Research Article

Zero-Divisor Graphs of ℤ_n, their products and D_n

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This paper is an endeavor to discuss some properties of zero-divisor graphs of the ring $\mathbb{Z}_n$, the ring of integers modulo n. The zero divisor graph of a commutative ring R, is an undirected graph whose vertices are the nonzero zero-divisors of R, where two distinct vertices are adjacent if their product is zero. The zero-divisor graph of R is denoted by $\Gamma(R)$. We discussed $\Gamma(\mathbb{Z}_n)$'s by the attributes of completeness, k-partite structure, complete k-partite structure, regularity, chordality, $\gamma - \beta$ perfectness, simplicial vertices. The clique number for arbitrary $\Gamma(\mathbb{Z}_n)$ was also found. This work also explores related attributes of finite products $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$, seeking to extend certain results to the product rings. We find all $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ that are perfect. Likewise, a lower bound of clique number of $\Gamma(\mathbb{Z}_m \times \mathbb{Z}_n)$ was found. Later, in this paper, we discuss some properties of the zero divisor graph of the poset D_n, the set of positive divisors of a positive integer n partially ordered by divisibility.

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1. Introduction

Zero-divisor graphs were first discussed by Beck [1] as a way to color commutative rings. They were further discussed by Livingston and Anderson in [2] and [3]. A zero-divisor graph of a ring R, denoted by $\Gamma(R)$, is a graph whose vertices are all the zero-divisors of R. Two distinct vertices u and v are adjacent if $uv = 0$. Beck [1] considered every element of R a vertex, with 0 sharing an edge with all other vertices. Since then, others have chosen to omit 0 from zero-divisor graphs [2, 3, 4, 5]. For our purposes, we omit 0 so that the vertex set of $\Gamma(\mathbb{Z}_n)$ denoted by $ZD(\mathbb{Z}_n)$ will only be the non-zero zero-divisors.

In the first section, we explore a concept explored by Smith [4] called type graphs. In [4], type graphs were used to find all perfect $\Gamma(\mathbb{Z}_n)$. We extended the notion of type graphs for $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ to find all perfect zero-divisor graphs of such products, where $n_1, n_2, \cdots, n_k$ are positive integers and $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ is the direct product of $Z'_{n_i}s$, $1 \leq i \leq k$. We then move on to various properties of $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. In the last section, we explore zero divisor graphs of the poset D_n, the set of positive divisors of a positive integer n partially ordered by divisibility and we catalog them in a similar way. Zero divisor graph of poset is studied in [5], [6], [7].

2. Type Graphs

When we consider zero-divisor graphs of $\Gamma(\mathbb{Z}_n)$, it is useful to consider the type graphs of these rings. A type graph has vertices of T_a where a is a factor of n that is neither 1 nor 0. The set of all such T_a forms a partition of the vertex set of $\Gamma(\mathbb{Z}_n)$ where $T_a = \{x \in ZD(\mathbb{Z}_n) | gcf(x, n) = a\}$. This concept was shown by Smith [4], where the type graph was used to find all perfect $\Gamma(\mathbb{Z}_n)$. Smith used the notation $\Gamma^T(\mathbb{Z}_n)$ to denote the type graph. In that paper, four key observations were shown to be true regarding the type graphs on $\mathbb{Z}_n$. In this section, we modify the definition of type graph to fit the zero divisor graph of the finite

 doi.org/10.32388/SA4XI4

direct product $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, $n_1, n_2, \cdots n_k$ being k many positive integers. Additionally, we show these observations to be true over this type graph as well. We then use analogues of some theorems from [4] to characterize the perfectness of $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$.

The following are two important theorems from [4].

Theorem 2.1. (Smith's Main Theorem). [4] *A graph $\Gamma(\mathbb{Z}_n)$ is perfect iff n is of one of the following forms:*

1. *$n = p^a$ for prime p and positive integer a.*
2. *$n = p^a q^b$ for distinct primes p, q and positive integers a, b.*
3. *$n = p^a qr$ for distinct primes p, q, r and positive integer a.*
4. *$n = pqrs$ for distinct primes p, q, r, s.*

Theorem 2.2. (Simth's Theorem 4.1). [4] *$\Gamma(\mathbb{Z}_n)$ is perfect iff its type graph $\Gamma^T(\mathbb{Z}_n)$ is perfect.*

Definition 2.3. (Type graph of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$). *The type graph of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ denoted by $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ has a vertex set of the type classes $T(x_1, x_2, \cdots, x_k)$ where $(x_1, x_2, \cdots, x_k) \neq (0, 0, \cdots, 0)$ nor $(1, 1, \cdots, 1)$, and x_i is a divisor of n_i, 1, or 0.*
$T(x_1, x_2, \cdots, x_k) = \{(a_1, a_2, \cdots, a_k) \mid |a_i \in \mathbb{Z}_{n_i}/0$ and $gcf(a_i, n_i) = x_i$ or $a_i = 0$ if $x_i = 0\}$. Arbitrary $T(x_1, x_2, \cdots, x_k)$ shares an edge with arbitrary $T(y_1, y_2, \cdots, y_k)$ iff $x_i y_i = 0$ for all i.

Smith [4] gave the following four observations for the type graph of $\Gamma(\mathbb{Z}_n)$.

Theorem 2.4. *Each vertex of $\Gamma(\mathbb{Z}_n)$ is in exactly one type class.*

Theorem 2.5. *Arbitrary distinct vertices T_x and T_y share an edge in $\Gamma^T(\mathbb{Z}_n)$ iff each $a \in T_x$ shares an edge with each $b \in T_y$ in $\Gamma(\mathbb{Z}_n)$.*

Theorem 2.6. *Arbitrary distinct vertices T_x and T_y don't share an edge in $\Gamma^T(\mathbb{Z}_n)$ iff each $a \in T_x$ doesn't share an edge with each $b \in T_y$ in $\Gamma(\mathbb{Z}_n)$.*

Theorem 2.7. *In $\Gamma(\mathbb{Z}_n)$ consider arbitrary a and b in the same type class. An arbitrary vertex c in $\Gamma(\mathbb{Z}_n)$ shares an edge with b iff it shares an edge with a also.*

Following are the four analogues to the above results for $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$.

Theorem 2.8. *Each vertex of $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ is in exactly one type class.*

Theorem 2.9. *Arbitrary distinct vertices $T_x = T(x_1, x_2, \cdots, x_k)$ and $T_y = T(y_1, y_2, \cdots, y_k)$ share an edge in $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ iff each $a \in T_x$ shares an edge with each $b \in T_y$ in $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$.*

Theorem 2.10. *Arbitrary distinct vertices $T_x = T(x_1, x_2, \cdots, x_k)$ and $T_y = T(y_1, y_2, \cdots, y_k)$ don't share an edge in $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ iff each $a \in T_x$ doesn't share an edge with each $b \in T_y$ in $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$.*

Theorem 2.11. *In $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ consider arbitrary $a = (a_1, a_2, \cdots, a_k)$ and $b = (b_1, b_2, \cdots, b_k)$ in the same type class $T(t_1, t_2, \cdots, t_k)$. An arbitrary vertex $c = (c_1, c_2, \cdots, c_k)$ shares an edge with b iff it shares an edge with a also.*

Proof. Follows from Theorem 2.5 and 2.6. □

Next, we have the following theorem:

doi.org/10.32388/SA4XI4

Theorem 2.12. $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ *is perfect iff its type graph* $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ *is perfect.*

To show this, we will use the following three theorems, whose proofs are analogous to the corresponding proofs in [4].

Theorem 2.13. *Given arbitrary hole or antihole H of length greater than 4 in $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$, every vertex in H belongs to a different type class.*

Theorem 2.14. *Let there be a hole or antihole H length $l > 4$ in $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$. Then the type graph $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ must also contain a hole or antihole length l.*

Theorem 2.15. *Let there be a hole or antihole H length $l > 4$ in the type class $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$. Then the graph $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ must also contain a hole or antihole length l.*

Using these theorems, now we can establish the following proof of Theorem 2.12.

Proof. The proof is analogous to the proof in [4]. □

Now that we know perfectness in the type graph implies perfectness in the zero-divisor graph, it is possible to find all such perfect $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$. As it turns out, for both $\Gamma^T(\mathbb{Z}_n)$ and $\Gamma^T(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$, we can exchange the primes of each n_i, and as long as the form of the primes (the amount of distinct primes and the power of each prime) stays the same, the type graph will be isomorphic. To illustrate this, consider $\Gamma^T(\mathbb{Z}_{p^2q} \times \mathbb{Z}_p)$ where p, q are prime. This type graph is isomorphic to $\Gamma^T(\mathbb{Z}_{r^2s} \times \mathbb{Z}_t)$ where r, s, t are prime, even if the value of the primes are different. We will use this to find all perfect $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$.

