists \) a cycle \( C \) of length > 3 that has no chord.

Let \( a \) be a vertex of \( C \) in the type class \( T_p \). Each neighbor of \( a \) must be a multiple of \( 2p \), and therefore, is in the type class \( T_{2p} \). Each multiple of \( 2p \) shares an edge, so there exists a chord in \( C \). So there can be no vertices in the type class \( T_p \) in \( C \).

Let \( b \) be a vertex of \( C \) that is in the type class \( T_{2p} \). Every neighbor of \( b \) must be in the type class \( T_{p^2} \). But there is only one element in \( T_{p^2} \) so \( b \) cannot have two distinct neighbors. So \( b \) is not a vertex of \( C \).

So each vertex of \( C \) must be in either \( T_p \) or \( T_{2p} \). Then since there is only one element of \( T_{p^2} \), and the magnitude of \( C \) is at least 4, there are at least 3 elements of \( T_{2p} \) in \( C \). Those 3 elements form a triangle since each multiple of \( 2p \) annihilates each other multiple of \( 2p \). But \( C \) can’t have a triangle since it is chord-less. This is a contradiction.

Let \( n \) not be \( p^x, 2p \) or \( 2p^2 \).

Case 1: \( n = 2^x p^y \) where \( y \geq 3 \), \( x \geq 1 \) and \( p \) is an odd prime.

Then \( 2^x p - p^y - 2^{x+1} p - p^{y-1} \) is a chord-less cycle.

Case 2: \( n = 2^x p^y \) where \( x \geq 2 \), \( y \geq 1 \) and \( p \) is an odd prime.

Then \( 2p^y - 2^x - p^y - 2^{x+1} \) is a chord-less cycle.

Case 3: \( n = p^x q^y \) where \( p, q \geq 3 \) where \( p \neq q \) are primes and \( x, y \) are non-zero.

Then \( p^x - q^y - 2p^x - 2q^y \) is a chord-less cycle.

Case 4: \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) where \( k \geq 3 \) and \( \alpha_i \) is non-zero.

Since \( k \geq 3 \), \( n \) has an odd prime factor \( p_1 \). Then \( p_1^{\alpha_1} - n/p_1^{\alpha_1} - 2p_1^{\alpha_1} - 2n/p_1^{\alpha_1} \) is a chord-less cycle.

So \( \Gamma(\mathbb{Z}_n) \) is non-chordal if \( n \) is not \( p^x, 2p \) or \( 2p^2 \). \hfill \Box

**Lemma 3.20.** If \( n^* \neq n \), \( \Gamma(\mathbb{Z}_n) \) has a simplicial vertex.

**Proof.** Let \( n^* \neq n \). Then \( n/n^* \) is a vertex since \( n/n^* \) shares an edge with \( n^* \) which is not a multiple of \( n \). Since every neighbor of \( n/n^* \) is
a multiple of \( n^* \) and any two multiples of \( n^* \) share an edge, \( n/n^* \) is a simplicial vertex. So \( \Gamma(\mathbb{Z}_n) \) has a simplicial vertex. \( \square \)

Another construction \( n_* \) can be useful. It is similar to \( n^* \), but for the odd powered primes, round down instead of up. Consider \( \Gamma(\mathbb{Z}_n) \) where 
\[ n = p_1^a_1 p_2^a_2 \cdots p_k^a_k \] Define \( n_* \) as 
\[ n_* = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \] and \( \beta_i = \lfloor \alpha_i/2 \rfloor \) for all \( i \).

Note that \( n_* n^* = n \) and if \( n \) is square-free, \( n_* = 1 \).

**Lemma 3.21.** Arbitrary vertex \( v \) in \( \Gamma(\mathbb{Z}_n) \) is a simplicial vertex iff \( v \in T_2 \) or \( v \in T_g \) where \( g \) is a factor of \( n_* \).

**Proof.** Take arbitrary \( v \) in \( \Gamma(\mathbb{Z}_n) \). Let \( v \in T_2 \). Then \( v \) only shares an edge with vertices in \( T_{n/2} \). By Lemma 3.9, \( T_{n/2} \) has only one element, which makes a clique. So \( v \) is simplicial.

Let \( v \in T_g \) where \( g \) is a factor of \( n_* \). So \( n_* = ag \) where \( a \) is a positive integer. Consider some vertex \( h \) in \( T_j \) that shares an edge with \( v \). Then \( jg = bn \) for some positive integer \( b \). \( \frac{bn}{a} = bn_* \). \( \frac{j}{a} = bn^* \). Then \( j = abn^* \). So \( j \) is a multiple of \( n^* \) and therefore, \( h \) is a multiple of \( n^* \). Since every multiple of \( n^* \) shares an edge with every other such multiple, \( v \) is a simplicial vertex.

Conversely, let \( v \) be neither in \( T_2 \) nor in any \( T_g \) where \( g \) is a factor of \( n_* \). Then, since \( v \) is not in any \( T_g \), \( v \) has some prime with a power greater than half of that in \( n \). Call that prime \( p_x \) and its power in \( v \), \( \alpha_x \). Let the type class of \( v \) be called \( T_w \). Consider the type class \( T_{n/w} \). Each vertex in \( T_{n/w} \) shares an edge with \( v \). Since \( v \notin T_2 \), \( T_{n/w} \neq T_{n/2} \). So by Lemma 3.9, \( T_{n/w} \) has more than one element. Since \( n/w \) has a power of \( p_x \) less than that of half in \( n \), none of the vertices in \( T_{n/w} \) share an edge with each other. So the neighbors of \( v \) do not form a clique. Hence, \( v \) is not simplicial. \( \square \)

**Theorem 3.22.** \( \Gamma(\mathbb{Z}_n) \) has a simplicial vertex iff the prime factorization of \( n \) is not square free or \( n \) is an even greater than 2.