Theorem 2.16. *Consider some $\Gamma^T(\mathbb{Z}_n)$ and $\Gamma^T(\mathbb{Z}_m)$ such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$. Then $\Gamma^T(\mathbb{Z}_n) \cong \Gamma^T(\mathbb{Z}_m)$.*

Proof. Consider arbitrary vertex u in $\Gamma^T(\mathbb{Z}_n)$. u is a factor of n, so we can write $u = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$. Note that $0 \leq x_i \leq \alpha_i$, $\forall i$. Define a function $f : \Gamma^T(\mathbb{Z}_n) \to \Gamma^T(\mathbb{Z}_m)$ as $f(u) = f(p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}) = q_1^{x_1} q_2^{x_2} \cdots q_k^{x_k}$. Since n and m both have the same amount of prime factors, and each corresponding prime has the same power α_i, the result follows. □

Theorem 2.17. *Consider $\Gamma^T(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ and $\Gamma^T(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k})$ where the prime factorization of n_i has the same form as m_i for each i. That is, n_i and m_i have the same amount of prime factors and the same power for each prime. Then $\Gamma^T(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \cong \Gamma^T(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k})$.*

Proof. Take arbitrary n_i.

Denote the prime factorization of $n_i = p_{i,1}^{\alpha_{i,1}} \cdots p_{i,j_i}^{\alpha i,j_i}$ where j_i is the amount of prime of n_i. Likewise, $m_i = q_{i,1}^{\alpha_{i,1}} \cdots q_{i,j_i}^{\alpha i,j_i}$. Note that the only difference between these factorizations is the values of the primes that are used. The powers and the number of distinct primes in the respective factorizations are the same. Consider arbitrary $(u_1, \cdots, u_k) \in \Gamma^T(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. Each u_i is a factor of n_i or 0. We can write $u_i = p_{i,1}^{x_{i,1}} \cdots p_{i,j_i}^{x_{i,j_i}}$ where $0 \leq x_{i,l} \leq \alpha_{i,l}$. Note that if u_i is 1, each $x_{i,l}$ is 0 and if u_i is 0, $x_{i,l} = \alpha_{i,l}$ for every l.

Defining a function $f : \Gamma^T(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \to \Gamma^T(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k})$ in a natural way component-wise, by using the bijective function in the proof of the last theorem we get the desired bijection. □

Theorem 2.18. $\Gamma^T(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ *is isomorphic to* $\Gamma^T(\mathbb{Z}_{n_1 \cdots n_k})$ *if all n_i's are mutually co-prime.*

Proof. The proof follows by Chinese Remainder theorem. □

The next theorem will show how we can characterize the perfectness of $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. Because now by the above three theorems, without loss of generality, we can simply choose primes that will make the n_i's mutually co-prime. Then we know the type graph will be isomorphic to $\Gamma(\mathbb{Z}_n)$ where n is the product of all such co-prime n_i. So, we have the following theorem.

Theorem 2.19. $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \cdots \mathbb{Z}_{n_k})$ *is perfect iff it is possible to find mutually coprime positive integers* $m_1, m_2 \cdots m_k$*, so that each* m_i *has same amount of prime factors with same exponent in its prime factorization as that in* n_i *and* $\Gamma(\mathbb{Z}_{m_1 m_2 \cdots m_k})$ *is perfect.*

Example 2.20. *For example,* $\Gamma(\mathbb{Z}_{p^2 q} \times \mathbb{Z}_p)$ *is perfect because* $\Gamma(\mathbb{Z}_{a^2 bc})$ *is perfect as shown by* [4]*. Also note, no product with a dimension greater than four can be perfect.* $\Gamma(\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_5})$ *is not perfect since no* $\Gamma(\mathbb{Z}_{p_1 \cdots p_5})$ *is perfect as shown by* [4]*.*

3. Some properties of $\Gamma(\mathbb{Z}_n)$

In this section, we characterize $\Gamma(\mathbb{Z}_n)$ by various qualities such as completeness, cordiality and clique number. A helpful construction used is the strong type graph. We define the strong type graph as the type graph with self-loops. We normally do not consider self-loops, in zero-divisor graphs and type graphs, but in the strong type graph, a vertex has a loop at it if it annihilates itself. We denote the strong type graph of $\Gamma(\mathbb{Z}_n)$ as $\Gamma^S(\mathbb{Z}_n)$.

Another construction used commonly in this section is n^*. Consider some $\Gamma(\mathbb{Z}_n)$. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, $n^* = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ where $\beta_i = \lceil \frac{\alpha_i}{2} \rceil$.

Lemma 3.1. *Two arbitrary vertices* u *and* v *in* $\Gamma(\mathbb{Z}_n)$ *that are both in the same type class* T_i *share an edge iff* T_i *has a self-loop in the strong type graph.*

Proof. Let T_i have a self-loop. Then $i^2 = 0$. Since every $u, v \in T_i$ are multiples of i, u and v will share an edge.

Conversely, let T_i does not have a self-loop. Take arbitrary u and v in T_i. According to the definition of type class, u and v are some multiple of i where $gcf(u,n) = i$ and likewise for v. We can write $u = ai$ and $v = bi$ where $gcf(a, n/i) = 1$ and $gcf(b, n/i) = 1$. Assume u and v share an edge. Then $uv = cn$, $abi^2 = cn$ where c is a natural number. So $\frac{abi^2}{n} = c$. Since T_i does not have a self-loop, $i^2 \neq 0$ which means n has a factor that is not a factor of i^2. Let this factor be called d. Let $\frac{g}{d}$ represent the simplified form of the fraction $\frac{i^2}{n}$ where d is not 1. By substitution, $\frac{abg}{d} = c$. This is a contradiction since a, b and g do not share a factor with n/i, so cannot cancel the d out of the denominator. Therefore, the expression cannot be equal to c, a natural number. So, u and v do not share an edge. □

As a result, we have

Theorem 3.2. $\Gamma(\mathbb{Z}_{p^2})$ *is complete where* p *is prime.*

Theorem 3.3. $\Gamma(\mathbb{Z}_{p^x})$ *where* p *is prime and* $x \geq 3$ *is not complete.*

Proof. Let $x \geq 3$.

Case 1: $p = 2$: p and $3p$ are distinct non-zero zero-divisors that are not connected.

Case 2: $p \neq 2$: p and $2p$ are distinct non-zero zero-divisors that are not connected. □

Theorem 3.4. $\Gamma(\mathbb{Z}_n)$*, where* $n \geq 2$ *is complete iff* $n = p^2$*.*

Proof. Let $\Gamma(\mathbb{Z}_n)$ be complete. Assume n has two or more distinct prime factors. Label the smallest such factor by p. Now choose another one as q. p is a zero divisor and shares an edge with n/p. Since p and q are both prime factors of n, $pq \leq n$. Also, since $p < q$, $p^2 < pq$. So $p^2 < pq \leq n$ which means p^2 is non-zero and distinct from p. p^2 shares an edge with n/p so p^2 is a distinct zero-divisor that does not share an edge with p, making $\Gamma(\mathbb{Z}_n)$ not complete. The converse follows by the above two Theorems.

□

Theorem 3.5. $\Gamma(\mathbb{Z}_n)$ *is k-partite if* $\Gamma^S(\mathbb{Z}_n)$ *is k-partite.*

 doi.org/10.32388/SA4XI4

Proof. Let $\Gamma^S(\mathbb{Z}_n)$ be k-partite. Then $\Gamma^S(\mathbb{Z}_n)$ can be partitioned into k disjoint subsets $S_1, S_2, \cdots, S_k$ such that no vertex in the same set share an edge. Partition $\Gamma(\mathbb{Z}_n)$ into a similar grouping $Q_1, Q_2, \cdots, Q_k$ where $u \in Q_i$ iff $u \in T_u \in S_i$. Consider arbitrary u and v, vertices of $\Gamma(\mathbb{Z}_n)$ that are in the same partitioned set Q_i.

Case 1: u and v are in different type classes.

Call such classes T_u and T_v. Then since u and v are both in Q_i, T_u and T_v are both in S_i which means T_u does not share an edge with T_v. So, by [4] u and v do not share an edge.

Case 2: u and v are in the same type class.

Call this class T_u. Then since $\Gamma^S(\mathbb{Z}_n)$ is k-partite, T_u does not form a loop with itself. Hence, by Lemma 3.1, u and v do not share an edge. □

Theorem 3.6. *$\Gamma(\mathbb{Z}_n)$ is complete k-partite if $\Gamma^S(\mathbb{Z}_n)$ is complete k-partite.*

Proof. Let $\Gamma^S(\mathbb{Z}_n)$ be complete k-partite. Then by the above theorem $\Gamma(\mathbb{Z}_n)$ is k-partite. Using the partition used in the above Theorem, if we let $\Gamma^S(\mathbb{Z}_n)$ be partitioned into k disjoint subsets $S_1, S_2, \cdots, S_k$, then $\Gamma(\mathbb{Z}_n)$ can be partitioned into k disjoint subsets $Q_1, Q_2, \cdots, Q_k$, where arbitrary vertex of $\Gamma(\mathbb{Z}_n)$ is in Q_i if its type class is in S_i. Consider arbitrary vertices in $\Gamma(\mathbb{Z}_n)$, u and v, that are not in the same Q_i. Then u and v must be in different type classes in two different S_i's. Call these classes T_u and T_v. Since $\Gamma^S(\mathbb{Z}_n)$ is complete k-partite, T_u and T_v share an edge. Then u and v share an edge by [4]. □

Remark 3.7. *The converse of Theorem 3.5 and 3.6 is not always true. If the zero-divisor graph is k-partite, but has a self-annihilating vertex, the strong type graph will have a self-loop, which prevents it from being k-partite. For example, $\Gamma(Z_9)$ is complete bi-partite, whereas $\Gamma^S(Z_9)$ is not.*

Theorem 3.8. *If n is square free, $\Gamma(\mathbb{Z}_n)$ is k-partite, where k is the number of distinct prime factors of n.*

Proof. Consider the strong type graph $\Gamma^S(\mathbb{Z}_n)$. Let, $n = p_1p_2\cdots p_k$. Partition the graph into k sets $S_1, S_2, \cdots, S_k$. A vertex T_a in the strong type graph is in S_i if $gcf(a, p_i) = 1$ and $gcf(a, p_h) > 1$ for all $h < i$.

We now claim that $S_1, S_2, \cdots, S_k$ covers all the vertices of $\Gamma^S(\mathbb{Z}_n)$.

Assume there is a T_a that is not in any S_i. Since T_a is a vertex, a must be a factor of n that is also less than n. So a must omit at least one p_i. So $gcf(a, p_i) = 1$. Since T_a is not in any S_i, there must exist some $h < i$ such that $gcf(a, p_h) = 1$. Choose the smallest index h of such p_h. Then T_a must be in S_h which is a contradiction.

Our next claim is any two vertices u and v in the same partition do not share an edge.

Consider arbitrary u and v in S_i. Both u and v do not contain p_i so they do not share an edge. So the strong type graph is k-partite.

By Theorem 3.5, $\Gamma(\mathbb{Z}_{p_1p_2\cdots p_k})$ is k-partite. □

Lemma 3.9. *Arbitrary type class T_a in $\Gamma^T(\mathbb{Z}_n)$ contains only one element iff $a = \frac{n}{2}$.*

Proof. Let $T_a \in \Gamma(\mathbb{Z}_n)$ have a type class that has only one element. Assume $a \neq \frac{n}{2}$. Since a is a factor of n, $\frac{n}{a} = f$ is also a factor of n. Note that $f \geq 3$.

Consider the vertex $a(f-1)$ of $\Gamma(Z_n)$. The quantity $(f-1)$ does not share any factors with f. Since $af = n$, $gcf(a(f-1), n) = a$. So $a(f-1) \in T_a$. Also note that $a < a(f-1) < n$. So $a(f-1)$ is a distinct vertex in T_a which is a contradiction. So $a = \frac{n}{2}$

Let $a = \frac{n}{2}$. Then a is the only element in T_a since $2a = n$. □

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Corollary 3.10. *Analogous to above, $T_{n/p}$ in $\Gamma^T(\mathbb{Z}_n)$ contains exactly $p-1$ elements if p is the smallest prime factor of n.*

Lemma 3.11. *There is at most one type class with only one element.*

Proof. Assume there are two or more distinct type classes that have only one element. Call two of these classes T_u and T_v. By Lemma 3.9, $u = v = \frac{n}{2}$ which is a contradiction. □

Theorem 3.12. *$\Gamma(\mathbb{Z}_n)$ is k-partite if $\Gamma^S(\mathbb{Z}_n)$ is k-partite or $\Gamma^T(\mathbb{Z}_n)$ is k-partite and the only self-connected vertex of $\Gamma(\mathbb{Z}_n)$ is $T_{\frac{n}{2}}$.*

Proof. Let $\Gamma^S(\mathbb{Z}_n)$ be k-partite. By Theorem 3.5, $\Gamma(\mathbb{Z}_n)$ is k-partite. Let $\Gamma^T(\mathbb{Z}_n)$ be k-partite and let $\Gamma^S(\mathbb{Z}_n)$ have only one self-connected vertex, $T_{\frac{n}{2}}$. Consider arbitrary distinct u and v, zero divisors of $\Gamma(\mathbb{Z}_n)$, that are in the same partition.

Case 1: u and v are in the same type class.

By Lemma 3.9, $T_{\frac{n}{2}}$ has only one element, so if u and v are distinct, they cannot be in $T_{\frac{n}{2}}$. Then, the type class they are in is not self-connected, so u and v do not share an edge.

Case 2: u and v are in different type classes.

Since u and v are in the same partition, their type classes are in the same partition and do not share an edge. Thus, u and v do not share an edge. □

Lemma 3.13. *A vertex in $\Gamma(\mathbb{Z}_n)$ annihilates itself iff it is a multiple of n^*.*

Lemma 3.14. *Consider two arbitrary vertices in $\Gamma(\mathbb{Z}_n)$, u and v such that u is a factor of v. The largest clique containing v, M_v has a magnitude greater than or equal to the M_u, the largest clique containing u.*

Proof. Take arbitrary vertices u and v in $\Gamma(\mathbb{Z}_n)$. Let u be a factor of v. Every element e in $M_u \setminus u$ has the property $eu = 0$. Then $\forall e \in M_u$, $ev = 0$. So a clique C exists with v and each e in $M_u \setminus u$. So, C is a click containing v magnitude of at least M_u. □

Theorem 3.15. *$cl(\Gamma(\mathbb{Z}_n)) \geq \frac{n}{n^*} + k - 1$ where k is the number of distinct primes having odd power in the prime factorization of n.*

Proof. The multiples of n^* form a clique. Call it C. An arbitrary vertex of C will be of the form an^* for $1 < a < \frac{n}{n^*}$. The number of elements in this clique is $\frac{n}{n^*} - 1$, so the clique number of the graph is at least $\frac{n}{n^*} - 1$. Now consider all vertices of the form n^*/q where q is an arbitrary odd-power prime in the prime factorization of n. Arbitrary n^*/q shares an edge with each an^* in C. Also, each n^*/q_1 shares an edge to each other n^*/q_2. Since k is the number of distinct odd powered primes in the prime factorization of n, $cl(\Gamma(\mathbb{Z}_n)) \geq \frac{n}{n^*} + k - 1$. □

Theorem 3.16. *$cl(\Gamma(\mathbb{Z}_n)) \leq \frac{n}{n^*} + k - 1$ where k is the number of odd-power primes in the prime factorization of n.*

Proof. Consider arbitrary clique C. Partition C into sets L and N where L is the set of vertices of C that are not multiples of n^* and N is the set of vertices of C that are multiples of n^*. Consider arbitrary vertex l_1 in L. Since l_1 is not a multiple of n^*, there must be some prime factor p_1 of n whose power in l_1 is less than half of its power in n. Every other l_i in L must have its p_1 factor with a power greater than or equal to half its power in n for it to share an edge with l_1. Consider another vertex l_2 in L. l_2 must also have a prime factor whose power is less than half its power in n, but it cannot be p_1. Call it p_2. So each l_i in L must have a distinct prime factor p_i that has a power less than or equal to half its power in n. Let m be the number of distinct prime factors of n. Then there can be a maximum of m many l_i in L. N has a maximum size of $\frac{n}{n^*} - 1$, so the clique number is at most $\frac{n}{n^*} + m - 1$.

Consider some e_1, a vertex in L whose corresponding p_1, has an even power in n. e_1 does not share an edge with n^*. This means the clique number is one less if n has an even-powered prime. Consider another e_2 that has an even p_2 whose power is less than half. Then e_2 does not share an edge with $p_1 n^*$. In general, a vertex e_i whose corresponding p_i has an even power does not share

doi.org/10.32388/SA4XI4

an edge with distinct vertices $p_1p_2\cdots p_{i-1}n^*$. So the size of C is reduced by the number of even powered primes of n. This value can be represented by $m-k$ where k is the number of odd-powered primes of n. Hence, since C is arbitrary, $cl(\Gamma(\mathbb{Z}_n)) \leq \frac{n}{n^*} + m - (m-k) - 1$. $cl(\Gamma(\mathbb{Z}_n)) \leq \frac{n}{n^*} + k - 1$. □

Theorem 3.17. $cl(\Gamma(\mathbb{Z}_n)) = \frac{n}{n^*} + k - 1$.