**Proof.** Let \( n \) not be square-free. Then, \( n^* \neq n \). So by Lemma 3.20, \( \Gamma(\mathbb{Z}_n) \) has a simplicial vertex.

Let \( n \) be even. Then, 2 divides \( n \). So, by the above lemma any \( v \in T_2 \) is a simplicial vertex.

Let \( n \) be square free and odd. 2 is therefore not a factor of \( n \). Then consider arbitrary vertex \( x \). \( x \) shares an edge with both \( n/x \) and \( 2n/x \). \( 2n/x \) is non-zero since \( x \) is necessarily odd, and \( n/x \) and \( 2n/x \)
do not share an edge since \( n \) is odd. For if \( \frac{2n}{x} = ny, 2n = yx^2 \) and \( n = \frac{yx^2}{2} \) which is a contradiction. So there are no simplicial vertices of \( \Gamma(Z_n) \). \( \square \)

Note: It follows by [3], (observation 3.2), if in \( \Gamma(Z_n) \) a vertex \( u \) is simplicial then \( T_u \) is simplicial in \( \Gamma^T(Z_n) \). But, not conversely. For example, in \( \Gamma^T(Z_{12}) \), \( T_3 \) is simplicial, where as \( 3 \) is not so in \( \Gamma(Z_{12}) \).

**Lemma 3.23.** If \( n \) has three or more distinct prime factors then, \( \Gamma(Z_n) \) is not \( \gamma - \beta \) perfect.

**Proof.** Let \( n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k} \) where \( k \geq 3 \). By [2], the domination number of \( \Gamma(Z_n) \) is \( k \). If there is a vertex cover \( V \) whose size is \( k \), we claim that \( V \) must contain the vertex \( n/p_x \) for every \( p_x \) prime factor of \( n \). Consider the vertex \( n/p_x \) for some \( p_x \) prime factor of \( n \). Let \( n/p_x \) not be in \( V \). Construct set \( C = \{p_xp_i | 1 \leq i \leq k\} \). Since \( n/p_x \notin V \), every element of \( C \) is in \( V \). \( C \) has \( k \) many vertices, so \( V \) has at least \( k \) many vertices. Consider vertex \( p_x \). \( p_x \) shares an edge with \( n/p_x \) which is not covered by \( V \), so \( V \) has at least \( k + 1 \) vertices. That is a contradiction since the size of \( V \) is \( k \). So each \( n/p_x \) is in \( V \).

Consider the type classes \( T_{n/p_1}, T_{n/p_2} \) and \( T_{n/p_3} \). By Lemma 3.11, there can be at most one type class with only one element. At least two of these type classes have more than one element. Without loss of generality, let them be \( T_{n/p_1} \) and \( T_{n/p_2} \). Since \( n/p_1 \) and \( n/p_2 \) are both in \( V \), choose different vertices in the type classes \( u \) and \( v \). \( u \) and \( v \) share an edge since they are multiples of \( n/p_1 \) and \( n/p_2 \) respectively, so they share an edge, but are not in \( V \), as the size of \( V \) is \( k \). This is a contradiction to the assumption that \( V \) is a vertex cover. So, \( \Gamma(Z_n) \) is not \( \gamma - \beta \) perfect. \( \square \)

**Theorem 3.24.** \( \Gamma(Z_n) \) is \( \gamma - \beta \) perfect if \( n = 2^3, 3^2, p, 2p \) and \( 3p \) for prime \( p \).

**Proof.** Let \( n = 2^3 \) or \( n = 3^2 \). The domination number clearly equals the size of the smallest vertex cover.

Let \( n = p \). Then both the domination number and the smallest vertex map are 0 since the graph is empty.

Let \( n = 2p \). Then the graph is a star, so the domination number and the vertex covering numbers of any of its induced subgraphs are both 1. Let \( n = 3p \). Then \( V = \{p, 2p\} \) is both a minimal dominating set and a minimal vertex cover of the original graph.
Now, we will show that all other $\Gamma(\mathbb{Z}_n)$ are not $\gamma - \beta$ perfect.

Let $n = 2^x$, $x \geq 4$. Then $2^{x-1} - 2^{x-2} - 3 \cdot 2^{x-2}$ is a triangle. Triangles prevent vertex covers of size 1, and by [2] the domination number is 1, so the values do not match.
Let $n = 3^x$, $x \geq 3$. Then $3^{x-1} - 2 \cdot 3^{x-1} - 3^{x-2}$ is a triangle that prevents vertex maps of size 1.
Let $n = p^x$, $p \geq 5$, $x \geq 2$. Then $p^{x-1} - 2 \cdot p^{x-1} - 3 \cdot p^{x-1}$ is a triangle.

**Theorem 3.25.** For $n = p^2 q$, the $\Gamma(\mathbb{Z}_n)$ is not domination perfect.

Case 1: $p = 2$.
Then $p^{x-1}q - p^x - q - p^{x+1} - pq$ is a non-induced sub-graph that cannot be covered by a vertex map size 2.

Case 2: $p \neq 2$.
Then $p^x - p^{x-1}q - p - 2p^{x-1}q - 2p$ is a non-induced sub-graph that cannot be covered by a vertex map size 2. The smallest vertex map is larger than 2 making the graph not $\gamma - \beta$ perfect.
Let $n = p^x q^y$, $x, y \geq 2$. The domination number is 2 by [2]. Assume there is a vertex map $V$ size 2. Consider the edges $p - p^x q^y$ and $q - p^x q^{y-1}$. $V$ must contain at least vertex one of each edge. By Lemma 3.11 only one type class can have only one vertex. Consider the type classes $T_{p^x q^y-1}$ and $T_{p^x-1 q^y}$. At least one of them must contain more than one vertex. Without loss of generality let that be $T_{p^x-1 q^y}$. Then there exists some $u \in T_{p^x-1 q^y}$ that is not in $V$. The edge $p - u$ is not covered by $V$, so the size of $V$ is at least one more than 2 which is a contradiction.
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $k \geq 3$. Then by Lemma 3.23, the graph is not $\gamma - \beta$ perfect.
So the only $\gamma - \beta$ graphs $\Gamma(\mathbb{Z}_n)$ are $2^3, 3^2, p, 2p$ and $3p$.