Proof. The proof follows by Theorem 3.15 and Theorem 3.16. □

Theorem 3.18. *There are no non-empty, non-complete, regular* $\Gamma(\mathbb{Z}_n)$.

Proof. Consider all $\Gamma(\mathbb{Z}_n)$ that are non-empty and not complete. Assume $\exists$ some regular graph among these graphs.

Case 1: $n = p^x$ where p is prime

If $x = 1$, the graph is empty, and if $x = 2$, the graph is complete, so $x \geq 3$. Then p is a vertex that shares an edge with $p-1$ many other vertices, and p^2 is a vertex that shares an edge with $p^2 - 1$ many other vertices. Since the graph is regular, $p - 1 = p^2 - 1$, thus $p = p^2$, which means $p = 1$ – i.e., a contradiction.

Case 2: $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, $m \geq 2$ and p_i are all prime

Vertex p_1 shares an edge with $p_1 - 1$ many other vertices, and the vertex p_2 shares an edge with $p_2 - 1$ many other vertices. Since the graph is regular, $p_1 - 1 = p_2 - 1$, thus $p_1 = p_2$ which is a contradiction since p_1 and p_2 are distinct.

So the only non-empty regular graphs are complete. □

Theorem 3.19. $\Gamma(\mathbb{Z}_n)$ *is chordal iff* $n = p^x$, $2p$ *or* $2p^2$*, where* p *is prime and* x *is a positive integer.*

Proof. Let $n = p^x$. Assume that $\Gamma(\mathbb{Z}_{p^x})$ is not chordal. Then $\exists$ a cycle C of length > 3, that has no chord. Let y be a vertex of C that is not a multiple of n^*. Then, since the power of p in y has a power strictly less than $\frac{x}{2}$, each neighbor must be a multiple of n^*. Then the two neighbors of y in C share an edge which is a chord. So all vertices in C must be a multiple of n^* which also causes a chord. So $\Gamma(\mathbb{Z}_{p^x})$ is chordal.

Let $n = 2p$. Then, $\Gamma(\mathbb{Z}_{2p})$ is chordal.

Let $n = 2p^2$. Assume $\Gamma(\mathbb{Z}_{2p^2})$ is non-chordal. Then $\exists$ a cycle C of length > 3 that has no chord.

Let a be a vertex of C in the type class T_p. Each neighbor of a must be a multiple of $2p$, and therefore, is in the type class T_{2p}. Each multiple of $2p$ shares an edge, so there exists a chord in C. So there can be no vertices in the type class T_p in C.

Let b be a vertex of C that is in the type class T_2. Every neighbor of b must be in the type class T_{p^2}. But there is only one element in T_{p^2} so b cannot have two distinct neighbors. So b is not a vertex of C.

So each vertex of C must be in either T_{p^2} or T_{2p}. Then since there is only one element of T_{p^2}, and the magnitude of C is at least 4, there are at least 3 elements of T_{2p} in C. Those 3 elements form a triangle since each multiple of $2p$ annihilates each other multiple of $2p$. But C can't have a triangle since it is chord-less. This is a contradiction.

Let n not be p^x, $2p$ or $2p^2$.

Case 1: $n = 2^x p^y$ where $y \geq 3$, $x \geq 1$ and p is an odd prime.

Then $2^x p - p^y - 2^{x+1} p - p^{y-1}$ is a chord-less cycle.

Case 2: $n = 2^x p^y$ where $x \geq 2$, $y \geq 1$ and p is an odd prime.

Then $2p^y - 2^x - p^y - 2^{x+1}$ is a chord-less cycle.

Case 3: $n = p^x q^y$ where $p, q \geq 3$ where $p \neq q$ are primes and x, y are non-zero.

Then $p^x - q^y - 2p^x - 2q^y$ is a chord-less cycle.

doi.org/10.32388/SA4XI4

Case 4: $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $k \geq 3$ and α_i is non-zero. Since $k \geq 3$, n has an odd prime factor p_1.

Then $p_1^{\alpha_1} - n/p_1^{\alpha_1} - 2p_1^{\alpha_1} - 2n/p_1^{\alpha_1}$ is a chord-less cycle.

So $\Gamma(\mathbb{Z}_n)$ is non-chordal if n is not p^x, $2p$ or $2p^2$. □

Lemma 3.20. *If $n^* \neq n$, $\Gamma(\mathbb{Z}_n)$ has a simplicial vertex.*

Proof. Let $n^* \neq n$. Then n/n^* is a vertex since n/n^* shares an edge with n^* which is not a multiple of n. Since every neighbor of n/n^* is a multiple of n^* and any two multiples of n^* share an edge, n/n^* is a simplicial vertex. So $\Gamma(\mathbb{Z}_n)$ has a simplicial vertex. □

Another construction n_* can be useful. It is similar to n^*, but for the odd powered primes, round down instead of up. Consider $\Gamma(\mathbb{Z}_n)$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Define n_* as $n_* = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ and $\beta_i = \lfloor \frac{\alpha_i}{2} \rfloor$ for all i.

Note that $n_* n^* = n$ and if n is square-free, $n_* = 1$.

Lemma 3.21. *Arbitrary vertex v in $\Gamma(\mathbb{Z}_n)$ is a simplicial vertex iff $v \in T_2$ or $v \in T_g$ where g is a factor of n_*.*

Proof. Take arbitrary v in $\Gamma(\mathbb{Z}_n)$. Let $v \in T_2$. Then v only shares an edge with vertices in $T_{n/2}$. By Lemma 3.9, $T_{n/2}$ has only one element, which makes a clique. So v is simplicial.

Let $v \in T_g$ where g is a factor of n_*. So $n_* = ag$ where a is a positive integer. Consider some vertex h in T_j that shares an edge with v. Then $jg = bn$ for some positive integer b. $\frac{jn_*}{a} = bn_* n^*$. $\frac{j}{a} = bn^*$. Then $j = abn^*$. So j is a multiple of n^* and therefore, h is a multiple of n^*. Since every multiple of n^* shares an edge with every other such multiple, v is a simplicial vertex.

Conversely, let v be neither in T_2 nor in any T_g where g is a factor of n_*. Then, since v is not in any T_g, v has some prime with a power greater than half of that in n. Call that prime p_x and its power in v, α_x. Let the type class of v be called T_w. Consider the type class $T_{n/w}$. Each vertex in $T_{n/w}$ shares an edge with v. Since $v \notin T_2$, $T_{n/w} \neq T_{n/2}$. So by Lemma 3.9, $T_{n/w}$ has more than one element. Since n/w has a power of p_x less than that of half in n, none of the vertices in $T_{n/w}$ share an edge with each other. So the neighbors of v do not form a clique. Hence, v is not simplicial. □

Theorem 3.22. *$\Gamma(\mathbb{Z}_n)$ has a simplicial vertex iff the prime factorization of n is not square free or n is an even greater than 2.*

Proof. Let n not be square-free. Then, $n^* \neq n$. So by Lemma 3.20, $\Gamma(\mathbb{Z}_n)$ has a simplicial vertex.

Let n be even. Then, 2 divides n. So, by the above lemma any $v \in T_2$ is a simplicial vertex.

Let n be square free and odd. 2 is therefore not a factor of n. Then consider arbitrary vertex x. x shares an edge with both n/x and $2n/x$. $2n/x$ is non-zero since x is necessarily odd, and n/x and $2n/x$ do not share an edge since n is odd. For if $\frac{n}{x}\frac{2n}{x} = ny$, $2n = yx^2$ and $n = \frac{yx^2}{2}$ which is a contradiction. So there are no simplicial vertices of $\Gamma(\mathbb{Z}_n)$. □

Note: It follows by [4], (observation 3.2), if in $\Gamma(\mathbb{Z}_n)$ a vertex u is simplicial then T_u is simplicial in $\Gamma^T(\mathbb{Z}_n)$. But, not conversely. For example, in $\Gamma^T(\mathbb{Z}_{12}), T_3$ is simplicial, where as 3 is not so in $\Gamma(\mathbb{Z}_{12})$.

Lemma 3.23. *If n has three or more distinct prime factors then, $\Gamma(\mathbb{Z}_n)$ is not $\gamma - \beta$ perfect.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $k \geq 3$. By [8], the domination number of $\Gamma(Z_n)$ is k. If there is a vertex cover V whose size is k, we claim that V must contain the vertex n/p_x for every p_x prime factor of n. Consider the vertex n/p_x for some p_x prime factor of n. Let n/p_x not be in V. Construct set $C = \{p_x p_i | 1 \leq i \leq k\}$. Since $n/p_x \notin V$, every element of C is in V. C has k many vertices, so V has at least k many vertices. Consider vertex p_x. p_x shares an edge with n/p_x which is not covered by V, so V has at least $k+1$ vertices. That is a contradiction since the size of V is k. So each n/p_x is in V.