4. Some properties of $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$

In this section, we discuss some facts about $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. It is often possible to relate some properties of the individual $\Gamma(\mathbb{Z}_{n_i})$ to the graph of the product. One example is that the domination number of $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ has an upper and lower bound corresponding to the domination number of each $\Gamma(\mathbb{Z}_{n_i})$. 

Theorem 4.1. Consider two arbitrary commutative rings with unity, \( R \) and \( S \). \( \Gamma(R \times S) \) is complete iff \( |R| = |S| = 2 \).

Proof. Consider some \( R \) and \( S \) such that \( |R| = |S| = 2 \). Since both \( R \) and \( S \) have 1, the only elements of \( R \) and \( S \) are 0 and 1, where by 1 we denote the unity of the respective ring. Then the zero divisor graph is \((0,1) - (1,0)\) which is complete.

Conversely, let \( R \) or \( S \) have more than 2 elements. Without loss of generality, let \( R \) have more than 2 elements. Then \( R \) has some element \( a \) that is neither 1 nor 0. The graph \( \Gamma(R \times S) \) has vertices \((1,0), (a,0)\). These vertices do not share an edge because \( 1 \cdot a = a \) which is not zero. So \( \Gamma(R \times S) \) is not complete. \( \square \)

Theorem 4.2. \( \Gamma(R_1 \times \cdots \times R_k) \) where \( k \geq 2 \) and each \( R_i \) is a commutative ring with 1. This graph is complete iff \( k = 2 \) and \( |R_i| = 2 \) for all \( i \).

Proof. One direction follows from the last theorem. Consider some \( \Gamma(R_1 \times \cdots \times R_k) \) that does not meet these criteria. If \( k \geq 3 \), then \((1,0,1)\) and \((1,1,0)\) are two vertices that do not share an edge. If any \( |R_i| \geq 2 \), then \( R_i \) has an element \( a \) that is not 0 or 1. Then \((\cdots,a,\cdots)\) does not share an edge with \((\cdots,1,\cdots)\), where \( a \) and 1 are placed in the \( i \)-th entry of the respective elements. So \( \Gamma(R_1 \times \cdots \times R_k) \) is not complete. \( \square \)

Theorem 4.3. \( \Gamma(Z_n \times Z_m) \) where \( n,m \geq 2 \) is complete-bipartite iff \( n \) and \( m \) are prime.

Proof. Let \( m \) and \( n \) be prime. Then construct \( S_n \) and \( S_m \) such that \( S_n = \{(x,0)|0 < x < n\} \) and \( S_m = \{(0,y)|0 < y < m\} \).

We claim that \( S_n \cup S_m = V(\Gamma(Z_n \times Z_m)) \).

Assume, \( \exists \) a zero divisor \( a = (a_1,a_2) \) that is not in \( S_n \cup S_m \). Both \( a_1 \) and \( a_2 \) are non-zero. Since \( a \) is a zero-divisor, there must be some \( b = (b_1,b_2) \) that shares an edge with \( a \). So \( a_1b_1 = 0 \). Since \( Z_n \) has no non-zero divisors, and \( a_1 \) is not zero, \( b_1 = 0 \). In the same way, we find that \( b_2 \) is zero. This means \( a \) is not a zero-divisor because it only shares an edge with 0. So \( S_n \cup S_m = V(\Gamma(Z_n \times Z_m)) \).

Take arbitrary \( u,v \in S_n \). Then \( u = (u_1,0) \) and \( v = (v_1,0) \). Since \( u_1v_1 \neq 0, uv \neq (0,0) \) which means \( u \) and \( v \) do not share an edge. In the same way, \( u \) and \( v \) do not share an edge if they are both in \( S_m \). So \( u \) and \( v \) do not share an edge if they are in the same partition which is the definition of bipartite.

Thus, it follows from the construction of \( S_m \) and \( S_m \), that \( \Gamma(Z_n \times Z_m) \)
is complete bipartite. Conversely, let \( \Gamma(Z_n \times Z_m) \) be complete bipartite. Assume one or both \( n \) and \( m \) are not prime. Let the non-prime be \( n \). Then, there is a non-zero zero divisor of \( Z_n \). Call it \( k \). Since \( \Gamma(Z_n \times Z_m) \) is complete-bipartite, the vertices of \( \Gamma(Z_n \times Z_m) \) can be partitioned into 2 disjoint subsets such that no edges exist between two vertices in the same partition, and every pair of vertices in different partitions share an edge. \((1,0)\) is a zero divisor since it shares an edge with \((0,1)\). \((k,0)\) is also a zero divisor since it also shares an edge with \((0,1)\). Since \((k,0)\) does not share an edge with \((1,0)\), they must be in the same partition. Call it \( S_1 \) and let the other partition be \( S_2 \). Since \( k \) is a zero-divisor of \( Z_n \), \( \exists k' \) not necessarily distinct such that \( k \cdot k' = 0 \). Then \((k',1)\) shares an edge with \((k,0)\) which means \((k',1) \in S_2 \). Since \( \Gamma(Z_n \times Z_m) \) is complete-bipartite, \((1,0)\) must share an edge with \((k',1)\) since they are in opposite partitions, but their product is not 0, which is a contradiction. So both \( n \) and \( m \) must be prime. \( \square \)

**Corollary 4.4.** From this theorem it follows that \( \Gamma(Z_n \times Z_m) \) has a complete bipartite sub-graph.

One way to form this is by constructing \( S_n \) by taking all the non-zero elements in \( Z_n \) that are not zero divisors of \( n \) in the first entry and accordingly for \( S_m \).