Consider the type classes T_{n/p_1}, T_{n/p_2} and T_{n/p_3}. By Lemma 3.11, there can be at most one type class with only one element. At least two of these type classes have more than one element. Without loss of generality, let them be T_{n/p_1} and T_{n/p_2}. Since

n/p_1 and n/p_2 are both in V, choose different vertices in the type classes u and v. u and v share an edge since they are multiples of n/p_1 and n/p_2 respectively, so they share an edge, but are not in V, as the size of V is k. This is a contradiction to the assumption that V is a vertex cover. So, $\Gamma(\mathbb{Z}_n)$ is not $\gamma - \beta$ perfect. □

Theorem 3.24. *$\Gamma(\mathbb{Z}_n)$ is $\gamma - \beta$ perfect if $n = 2^3, 3^2, p, 2p$ and $3p$ for prime p.*

Proof. Let $n = 2^3$ or $n = 3^2$. The domination number clearly equals the size of the smallest vertex cover.

Let $n = p$. Then both the domination number and the smallest vertex map are 0 since the graph is empty.

Let $n = 2p$. Then the graph is a star, so the domination number and the vertex covering numbers of any of its induced subgraphs are both 1. Let $n = 3p$. Then $V = \{p, 2p\}$ is both a minimal dominating set and a minimal vertex cover of the original graph.

Now, we will show that all other $\Gamma(\mathbb{Z}_n)$ are not $\gamma - \beta$ perfect.

Let $n = 2^x$, $x \geq 4$. Then $2^{x-1} - 2^{x-2} - 3 \cdot 2^{x-2}$ is a triangle. Triangles prevent vertex covers of size 1, and by [8] the domination number is 1, so the values do not match.

Let $n = 3^x$, $x \geq 3$. Then $3^{x-1} - 2 \cdot 3^{x-1} - 3^{x-2}$ is a triangle that prevents vertex maps of size 1.

Let $n = p^x$, $p \geq 5$, $x \geq 2$. Then $p^{x-1} - 2 \cdot p^{x-1} - 3 \cdot p^{x-1}$ is a triangle.

Theorem 3.25. *For $n = p^x q$, the $\Gamma(Z_n)$ is not domination perfect.*

Case 1: $p = 2$.

Then $p^{x-1}q - p^x - q - p^{x+1} - pq$ is a non-induced sub-graph that cannot be covered by a vertex map size 2.

Case 2: $p \neq 2$.

Then $p^x - p^{x-1}q - p - 2p^{x-1}q - 2p$ is a non-induced sub-graph that cannot be covered by a vertex map size 2. The smallest vertex map is larger than 2 making the graph not $\gamma - \beta$ perfect.

Let $n = p^x q^y$, $x, y \geq 2$. The domination number is 2 by [8]. Assume there is a vertex map V size 2. Consider the edges $p - p^{x-}q^y$ and $q - p^x q^{y-1}$. V must contain at least vertex one of each edge. By Lemma 3.11 only one type class can have only one vertex. Consider the type classes $T_{p^x q^{y-1}}$ and $T_{p^{x-1}q^y}$. At least one of them must contain more than one vertex. Without loss of generality let that be $T_{p^{x-1}q^y}$. Then there exists some $u \in T_{p^{x-1}q^y}$ that is not in V. The edge $p - u$ is not covered by V, so the size of V is at least one more than 2 which is a contradiction.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $k \geq 3$. Then by Lemma 3.23, the graph is not $\gamma - \beta$ perfect.

So the only $\gamma - \beta$ graphs $\Gamma(\mathbb{Z}_n)$ are $2^3, 3^2, p, 2p$ and $3p$. □

4. Some properties of $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$

In this section, we discuss some facts about $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. It is often possible to relate some properties of the individual $\Gamma(\mathbb{Z}_{n_i})$ to the graph of the product. One example is that the domination number of $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ has an upper and lower bound corresponding to the domination number of each $\Gamma(\mathbb{Z}_{n_i})$.

Theorem 4.1. *Consider two arbitrary commutative rings with unity, R and S. $\Gamma(R \times S)$ is complete iff $|R| = |S| = 2$.*

Proof. Consider some R and S such that $|R| = |S| = 2$. Since both R and S have 1, the only elements of R and S are 0 and 1, where by 1 we denote the unity of the respective ring. Then the zero divisor graph is $(0, 1) - (1, 0)$ which is complete.

Conversely, let R or S have more than 2 elements. Without loss of generality, let R have more than 2 elements. Then R has some

doi.org/10.32388/SA4XI4

element a that is neither 1 nor 0. The graph $\Gamma(R \times S)$ has vertices $(1, 0)$ and $(a, 0)$. These vertices do not share an edge because $1 \cdot a = a$ which is not zero. So $\Gamma(R \times S)$ is not complete. □

Theorem 4.2. $\Gamma(R_1 \times \cdots \times R_k)$ *where* $k \geq 2$ *and each* R_i *is a commutative ring with* 1*. This graph is complete iff* $k = 2$ *and* $|R_i| = 2$ *for all* i.

Proof. One direction follows from the last theorem.

Consider some $\Gamma(R_1 \times \cdots \times R_k)$ that does not meet these criteria. If $k \geq 3$, then $(1, 0, 1)$ and $(1, 1, 0)$ are two vertices that do not share an edge. If any $|R_i| \geq 2$, then R_i has an element a that is not 0 or 1. Then $(\cdots, a, \cdots)$ does not share an edge with $(\cdots, 1, \cdots)$, where a and 1 are placed in the $i - th$ entry of the respective elements. So $\Gamma(R_1 \times \cdots \times R_k)$ is not complete. □

Theorem 4.3. $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ *where* $n, m \geq 2$ *is complete-bipartite iff* n *and* m *are prime.*

Proof. Let m and n be prime. Then construct S_n and S_m such that $S_n = \{(x, 0)|0 < x < n\}$ and $S_m = \{(0, y)|0 < y < m\}$.

We claim that $S_n \cup S_m = V(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$.

Assume, $\exists$ a zero divisor $a = (a_1, a_2)$ that is not in $S_n \cup S_m$. Both a_1 and a_2 are non-zero. Since a is a zero-divisor, there must be some $b = (b_1, b_2)$ that shares an edge with a. So $a_1 b_1 = 0$. Since $\mathbb{Z}_n$ has no non-zero divisors, and a_1 is not zero, $b_1 = 0$. In the same way, we find that b_2 is zero. This means a is not a zero-divisor because it only shares an edge with 0. So $S_n \cup S_m = V(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$.

Take arbitrary $u, v \in S_n$. Then $u = (u_1, 0)$ and $v = (v_1, 0)$. Since $u_1 v_1 \neq 0$, $uv \neq (0, 0)$ which means u and v do not share an edge. In the same way, u and v do not share an edge if they are both in S_m. So u and v do not share an edge if they are in the same partition which is the definition of bipartite.

Thus, it follows from the construction of S_m and S_m, that $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is complete bipartite.

Conversely, let $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ be complete bipartite. Assume one or both n and m are not prime. Let the non-prime be n. Then, there is a non-zero zero divisor of $\mathbb{Z}_n$. Call it k. Since $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is complete-bipartite, the vertices of $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ can be partitioned into 2 disjoint subsets such that no edges exist between two vertices in the same partition, and every pair of vertices in different partitions share an edge. $(1, 0)$ is a zero divisor since it shares an edge with $(0, 1)$. $(k, 0)$ is also a zero divisor since it also shares an edge with $(0, 1)$. Since $(k, 0)$ does not share an edge with $(1, 0)$, they must be in the same partition. Call it S_1 and let the other partition be S_2. Since k is a zero-divisor of $\mathbb{Z}_n$, $\exists k'$ not necessarily distinct such that $k \cdot k' = 0$. Then $(k', 1)$ shares an edge with $(k, 0)$ which means $(k', 1) \in S_2$. Since $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is complete-bipartite, $(1, 0)$ must share an edge with $(k', 1)$ since they are in opposite partitions, but their product is not 0, which is a contradiction. So both n and m must be prime. □

Corollary 4.4. *From this theorem it follows that* $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ *has a complete bipartite sub-graph.*

One way to form this is by constructing S_n by taking all the non-zero elements in Z_n that are not zero divisors of n in the first entry and accordingly for S_m.

Theorem 4.5. $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ *where* $\forall n_i \geq 2$ *and* $k \geq 2$ *is bipartite iff* $k = 2$ *and both* n_i *are prime, or one* n_x *is prime and the other is* 4.

Proof. Let $k = 2$ and both n_1 and n_2 be prime. By Theorem 4.3, $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2})$ is bipartite.