**Theorem 4.5.** \( \Gamma(Z_{n_1} \times \cdots \times Z_{n_k}) \) where \( \forall n_i \geq 2 \) and \( k \geq 2 \) is bipartite iff \( k = 2 \) and both \( n_i \) are prime, or one \( n_x \) is prime and the other is 4.

**Proof.** Let \( k = 2 \) and both \( n_1 \) and \( n_2 \) be prime. By Theorem 4.3, \( \Gamma(Z_{n_1} \times Z_{n_2}) \) is bipartite.

Let \( k = 2 \) and let one of \( n_i \) be 4 and the other be prime. Without loss of generality, let \( n_1 = 4 \). Then \( n_2 \) is prime. Partition the vertices into sets \( A \) and \( B \) where \( A \) is the set of all vertices of the form \((a,0)\) where \( a \in Z_{n_1} \setminus \{0\} \) and \( B \) is everything else. Consider arbitrary, distinct elements of \( A \), \((a_1,0)\) and \((a_2,0)\). They do not share an edge. Consider all vertices in \( B \). Assume \( \exists u, v \in B \) such that \( u \) shares an edge with \( v \). Then, \( u = (u_1,u_2) \) and \( v = (v_1,v_2) \). Note that \( u_2v_2 \neq 0 \), as \( n_2 \) is a prime. So \( \Gamma(Z_{n_1} \times \cdots \times Z_{n_k}) \) is bipartite. Conversely, let \( \Gamma(Z_{n_1} \times \cdots \times Z_{n_k}) \) be bipartite.

We first claim, that \( k = 2 \). Assume \( k \geq 3 \). Then, \((1,0,0,\cdots,0) - (0,1,0,\cdots,0) - (0,0,1,\cdots,0)\) is a triangle which cannot exist in a bipartite graph. So \( k < 3 \). By our definition, \( k \geq 2 \), so \( k = 2 \).

We now claim no \( \Gamma(Z_{n_i}) \) can have two or more distinct zero divisors. Assume otherwise. Call two such divisors \( u \) and \( v \) that share an edge
in $\Gamma(\mathbb{Z}_n)$. Without loss of generality, let $u$ and $v$ be in the first slot (so $i = 1$). Then $(u, 0) - (v, 0) - (0, 1)$ is a triangle that cannot exist in a bipartite graph. The only $\Gamma(\mathbb{Z}_n)$ that has one element is $\Gamma(\mathbb{Z}_4)$. So all $n_i$ must be either 4 or prime.

Our final claim is it is not possible for both $n_i$ to be 4.

Assume otherwise. Then $(2, 0) - (2, 2) - (0, 2)$ is a triangle which cannot exist in a bipartite graph. So, because $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ is bipartite, $k = 2$ and either both $n_i$ are prime, or one is 4 and the other is prime. □

Theorem 4.6. $\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$ is k-partite where every $p_i$ is prime.

Proof. Consider some graph $\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$. Construct a collection of subsets $S_i$ which is the set of all vertices with a non-zero term in the $i$th slot and zero in any slot less than $i$.

$S_1 = \{ (a, \cdots) | a \in \mathbb{Z}_{p_1}, a \neq 0 \}$

$S_2 = \{ (0, a, \cdots) | a \in \mathbb{Z}_{p_2}, a \neq 0 \}$

$\cdots$

$S_k = \{ (0, 0, \cdots, 0, a) | a \in \mathbb{Z}_{p_k}, a \neq 0 \}$

No two vertices $u, v$ from the same subset $S_x$ share an edge.

All these $S_i$ form a partition of $\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$.

So $\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$ is k-partite. □

Theorem 4.7. $\Gamma(R_1 \times \cdots \times R_k)$ where each $R_i$ is a commutative ring with 1 is not perfect if some $\Gamma(R_i)$ is not perfect.

Proof. Let some $\Gamma(R_i)$ be non-perfect. Then by the Strong Perfect Graph theorem, there exists an odd hole or anti-hole $H$ of length 5 or greater. Let $H$ have a length $l$. Then we write it as, $v_1 - v_2 - \cdots - v_{l-1} - v_l - v_1$. Then to obtain a hole or antihole of length 5 or greater in $\Gamma(R_1 \times \cdots \times R_k)$ fill in the $i$th position with the vertices of $H$, and fill the rest in with zeros. $(0, \cdots, 0, v_1, 0, \cdots, 0) - (0, \cdots, 0, v_2, 0, \cdots, 0) - \cdots - (0, \cdots, 0, v_{l-1}, 0, \cdots, 0) - (0, \cdots, 0, v_1, 0, \cdots, 0) - (0, \cdots, 0, v_1, 0, \cdots, 0)$ is a hole or anti-hole of odd length of 5 or greater making the graph $\Gamma(R_1 \times \cdots \times R_k)$ non perfect. □

Note 4.8. The converse of Theorem 3.4 is not true. In the graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, every $\Gamma(\mathbb{Z}_2)$ is perfect, but we find the hole $(1, 1, 0, 0, 0) - (0, 0, 1, 1, 0) - (1, 0, 0, 0, 1) - (0, 1, 1, 0, 0) - (0, 0, 0, 1, 1)$.

Theorem 4.9. $\Gamma(R_1 \times \cdots \times R_k)$ where each $R_i$ is a commutative ring with 1 is not regular if any $\Gamma(R_i)$ is not empty.
Theorem 4.10. For arbitrary rings $R$ and $S$, $cl(\Gamma(R \times S)) \geq cl(\Gamma(R)) + cl(\Gamma(S)) + |R'||S'|$ where $R'$ and $S'$ are any set of self-annihilating vertices in a maximal clique of $\Gamma(R)$ and $\Gamma(S)$.

Proof. Let $C$ be a maximal clique in $\Gamma(R)$ and $D$ be a maximal clique in $\Gamma(S)$. Construct an induced subgraph $X = \{(c,0), (0,d) | c \in C, d \in D\}$. $X$ is a clique in $\Gamma(R \times S)$ with size $cl(\Gamma(R)) + cl(\Gamma(S))$.

Now consider $R'$, the set of all self-annihilating vertices in $C$ and $S'$, the set of all self-annihilating vertices in $D$. Define the induced subgraph $Y = \{(r,s) | r \in R', s \in S'\}$. Every vertex $(r,s) \in Y$ shares an edge with every other vertex in $Y$ and every vertex in $X$, so $X \cup Y$ forms a clique size $cl(\Gamma(R)) + cl(\Gamma(S)) + |R'||S'|$. 

Corollary 4.11. Consider $n$ many arbitrary rings $R_1, R_2, \cdots R_n$. Then, $cl(\Gamma(R_1 \times R_2 \cdots R_n)) \geq \sum_{i=1}^{n} cl(\Gamma(R_i)) + \sum_{i \neq j, i,j \in \{1,2,\ldots,n\}} |R'_i||R'_j| + \sum_{i \neq j, k \in \{1,2,\ldots,n\}} |R'_i||R'_j||R'_k| + \cdots + |R'_i||R'_2| \cdots |R'_k|$, where each $R'_i$ is any set of self-annihilating vertices in a maximal clique in $\Gamma(R_i)$.

Lemma 4.12. Consider $\Gamma(\mathbb{Z}_n)$ for arbitrary $n$. There is a maximal clique $M$ that contains all self-annihilating vertices.

Proof. Follows from Theorem 3.15 and Lemma 3.13.

Theorem 4.13. The clique number of $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ has a lower bound of $cl(\Gamma(\mathbb{Z}_n)) + cl(\Gamma(\mathbb{Z}_m)) + (\frac{n}{m} - 1)(\frac{m}{m} - 1)$.

Proof. Follows from Theorem 4.10 and the proof of Theorem 3.15 and Lemma 3.13.

Theorem 4.14. $\Gamma(R_1 \times \cdots \times R_k)$ where $k \geq 2$ and $R_i$ is a commutative ring with 1 has a simplicial vertex iff some $\Gamma(R_i)$ has a simplicial vertex or some $|R_i| = 2$. 

Proof. Take $\Gamma(R_1 \times \cdots \times R_x)$. Let some $\Gamma(R_i)$ be non-empty. Consider the vertex $g = (0, \ldots, 0, 1, 0, \ldots, 0)$ that has a 1 at the $i$th index and 0 filled in all other indices. All neighbors of $g$ must be of the form $(a_1, a_2, \cdots, a_{i-1}, 0, a_{i+1} \cdots, a_x)$, with a zero at the $i$th index and any value in the other indices, not all zero. Let there be $f$ many such vertices. Since $\Gamma(R_i)$ is non-empty, $\exists k \in \Gamma(R_i)$. Since $k$ is a zero divisor, there must be some $k' \in \Gamma(R_i)$, not necessarily distinct, such that $k \cdot k' = 0$. Consider the vertex $h = (0, \ldots, 0, k, 0, \ldots, 0)$ with $k$ in the $i$th index. This vertex shares an edge with all vertices that share an edge with $g$. So $h$ shares an edge with at least $f$ vertices. But it also shares an edge with $(1, \cdots, 1, k', 1 \cdots, 1)$ which means $h$ shares an edge with at least $f + 1$ vertices. This means $g$ and $h$ have a different number of neighbors, so $\Gamma(R_1 \times \cdots \times R_x)$ is not regular. 

$\square$
Proof. Take arbitrary $\Gamma(R_1 \times \cdots \times R_k)$. Let some $\Gamma(R_i)$ have a simplicial vertex $c$. Then the vertex $(1, \cdots, 1, c, 1, \cdots, 1)$ where $c$ is in the $i$th slot is a simplicial vertex of $\Gamma(R_1 \times \cdots \times R_k)$.

Let some $|R_i| = 2$. Then $(1, \cdots, 1, 0, 1, \cdots, 1)$ only shares an edge with $(0, \cdots, 0, 1, 0, \cdots, 0)$ making $(1, \cdots, 1, 0, 1, \cdots, 1)$ simplicial.

Let $\Gamma(R_1 \times \cdots \times R_k)$ have a simplicial vertex $v$. Also, assume all $|R_i| > 2$ and no $\Gamma(R_i)$ have any simplicial vertices. Consider arbitrary $v$ in $\Gamma(R_1 \times \cdots \times R_k)$. Let $v$ have 0 at some index, $v = (\cdots, 0, \cdots)$. Then since no $|R_i| = 2$, there exists some vertex $a \in R_i$ that is not 0 or 1. $v$ then shares an edge with $(0, \cdots, 0, 1, 0, \cdots, 0)$ and $(0, \cdots, 0, a, 0, \cdots, 0)$ and they do not share an edge. So for $v$ to be simplicial, it cannot contain any 0. Let $v$ have a at some index, where $a$ is a zero divisor in its respective $\Gamma(R_i)$. $v = (\cdots, a, \cdots)$. Then $v$ shares an edge with every $(0, \cdots, 0, a', 0, \cdots, 0)$ where $a \cdot a' = 0$ in $\Gamma(R_i)$. $a$ is not simplicial since no $\Gamma(R_i)$ have any simplicial vertex, so some neighbor $(0, \cdots, 0, a', 0, \cdots, 0)$ will not share an edge with another neighbor of the same form. So $v$ is not simplicial if it has any zero-divisors in its slots. For $v$ to be simplicial, every slot must be a non-zero, non-zero-divisor. However, elements of that form are not vertices. So $\Gamma(R_1 \times \cdots \times R_k)$ has no simplicial vertices, which is a contradiction. The assumption that all $|R_i| > 2$ and no $\Gamma(R_i)$ have any simplicial vertices is false. So some $|R_i| > 2$ or some $\Gamma(R_i)$ has a simplicial vertex. \hfill $\square$

**Theorem 4.15.** $\Gamma(R_1 \times \cdots \times R_k)$ where $R_i$ is a commutative ring with 1 is non-chordal if any $\Gamma(R_i)$ is non-chordal.