Let $k = 2$ and let one of n_i be 4 and the other be prime. Without loss of generality, let $n_1 = 4$. Then n_2 is prime. Partition the vertices into sets A and B where A is the set of all vertices of the form $(a, 0)$ where $a \in \mathbb{Z}_{n_1} \setminus \{0\}$ and B is everything else. Consider arbitrary, distinct elements of A, $(a_1, 0)$ and $(a_2, 0)$. They do not share an edge. Consider all vertices in B. Assume

 doi.org/10.32388/SA4XI4

$\exists u, v \in B$ such that u shares an edge with v. Then, $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Note that $u_2 v_2 \neq 0$, as n_2 is a prime. So $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ is bipartite.

Conversely, let $\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ be bipartite.

We first claim, that $k = 2$.

Assume $k \geq 3$. Then, $(1, 0, 0, \cdots, 0) - (0, 1, 0, \cdots, 0) - (0, 0, 1, \cdots, 0)$ is a triangle which cannot exist in a bipartite graph. So $k < 3$. By our definition, $k \geq 2$, so $k = 2$

We now claim no $\Gamma(\mathbb{Z}_{n_i})$ can have two or more distinct zero divisors.

Assume otherwise. Call two such divisors u and v that share an edge in $\Gamma(\mathbb{Z}_{n_i})$. Without loss of generality, let u and v be in the first slot (so $i = 1$). Then $(u, 0) - (v, 0) - (0, 1)$ is a triangle that cannot exist in a bipartite graph. The only $\Gamma(\mathbb{Z}_{n_i})$ that has one element is $\Gamma(\mathbb{Z}_4)$. So all n_i must be either 4 or prime.

Our final claim is it is not possible for both n_i to be 4.

Assume otherwise. Then $(2, 0) - (2, 2) - (0, 2)$ is a triangle which cannot exist in a bipartite graph. So, because $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ is bipartite, $k = 2$ and either both n_i are prime, or one is 4 and the other is prime. □

Theorem 4.6. *$\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$ is k-partite where every p_i is prime.*

Proof. Consider some graph $\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$. Construct a collection of subsets S_i which is the set of all vertices with a non-zero term in the ith slot and zero in any slot less than i.

$S_1 = \{(a, \cdots) | a \in \mathbb{Z}_{p_1}, a \neq 0\}$

$S_2 = \{(0, a, \cdots) | a \in \mathbb{Z}_{p_2}, a \neq 0\}$

$\cdots$

$S_k = \{(0, 0, \cdots, 0, a) | a \in \mathbb{Z}_{p_k}, a \neq 0\}$

No two vertices u, v from the same subset S_x share an edge.

All these S_i form a partition of $\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$.

So $\Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k})$ is k-partite. □

Theorem 4.7. *$\Gamma(R_1 \times \cdots \times R_k)$ where each R_i is a commutative ring with 1 is not perfect if some $\Gamma(R_i)$ is not perfect.* □

Proof. Let some $\Gamma(R_i)$ be non-perfect. Then by the Strong Perfect Graph theorem, there exists an odd hole or anti-hole H of length 5 or greater. Let H have a length l. Then we write it as, $v_1 - v_2 - \cdots - v_{l-1} - v_l - v_1$. Then to obtain a hole or antihole of length 5 or greater in $\Gamma(R_1 \times \cdots \times R_k)$ fill in the ith position with the vertices of H, and fill the rest in with zeros. $(0, \cdots, 0, v_1, 0, \cdots, 0) - (0, \cdots, 0, v_2, 0, \cdots, 0) - \cdots - (0, \cdots, 0, v_{l-1}, 0, \cdots, 0) - (0, \cdots, 0, v_l, 0, \cdots, 0) - (0, \cdots, 0, v_1, 0, \cdots, 0)$ is a hole or antihole of odd length of 5 or greater making the graph $\Gamma(R_1 \times \cdots \times R_k)$ non perfect. □

Note 4.8. *The converse of Theorem 3.4 is not true. In the graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, every $\Gamma(\mathbb{Z}_2)$ is perfect, but we find the hole $(1, 1, 0, 0, 0) - (0, 0, 1, 1, 0) - (1, 0, 0, 0, 1) - (0, 1, 1, 0, 0) - (0, 0, 0, 1, 1)$.*

Theorem 4.9. *$\Gamma(R_1 \times \cdots \times R_x)$ where each R_i is a commutative ring with 1 is not regular if any $\Gamma(R_i)$ is not empty.*

Proof. Take $\Gamma(R_1 \times \cdots \times R_x)$. Let some $\Gamma(R_i)$ be non-empty. Consider the vertex $g = (0, \cdots, 0, 1, 0, \cdots, 0)$ that has a 1 at the i^{th} index and 0 filled in all other indices. All neighbors of g must be of the form $(a_1, a_2, \cdots, a_{i-1}, 0, a_{i+1} \cdots, a_{x-1}, a_x)$, with a zero at the i^{th} index and any value in the other indices, not all zero. Let there be f many such vertices. Since $\Gamma(R_i)$ is non-empty, $\exists k \in \Gamma(R_i)$. Since k is a zero divisor, there must be some $k' \in \Gamma(R_i)$, not necessarily distinct, such that $k \cdot k' = 0$. Consider the vertex $h = (0, \cdots, 0, k, 0, \cdots, 0)$ with k in the i^{th} index. This vertex shares an edge with all vertices that share an edge with g. So

doi.org/10.32388/SA4XI4

h shares an edge with at least f vertices. But it also shares an edge with $(1, \cdots, 1, k', 1 \cdots, 1)$ which means h shares an edge with at least $f + 1$ vertices. This means g and h have a different number of neighbors, so $\Gamma(R_1 \times \cdots \times R_x)$ is not regular. □

Theorem 4.10. *For arbitrary rings R and S, $cl(\Gamma(R \times S)) \geq cl(\Gamma(R)) + cl(\Gamma(S)) + |R'||S'|$ where R' and S' are any set of self-annihilating vertices in a maximal clique of $\Gamma(R)$ and $\Gamma(S)$.*

Proof. Let C be a maximal clique in $\Gamma(R)$ and D be a maximal clique in $\Gamma(S)$. Construct an induced sub graph $X = \{(c, 0), (0, d) | c \in C, d \in D\}$. X is a clique in $\Gamma(R \times S)$ with size $cl(\Gamma(R)) + cl(\Gamma(S))$.

Now consider R', the set of all self-annihilating vertices in C and S', the set of all self-annihilating vertices in D. Define the induced sub-graph $Y = \{(r, s) | r \in R', s \in S'\}$. Every vertex $(r, s) \in Y$ shares an edge with every other vertex in Y and every vertex in X, so $X \cup Y$ forms a clique size $cl(\Gamma(R)) + cl(\Gamma(S)) + |R'||S'|$. □

Corollary 4.11. *Consider n many arbitrary rings $R_1, R_2, \cdots R_n$. Then,*

$cl(\Gamma(R_1 \times R_2 \cdots R_n)) \geq \sum_{i=1}^{n} cl(\Gamma(R_i)) + \sum_{i \neq j, i, j \in \{1, 2, \cdots n\}} |R'_i||R'_j| + \sum_{i \neq j \neq k; i, j, k \in \{1, 2, \cdots n\}} |R'_i||R'_j||R'_k| + \cdots + |R'_1||R'_2| \cdots |R'_k|$,

where each R'_i is any set of self-annihilating vertices in a maximal clique in $\Gamma(R_i)$.

Lemma 4.12. *Consider $\Gamma(\mathbb{Z}_n)$ for arbitrary n. There is a maximal clique M that contains all self-annihilating vertices.*

Proof. Follows from Theorem 3.15 and Lemma 3.13. □

Theorem 4.13. *The clique number of $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ has a lower bound of $cl(\Gamma(\mathbb{Z}_n) + cl(\Gamma(\mathbb{Z}_m)) + (\frac{n}{n^*} - 1)(\frac{m}{m^*} - 1)$.*

Proof. Follows from Theorem 4.10 and the proof of Theorem 3.15 and Lemma 3.13. □

Theorem 4.14. *$\Gamma(R_1 \times \cdots \times R_k)$ where $k \geq 2$ and R_i is a commutative ring with 1 has a simplicial vertex iff some $\Gamma(R_i)$ has a simplicial vertex or some $|R_i| = 2$.*

Proof. Take arbitrary $\Gamma(R_1 \times \cdots \times R_k)$. Let some $\Gamma(R_i)$ have a simplicial vertex c. Then the vertex $(1, \cdots, 1, c, 1, \cdots, 1)$ where c is in the ith slot is a simplicial vertex of $\Gamma(R_1 \times \cdots \times R_k)$.

Let some $|R_i| = 2$. Then $(1, \cdots, 1, 0, 1, \cdots, 1)$ only shares an edge with $(0, \cdots, 0, 1, 0, \cdots, 0)$ making $(1, \cdots, 1, 0, 1, \cdots, 1)$ simplicial.