*Proof.* Consider arbitrary $\Gamma(R_1 \times \cdots \times R_k)$. Then let some $\Gamma(R_i)$ be non-chordal. So there exists a cycle $a_1 - a_2 - \cdots - a_k - a_1$ greater than 3 with no chords. Then in $\Gamma(R_1 \times \cdots \times R_k)$, there is a cycle $(0, \cdots, a_1, \cdots, 0) - (0, \cdots, a_2, \cdots, 0) - \cdots - (0, \cdots, a_k, \cdots, 0) - (0, \cdots, a_1, \cdots, 0)$, which makes it non-chordal. \hfill $\square$

**Lemma 4.16.** $\Gamma(R_1 \times \cdots \times R_k)$ where $R_i$ is a commutative ring with 1 and $k \geq 2$ is non-chordal if more than one $|R_i| \geq 3$.

*Proof.* In $\Gamma(R_1 \times \cdots \times R_k)$, let two or more $|R_i| \geq 3$. Without loss of generality, let the first two slots be the $R_i$ with a magnitude greater than or equal to 3. Then $(1, 0, \cdots, 0) - (0, 1, \cdots, 0) - (a, 0, \cdots, 0) - (0, b, \cdots, 0)$ where $a$ is a non-trivial element of $R_1$ and $b$ is a non-trivial element of $R_2$, is a cycle with no chord. So $\Gamma(R_1 \times \cdots \times R_k)$ is non-chordal. \hfill $\square$
Lemma 4.17. $\Gamma(R_1 \times \cdots \times R_k)$ where $R_i$ is a commutative ring with 1 is non-chordal if $k \geq 4$.

Proof. Let $k \geq 4$. Then $(1, 1, 0, 0, \cdots, 0) - (0, 0, 1, 1, \cdots, 0) - (1, 0, 0, 0, \cdots, 0) - (0, 0, 0, 1, \cdots, 0)$ is a chord-less cycle. So $\Gamma(R_1 \times \cdots \times R_k)$ is non-chordal. □

Lemma 4.18. $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3})$ where at least one $n_i > 2$ is non-chordal.

Proof. Without loss of generality, let $n_3 > 2$. Then, $(1, 0, 0) - (0, 0, 2) - (1, 1, 0) - (0, 0, 1)$ is a chord-less cycle. □

Theorem 4.19. The only chordal $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ where $n_i \geq 2$ and $k \geq 2$ are $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p^2)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

Proof. Consider $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$. Since $\Gamma(\mathbb{Z}_p)$ has no vertices, the only vertices of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ are $(1, 0)$ or of the form $(0, x)$ where $0 < x < p$. So the graph is a star making it chordal.

Consider $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p^2)$. Assume that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p^2)$ is non-chordal. Then there exists a cycle $C$ length greater than 3 that has no chord. Let $v$ be an arbitrary vertex in $C$.

Let $v$ have a multiple of $p$ as its second entry, $v = (a, bp)$. Then every vertex that is not a neighbor of $v$ in $C$ must have a non-zero non-multiple of $p$ as its second element. Therefore, both neighbors of $v$ must have 0 as their second element so that they share an edge with their other neighbor. So both neighbors of $v$ are $(1, 0)$. We cannot repeat vertices so $v$ cannot have a multiple of $p$ as its second element. That means the only possible vertices in $C$ are $(1, 0)$ and $(0, b)$ where $b$ is a non-zero non-multiple of $p$. A cycle of size 4 or greater cannot be constructed out of these vertices since we cannot write $(1, 0)$ more than once and a vertex of such form $(0, b)$ does not share an edge with a vertex of the same form. $C$ cannot be constructed, so $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p^2)$ is chordal.

Consider $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. The graph of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is shown below and is chordal.
(0, 1, 1)
  /
(1, 0, 0)
  /
(0, 1, 0) - (0, 0, 1)
  /
(1, 0, 1) - (1, 1, 0)

To prove the converse, let’s assume the opposite. Let there be a chordal $\Gamma(Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k})$ not listed. By Lemma 4.16, only one $n_i$ can be greater than 2. By Lemma 4.17, $k \leq 3$. By Theorem 4.15, if any $n_i$ are non-chordal, $\Gamma(Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k})$ will be non-chordal. So every $n_i$ must be $p^x$, $2p^y$, or $2p^2$ which was shown by Theorem 3.19.

So the only possible $\Gamma(Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k})$ are $\Gamma(Z_2 \times Z_{p^x})$, $\Gamma(Z_2 \times Z_{2p})$, $\Gamma(Z_2 \times Z_2 \times Z_{p^x})$, $\Gamma(Z_2 \times Z_2 \times Z_{2p})$ and $\Gamma(Z_2 \times Z_2 \times Z_{2p^2})$.

In $\Gamma(Z_2 \times Z_{p^x})$ where $x \geq 3$ and $p$ is prime, $(1, p^{x-1}) - (0, (p-1)p) - (1, 0) - (0, p)$ is a chord-less cycle.

In $\Gamma(Z_2 \times Z_{2p})$ where $p \geq 3$ is a prime, $(1, 0) - (0, 4) - (1, p) - (0, 2)$ is a chord-less cycle.