Let $\Gamma(R_1 \times \cdots \times R_k)$ have a simplicial vertex v. Also, assume all $|R_i| > 2$ and no $\Gamma(R_i)$ have any simplicial vertices. Consider arbitrary v in $\Gamma(R_1 \times \cdots \times R_k)$. Let v have 0 at some index, $v = (\cdots, 0, \cdots)$. Then since no $|R_i| = 2$, there exists some vertex $a \in R_i$ that is not 0 or 1. v then shares an edge with $(0, \cdots, 0, 1, 0, \cdots, 0)$ and $(0, \cdots, 0, a, 0, \cdots, 0)$ and they do not share an edge. So for v to be simplicial, it cannot contain any 0. Let v have a at some index, where a is a zero divisor in its respective $\Gamma(R_i)$. $v = (\cdots, a, \cdots)$. Then v shares an edge with every $(0, \cdots, 0, a', 0, \cdots, 0)$ where $a \cdot a' = 0$ in $\Gamma(R_i)$. a is not simplicial since no $\Gamma(R_i)$ have any simplicial vertex, so some neighbor $(0, \cdots, 0, a', 0, \cdots, 0)$ will not share an edge with another neighbor of the same form. So v is not simplicial if it has any zero-divisors in its slots. For v to be simplicial, every slot must be a non-zero, non-zero-divisor. However, elements of that form are not vertices. So $\Gamma(R_1 \times \cdots \times R_k)$ has no simplicial vertices, which is a contradiction. The assumption that all $|R_i| > 2$ and no $\Gamma(R_i)$ have any simplicial vertices is false. So some $|R_i| > 2$ or some $\Gamma(R_i)$ has a simplicial vertex. □

Theorem 4.15. *$\Gamma(R_1 \times \cdots \times R_k)$ where R_i is a commutative ring with 1 is non-chordal if any $\Gamma(R_i)$ is non-chordal.*

Proof. Consider arbitrary $\Gamma(R_1 \times \cdots \times R_k)$. Then let some $\Gamma(R_i)$ be non-chordal. So there exists a cycle $a_1 - a_2 - \cdots - a_k - a_1$ greater than 3 with no chords. Then in $\Gamma(R_1 \times \cdots \times R_k)$, there is a cycle $(0, .., a_1, \cdots, 0) - (0, \cdots, a_2, \cdots, 0) - \cdots - (0, \cdots, a_k, \cdots, 0) - (0, \cdots, a_1, \cdots, 0)$, which makes it non-chordal. □

Lemma 4.16. $\Gamma(R_1 \times \cdots \times R_k)$ *where* R_i *is a commutative ring with 1 and* $k \geq 2$ *is non-chordal if more than one* $|R_i| \geq 3$.

Proof. In $\Gamma(R_1 \times \cdots \times R_k)$, let two or more $|R_i| \geq 3$. Without loss of generality, let the first two slots be the R_i with a magnitude greater than or equal to 3. Then $(1,0,\cdots,0) - (0,1,\cdots,0) - (a,0,\cdots,0) - (0,b,\cdots,0)$ where a is a non-trivial element of R_1 and b is a non-trivial element of R_2, is a cycle with no chord. So $\Gamma(R_1 \times \cdots \times R_k)$ is non-chordal. □

Lemma 4.17. $\Gamma(R_1 \times \cdots \times R_k)$ *where* R_i *is a commutative ring with 1 is non-chordal if* $k \geq 4$.

Proof. Let $k \geq 4$. Then $(1,1,0,0,\cdots,0) - (0,0,1,1,\cdots,0) - (1,0,0,0,\cdots,0) - (0,0,0,1,\cdots,0)$ is a chord-less cycle. So $\Gamma(R_1 \times \cdots \times R_k)$ is non-chordal. □

Lemma 4.18. $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3})$ *where at least one* $n_i > 2$ *is non-chordal.*

Proof. Without loss of generality, let $n_3 > 2$. Then,

$(1,0,0) - (0,0,2) - (1,1,0) - (0,0,1)$ is a chord-less cycle. □

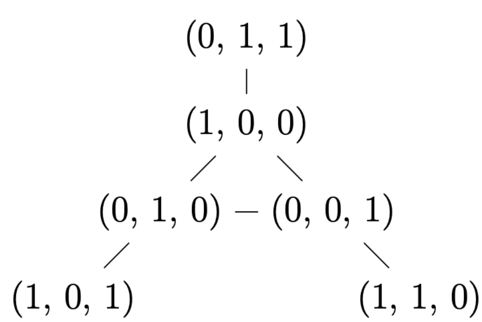
Theorem 4.19. *The only chordal* $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ *where* $n_i \geq 2$ *and* $k \geq 2$ *are* $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^2})$ *and* $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

Proof. Consider $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$. Since $\Gamma(\mathbb{Z}_p)$ has no vertices, the only vertices of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ are $(1,0)$ or of the form $(0,x)$ where $0 < x < p$. So the graph is a star making it chordal.

Consider $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^2})$. Assume that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^2})$ is non-chordal. Then there exists a cycle C length greater than 3 that has no chord. Let v be an arbitrary vertex in C.

Let v have a multiple of p as its second entry, $v = (a, bp)$. Then every vertex that is not a neighbor of v in C must have a non-zero non-multiple of p as its second element. Therefore, both neighbors of v must have 0 as their second element so that they share an edge with their other neighbor. So both neighbors of v are $(1,0)$. We cannot repeat vertices so v cannot have a multiple of p as its second element. That means the only possible vertices in C are $(1,0)$ and $(0,b)$ where b is a non-zero non-multiple of p. A cycle of size 4 or greater cannot be constructed out of these vertices since we cannot write $(1,0)$ more than once and a vertex of such form $(0,b)$ does not share an edge with a vertex of the same form. C cannot be constructed, so $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^2})$ is chordal.

Consider $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. The graph of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is shown below and is chordal.



To prove the converse, let's assume the opposite. Let there be a chordal $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ not listed. By Lemma 4.16, only one n_i can be greater than 2. By Lemma 4.17, $k \leq 3$. By Theorem 4.15, if any n_i are non-chordal, $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ will be non-chordal. So every n_i must be p^x, $2p$, or $2p^2$ which was shown by Theorem 3.19.

 doi.org/10.32388/SA4XI4

So the only possible $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$ are $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^x})$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{2p})$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{2p^2})$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^x})$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p})$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p^2})$.

In $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^x})$ where $x \geq 3$ and p is prime, $(1, p^{x-1}) - (0, (p-1)p) - (1, 0) - (0, p)$ is a chord-less cycle.

In $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{2p})$ where $p \geq 3$ is a prime, $(1, 0) - (0, 4) - (1, p) - (0, 2)$ is a chord-less cycle.

In $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{2p^2})$ where $p \geq 3$ is a prime, $(1, 2p) - (0, p) - (1, 4p) - (0, p^2)$ is a chord-less cycle.

By Lemma 4.17, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^x})$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p})$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p^2})$ are all non-chordal where $p \geq 3$.

So there are no other chordal $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k})$. □

5. Zero divisor graph of the poset D_n

Zero divisor graph of a poset has been studied in [5], [6], [7]. We always have the Clique number of the zero-divisor graph of a ring that does not exceed the Chromatic number of that. Beck conjectured that for an arbitrary ring R, they are the same. But Anderson and Naseer [9] have shown that this is not the case in general, namely, they presented an example of a commutative local ring R with 32 elements for which Chromatic number is strictly bigger than the clique number. In [9] Nimbhorkar, Wasadikar and DeMeyer have shown that Beck's conjecture holds for meet-semilattices with 0, i.e., commutative semigroups with 0 in which each element is idempotent. In fact, it is valid for a much wider class of relational structures, namely for partially ordered sets (posets, briefly) with 0. Now, to any poset $(P, \leq)$, with a least element 0 we can assign the graph G as follows: its vertices are the nonzero zero divisors of P, where a nonzero $x \in P$ is a called a zero divisor if there exists a non-zero $y \in P$, so that $L(x, y) = 0, L(x, y) = \{z \in P | z \leq x, y\}$. And x, y are connected by an edge if $L(x, y) = 0$. We discuss here some properties of the zero-divisor graph of a specific poset D_n. Very often we used the prime factorization of the positive integer n. By abuse of notation, let us call D_n as the zero-divisor graph of the poset D_n. Note that, the vertex set of D_n is the set of all factors of n that are not divisible by some prime factor of n. Also, note that two vertices in D_n are connected by an edge if and only if they are mutually co-prime.