In $\Gamma(Z_2 \times Z_{2p^2})$ where $p \geq 3$ is a prime, $(1, 2p) - (0, p) - (1, 4p) - (0, p^2)$ is a chord-less cycle.

By Lemma 4.17, $\Gamma(Z_2 \times Z_2 \times Z_{p^x})$, $\Gamma(Z_2 \times Z_2 \times Z_{2p})$ and $\Gamma(Z_2 \times Z_2 \times Z_{2p^2})$ are all non-chordal where $p \geq 3$.

So there are no other chordal $\Gamma(Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k})$. □

5. Zero divisor graph of the poset $D_n$

Zero divisor graph of a poset has been studied in [8], [9], [10]. We always have the Clique number of the zero-divisor graph of a ring that does not exceed the Chromatic number of that. Beck conjectured that for an arbitrary ring $R$, they are the same. But Anderson and Naseer [6] have shown that this is not the case in general, namely, they presented an example of a commutative local ring $R$ with 32 elements for which Chromatic number is strictly bigger than the clique number. In [6] Nimbhorkar, Wasadikar and DeMeyer have shown
that Beck’s conjecture holds for meet-semilattices with 0, i.e., commutative semigroups with 0 in which each element is idempotent. In fact, it is valid for a much wider class of relational structures, namely for partially ordered sets (posets, briefly) with 0. Now, to any poset \((P, \leq)\), with a least element 0 we can assign the graph \(G\) as follows: its vertices are the nonzero zero divisors of \(P\), where a nonzero \(x \in P\) is called a zero divisor if there exists a non-zero \(y \in P\), so that \(L(x, y) = 0\), \(L(x, y) = \{z \in P| z \leq x, y\}\). And \(x, y\) are connected by an edge if \(L(x, y) = 0\). We discuss here some properties of the zero-divisor graph of a specific poset \(D_n\). Very often we used the prime factorization of the positive integer \(n\). By abuse of notation, let us call \(D_n\) as the zero-divisor graph of the poset \(D_n\). Note that, the vertex set of \(D_n\) is the set of all factors of \(n\) that are not divisible by some prime factor of \(n\). Also, note that two vertices in \(D_n\) are connected by an edge if and only if they are mutually co-prime.

**Remark 5.1 (Properties of \(D_n\)).**

i. If \(n = p^m\) for some prime \(p\) and positive integer \(m\), then \(D_n\) is trivial.

So from now on consider \(D_n\) where \(n \neq p^m\) where \(p\) and \(m\) are as mentioned.

ii. The diameter of \(D_n\) is 3 iff \(n\) has three distinct prime factors namely \(p, q, r\). This is shown by the path \(pq - r - p - qr\). Otherwise, the diameter is 1 or 2, as \(D_{p^m q^n}\) is complete bipartite which has diameter 2 or in the case of \(m = n = 1\) has diameter 1. [7] shows zero divisors of a poset have diameter of 1, 2, or 3.

iii. \(D_n\) is complete only when \(n = pq\), where \(p\) and \(q\) are two distinct primes. \(D_n\) is complete bipartite iff \(n = p^m q^n\) where \(m\) and \(s\) are two positive integers.

iv. We have the clique number of \(D_n\) and a few coefficients of the clique polynomial. The clique number of \(D_n\) is the number of distinct prime factors of \(n\). For if \(n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}\) where \(p_i's\) are distinct primes \(\forall i\), any set of vertices \(\{p_1^{\beta_1}, p_2^{\beta_2}, p_3^{\beta_3} \cdots p_r^{\beta_r}\}\), where \(1 \leq \beta_i \leq \alpha_i \forall i\) forms a maximal clique. Hence the clique
number is \( r \), the number of distinct primes of \( n \). And the leading coefficient in the clique polynomial is \( \alpha_1 \alpha_2 \cdots \alpha_r \). The coefficient of \( x^{r-1} \) is \( \sum_{i=1}^{r} (\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_r) + \binom{r}{2} \alpha_1 \alpha_2 \cdots \alpha_r \).

Reason: Consider a clique of size \( r - 1 \). If all the vertices have single prime factors then, there are \( \sum_{i=1}^{r} (\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_r) \) many of this type, as a typical clique of this type is a set of the form \( \{ p_{\beta_1}, p_{\beta_2}, \ldots, p_{\beta_i-1}, p_{\beta_i+1}, \ldots, p_{\beta_r} \} \), where \( 1 \leq \beta_j \leq \alpha_j \) for all \( j \in \{1, 2, \ldots, r\} \). Otherwise, exactly one vertex will contain two primes. And in that case, we will obtain \( \binom{r}{2} \alpha_1 \alpha_2 \cdots \alpha_r \) many such clique sets with cardinality \( r - 1 \).

v. The domination number of \( D_n \) is the number of distinct prime factors of \( n \), the same as the clique number, as any dominating set must not omit a prime factor of \( n \). If some \( p_i \) is missing from a set of vertices \( V \), then the vertex \( p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_r \) is not adjacent to any vertex in \( V \). Furthermore, if we let \( V \) be the set of all distinct primes of \( n \), each vertex in \( D_n \) must share an edge with at least one vertex in \( V \) because each vertex in \( D_n \) must omit at least one prime of \( n \) from its prime factorization.

vi. \( D_n \) is regular iff \( n = (pq)^m \) for some positive integer \( m \). If \( n = p^m q^r \), \( m \neq r \), then \( D_n = K_{m,n} \), complete bipartite which is not regular. Then, if \( n \) has more than two distinct primes in its prime factorization, then for two distinct primes \( p \) and \( q \) in its prime factorization \( p \) and \( pq \) are vertices with different degrees making the graph non-regular.