Remark 5.1. (Properties of D_n).

i. *If $n = p^m$ for some prime p and positive integer m, then D_n is trivial.*

   So from now on consider D_n where $n \neq p^m$ where p and m are as mentioned.

ii. *The diameter of D_n is 3 iff n has three distinct prime factors namely p, q, r. This is shown by the path $pq - r - p - qr$. Otherwise, the diameter is 1 or 2, as $D_{p^m q^n}$ is complete bipartite which has diameter 2 or in the case of $m = n = 1$ has diameter 1.* [10] *shows zero divisors of a poset have diameter of 1, 2, or 3.*

iii. *D_n is complete only when $n = pq$, where p and q are two distinct primes. D_n is complete bipartite iff $n = p^m q^s$ where m and s are two positive integers.*

iv. *We have the clique number of D_n and a few coefficients of the clique polynomial. The clique number of D_n is the number of distinct prime factors of n. For if $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$ where p_i's are distinct primes $\forall i$, any set of vertices $\{p_1^{\beta_1}, p_2^{\beta_2}, p_3^{\beta_3} \cdots p_r^{\beta_r}\}$, where $1 \leq \beta_i \leq \alpha_i \ \forall i$ forms a maximal clique. Hence the clique number is r, the number of distinct primes of n. And the leading coefficient in the clique polynomial is $\alpha_1 \alpha_2 \cdots \alpha_r$. The coefficient of x^{r-1} is $\sum_{i=1}^{r} (\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_r) + \binom{r}{2} \alpha_1 \alpha_2 \cdots \alpha_r$. Reason: Consider a clique of size $r - 1$. If all the vertices have single prime factors then, there are $\sum_{i=1}^{r} (\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_r)$ many of this type, as a typical clique of this type is a set of the form*

doi.org/10.32388/SA4XI4

$\{p_1^{\beta_1}, p_2^{\beta_2}, \cdots p_{i-1}^{\beta_{i-1}}, p_{i+1}^{\beta_{i+1}}, \cdots p_r^{\beta_r}\}$, *where* $1 \le \beta_j \le \alpha_j \forall j \in \{1, 2, \cdots r\}$. *Otherwise, exactly one vertex will contain two primes. And in that case, we will obtain* $\binom{r}{2}\alpha_1\alpha_2\cdots\alpha_r$ *many such clique sets with cardinality* $r-1$.

v. *The domination number of* D_n *is the number of distinct prime factors of* n, *the same as the clique number, as any dominating set must not omit a prime factor of* n. *If some* p_i *is missing from a set of vertices* V, *then the vertex* $p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_r$ *is not adjacent to any vertex in* V. *Furthermore, if we let* V *be the set of all distinct primes of* n, *each vertex in* D_n *must share an edge with at least one vertex in* V *because each vertex in* D_n *must omit at least one prime of* n *from its prime factorization.*

vi. D_n *is regular iff* $n = (pq)^m$ *for some positive integer* m. *If* $n = p^m q^r, m \neq r$, *then* $D_n = K_{m,n}$, *complete bipartite which is not regular. Then, if* n *has more than two distinct primes in its prime factorization, then for two distinct primes* p *and* q *in its prime factorization* p *and* pq *are vertices with different degrees making the graph non-regular.*

vii. *In* [6], *it is discussed that the girth of the zero divisor graph of any poset is 3, 4, or* ∞. *The girth of* D_n *is* ∞ *iff* $n = p^m q$, *where* p *and* q *are two distinct primes and* m *is a positive integers bigger than* 1. *The girth of* D_n *is 4, if and only if* $n = p^m q^r$, *where* p *and* q *are two distinct primes and* m *and* r *are both positive integers bigger than* 1. *Otherwise, the girth of* D_n *is 3, because if* n *has at least 3 different prime factors* p, q *and* r, *then* $p - q - r - p$ *is a triangle in* D_n.

viii. D_n *is not perfect iff* n *is the product of least five different primes* p, q, r, s, t *in its prime factorization, then* $ps - qt - pr - qs - tr - ps$ *is a cycle of length five in* D_n. *Hence by Strong perfect graph theorem* D_n *is not perfect.*
Suppose n *has 4 distinct prime factors* p, q, r *and* s. *Assume there is an odd cycle of length 5 or greater that contains a vertex* v *that is the product of two such primes. Let* $v = p^x q^y$. *Then the two neighbors of* v *cannot be a multiple of* p *or* q. *Suppose the neighbors both consist of* r^a *for some positive integer* a. *Then, we get part of the cycle as* $r^a - p^x q^y - r^b$ *for another positive integer* b. *Then,* r^a *will necessarily share an edge with the other neighbors of* r^b *making the cycle length 4. So, the neighbors of* v *must have both* r *and* s. *Additionally, these parts of the cycle must be of the form* $r^a - p^x q^y - r^w s^z$; *otherwise, we get a cycle of length* 4 *again. But any vertex that shares an edge with* $r^w s^z$ *must also share an edge with* r^a *making such a cycle impossible. This means any odd cycle length greater than 5 cannot contain a vertex with two or more prime factors, making an odd cycle length greater than 4 impossible. The other two situations when* v *consists of only one prime, or three primes also give contradiction. Thus,* D *is perfect iff* n *has 4 or fewer prime factors.*

ix. D_n *is chordal iff* $n = p^m q$ *or* $n = pqr$ *where* p, q *and* r *are distinct primes and* $m \ge 1$. *For if* n *is not of that form,* $p - q - p^2 - q^2 - p$ *or* $p - q - p^2 - qr - p$ *or* $p - r - pq - rs - p$ *will give holes of length greater than 3 in respective* D_n*'s.*

x. *Let,* n *be a square free positive integer. Then, its simplicial vertices are precisely those factors of* n *that miss exactly one prime in its prime factorization. Now, suppose* n *is not square-free. Then, if all primes in its prime factorization are not square-free, it has no simplicial vertex. Otherwise, the simplicial vertices are precisely those that miss exactly one square free prime factor. For example, if* $n = p^2q^2r$, pq, p^2q, pq^2 *and* p^2q^2 *are the only simplicial vertices because* r *is the only square free prime factor.*

xi. *The only planar* D_n *has* n *of the form* $n = p^m q$, $p^m q^2$, pqr *or* p^2qr. *First, let* n *have only 2 prime factors. If* $n = p^m q^l$ *where* $l \ge 3$ *and* $m \ge 3$, *then* $K_{3,3}$ *is a subgraph of* D_n. *So, by Wagner's theorem,* D_n *is non-planar. But in the case of* $p^m q$, D_n *is a star, so it is planar. And, in the case of* $p^m q^2$, *the graph can be drawn without any crossing edges. Next, let's have three prime factors. If* $n = pqr$ *or* $n = p^2qr$ *the graph is clearly planar if drawn. If* $n = p^m qr$ *where* $m \ge 3$, *the subgraph consisting of* p, p^2, p^3, q, r *and* qr *form* $K_{3,3}$ *if we delete the edge between* q *and* r. *Then by Wagner's theorem, the graph is non-planar since* $K_{3,3}$ *is a minor. Next, if* $n = p^m q^l r$, *where* $m \ge 2$ *and* $l \ge 2$ *the set of vertices* q, q^2, p, p^2, r, pr *and* qr *is a subdivision of* K_5. *Then, by Kuratowski's theorem, the graph is non-planar. So the only planar* D_n *with only 3 primes in* n *are* pqr *and* p^2qr. *Lastly, consider the case where* n *has 4 primes in its prime factorization,* $n = pqrs$. *Then, the vertex set of* p, q, r, s, pq *and* rs *can be made isomorphic to* K_5 *by* *contracting the edge between pq and rs to make a single vertex. Therefore, K_5 is a minor of D_n for this case, and by Wagner's theorem the graph is non-planar.*

xii. *D_n is Eulerian iff the power of each prime in the prime factorization of n is even.*

For, if n has a prime p^α that appears in its prime factorization where α is odd, then the vertex $\frac{n}{p^\alpha}$ has an odd degree, otherwise every vertex has even degree.

xiii. *If n is square free, then we have the edge cardinality of D_n as $\sum_{i=1}^{r-1} 2^{r-i-1}\binom{r}{i} - 2^{r-1} - 1$, where r is the number of distinct primes of n.*

For, if we consider $n = p_1 p_2 \cdots p_r$, where p_i's are distinct primes, then the degree of each vertex p_i is $\sum_{i=1}^{r-1}\binom{r-1}{i} = 2^{r-1} - 1$ giving $r(2^{r-1} - 1)$ to the degree sum of the vertices. Similarly each vertex $p_i p_j$ is adjacent to $\sum_{i=1}^{r-2}\binom{r-2}{i} = 2^{r-2} - 1$ many vertices, giving $\binom{r}{2}(2^{r-2} - 1)$ in the degree sum. Proceeding in this way, we obtain the sum of the vertex degrees are $\sum_{i=1}^{r-1}\binom{r}{i}(2^{r-i} - 1) = \sum_{i=1}^{r-1}\binom{r}{i}2^{r-i} - 2^r - 2$. Then, as the sum of vertex degrees is twice the edge cardinalities the result follows.

Acknowledgments

The authors acknowledge Dr. Lisa DeMeyer for introducing this topic to us with an excellent presentation, which motivated us to work in this area.

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Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.