vii. In [9], it is discussed that the girth of the zero divisor graph of any poset is 3, 4, or \( \infty \). The girth of \( D_n \) is \( \infty \) iff \( n = p^m q \), where \( p \) and \( q \) are two distinct primes and \( m \) is a positive integers bigger than 1. The girth of \( D_n \) is 4, if and only if \( n = p^m q^r \), where \( p \) and \( q \) are two distinct primes and \( m \) and \( r \) are both positive integers bigger than 1. Otherwise, the girth of \( D_n \) is 3, because if \( n \) has at least 3 different prime factors \( p, q \) and \( r \), then \( p - q - r - p \) is a triangle in \( D_n \).

viii. \( D_n \) is not perfect iff \( n \) is the product of least five different primes \( p, q, r, s, t \) in its prime factorization, then \( ps - qt - pr - qs - tr - ps \) is a cycle of length five in \( D_n \). Hence by Strong perfect graph theorem \( D_n \) is not perfect. Suppose \( n \) has 4 distinct prime factors \( p, q, r \) and \( s \). Assume
there is an odd cycle of length 5 or greater that contains a vertex $v$ that is the product of two such primes. Let $v = p^a q^b$. Then the two neighbors of $v$ cannot be a multiple of $p$ or $q$. Suppose the neighbors both consist of $r^a$ for some positive integer $a$. Then, we get part of the cycle as $r^a - p^a q^b - r^b$ for another positive integer $b$. Then, $r^a$ will necessarily share an edge with the other neighbors of $r^b$ making the cycle length 4. So, the neighbors of $v$ must have both $r$ and $s$. Additionally, these parts of the cycle must be of the form $r^a - p^a q^b - r^b s^z$; otherwise, we get a cycle of length 4 again. But any vertex that shares an edge with $r^b s^z$ must also share an edge with $r^a$ making such a cycle impossible.

This means any odd cycle length greater than 5 cannot contain a vertex with two or more prime factors, making an odd cycle length greater than 4 impossible. The other two situations when $v$ consists of only one prime, or three primes also give contradiction. Thus, $D$ is perfect iff $n$ has 4 or fewer prime factors.

ix. $D_n$ is chordal iff $n = p^m q$ or $n = pqr$ where $p$, $q$ and $r$ are distinct primes and $m \geq 1$. For if $n$ is not of that form, $p - q - p^2 - q^2 - p$ or $p - q - p^2 - qr - p$ or $p - r - pq - rs - p$ will give holes of length greater than 3 in respective $D_n$'s.

x. Let, $n$ be a square free positive integer. Then, its simplicial vertices are precisely those factors of $n$ that miss exactly one prime in its prime factorization. Now, suppose $n$ is not square-free. Then, if all primes in its prime factorization are not square-free, it has no simplicial vertex. Otherwise, the simplicial vertices are precisely those that miss exactly one square free prime factor. For example, if $n = p^2 q^2 r$, $pq$, $p^2 q$, $pq^2$ and $p^2 q^2$ are the only simplicial vertices because $r$ is the only square free prime factor.

xi. The only planar $D_n$ has $n$ of the form $n = p^m q$, $p^m q^2$, $pqr$ or $p^2 qr$. First, let $n$ have only 2 prime factors. If $n = p^m q^l$ where $l \geq 3$ and $m \geq 3$, then $K_{3,3}$ is a subgraph of $D_n$. So, by Wagner’s theorem, $D_n$ is non-planar. But in the case of $p^m q$, $D_n$ is a star, so it is planar. And, in the case of $p^m q^2$, the graph can be drawn without any crossing edges. Next, let’s have three prime factors. If $n = pqr$ or $n = p^2 qr$ the graph is clearly planar if drawn. If $n = p^m qr$ where $m \geq 3$, the subgraph consisting of $p$, $p^2$, $p^3$, $q$, $r$ and $qr$ form $K_{3,3}$ if we delete the edge between $q$ and $r$. Then by Wagner’s theorem, the graph is non-planar since $K_{3,3}$ is a minor. Next, if $n = p^m q^l r$, where
m ≥ 2 and l ≥ 2 the set of vertices q, q^2, p, p^2, r, pr and qr is a subdivision of K_5. Then, by Kuratowski’s theorem, the graph is non-planar. So the only planar D_n with only 3 primes in n are pqr and p^2qr. Lastly, consider the case where n has 4 primes in its prime factorization, n = pqr. Then, the vertex set of p, q, r, s, pq and rs can be made isomorphic to K_5 by contracting the edge between pq and rs to make a single vertex. Therefore, K_5 is a minor of D_n for this case, and by Wagner’s theorem the graph is non-planar.

xii. D_n is Eulerian iff the power of each prime in the prime factorization of n is even. For, if n has a prime p^α that appears in its prime factorization where α is odd, then the vertex n/p^α has an odd degree, otherwise every vertex has even degree.

xiii. If n is square free, then we have the edge cardinality of D_n as \( \sum_{i=1}^{r-1} 2^{r-i-1}(r) - 2^{r-1} - 1 \), where r is the number of distinct primes of n. For, if we consider n = p_1p_2\cdots p_r, where p_i’s are distinct primes, then the degree of each vertex p_i is \( \sum_{i=1}^{r-1} \binom{r}{i} = 2^{r-1} - 1 \) giving r(2^{r-1} - 1) to the degree sum of the vertices. Similarly each vertex p_ip_j is adjacent to \( \sum_{i=1}^{r-2} \binom{r-2}{i} = 2^{r-2} - 1 \) many vertices, giving \( \binom{r}{2}(2^{r-2} - 1) \) in the degree sum. Proceeding in this way, we obtain the sum of the vertex degrees are \( \sum_{i=1}^{r-1} \binom{r}{i}(2^{r-i-1}) = \sum_{i=1}^{r-1} \binom{r}{i}2^{r-i} - 2^{r-2} - 2 \). Then, as the sum of vertex degrees is twice the edge cardinalities the result follows.

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