Research Article

The Measurement of Superpositions

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Superposition is a quantum mechanical concept currently without a recognized equivalent in metrology measurements. Relative Measurement Theory (RMT) developed the significance of unit standards to metrology measurements. This paper introduces a limit into Cantor's Transfinite Set Theory (CTST) and develops how it relates to a unit standard. Recognizing CTST also describes a superposition, this paper develops how a transfinite superposition transforms into a normal distribution of metrology measurement results, when measurement results include calibration to a unit standard. Then a superposition has an equivalent set of metrology measurements.

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1. Introduction

There are discrepancies and misunderstandings related to finite measurements (both empirical and theoretical) and their results across scientific disciplines:

- Gödel's incompleteness theorems[1] must apply to finite measurement systems.
- Finite measurement results must meet Tarski's undefinability requirement.[2]
- What is the "non-local structure" of a measurement that J. S. Bell identified?[3]
- How normal measurement distributions (without noise or distortion) occur?[4]
- The discrepancies between quantum and classical measurement results.[5]
- The paradoxes in CTST (see section 3.2).
- How can completely different properties (e.g., wave and particle) of the same entity appear?[6]

These can all be resolved with a deeper understanding of metrology unit standards and how they impact finite measurements.

J. C. Maxwell defined a finite measurement result as a quantity (1), where he assumed each *unit*[1] was equivalent to a *unit* standard.[7] In metrology (empirical measurement process), the measurement instrument[2] units Maxwell applies are calibrated to a unit standard, which establishes the precision[8] of the measurement instrument units relative to a unit standard.

$$\text{quantity } (Q) = \text{numerical value } (x) \text{ times } \textit{unit } (u) \tag{1}$$

Because Maxwell did not include accuracy or precision in (1), it implies that accuracy and precision are experimental artifacts of an empirical measurement process. This has become the current understanding of empirical measurement. Thus in representational measurement theory (the current theory of measurements in physics),[9] his implication has been formalized

and each *unit* is incorrectly assumed to equal a unit standard after empirical calibration. In effect, representational measurement theory requires that calibration occurs, but does not define it, which is backwards. A valid measurement theory must define calibration for measurement apparatus units to equal unit standards.

In addition the numerical value of a *unit* standard in this representational theory is often treated as unity (i.e., without formal effect) which ignores unit uncertainty. Relative Measurement Theory (RMT) [10] developed that measurement calibration to a unit standard must be included in a measurement theory. Because a unit standard makes independent measurement comparisons possible and because of measurement instrument unit uncertainty.

All finite measurement results have a variation due to the precision of the measurement instrument unit (see Fig. 2 below) relative to a unit standard as determined and corrected by calibration. In QM the minimum uncertainty of QM measurement results is limited by a Planck state.[11] There is a correlation between a Planck state as the lower limit of the possible QM uncertainty and a measurement instrument's calibration state as a lower limit of the possible precision.

This correlation is developed by first relating Maxwell's definition of a quantity to a finite subset of Cantor's Transfinite Set Theory (CSTS).[12] In this paper the finite and transfinite parts of CSTS are treated separately. This paper proposes that Maxwell's numerical value is represented by a finite *cardinal* which is the limit of one or more ordinals (see Fig. 1) and not a set member. And proposes in finite sets that Maxwell's *unit*, or the minimum factor thereof applied, is an ordinal (set member) representing a physical property between two contiguous cardinals.

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As an example, a physical meter stick measurement instrument (calibrated to a meter standard) is divided into 100 units of a centimeter each (a finite set) by black vertical marks. Each black mark is a cardinal. Between contiguous marks, each one centimeter length property is one minimum ordinal (u_i), which is one possible meter stick measurement result.

Sections 2.0 - 4.0 (below) introduce CTST, analyze finite sets as a subset of CTST, provide evidence for limits in finite sets, identify transfinite sets as superpositions, and give the definition of a unit standard based upon limits. Section 5.0 reviews the RMT development of property, accuracy and precision. Section 6.0 formalizes the relationship between uncertainty and precision. Section 7.0 explains how the discrepancies and misunderstandings can be resolved. Section 8.0 offers conclusions.

2. Cantor's set theory

Cantor's set theory is based upon elements with a common property that determines their set membership (he uses "aggregate", the modern usage is "set"). "Since every single element m, if we abstract from its nature, becomes a 'unit', the cardinal number $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M."[12] I.e., Cantor denotes a set by its highest cardinal number.

3. Finite sets and their limits

Changing the cardinal from a set member to the limit of a set supports a formal definition of a unit standard. A limit is defined as the numerical value that a function (or sequence) approaches as the argument (or index) approaches some value.[13]

Fig. 1 relates a measurement apparatus scale to a finite set defined by its ordinals (e.g., u_1 - u_8). Cantor described m as an element of the set $\overline{\overline{M}}$.[3] In this paper m is replaced with u_i to represent elements with both property (u) and numerical order (sub i, where i identifies any one well ordered ordinal in the set).

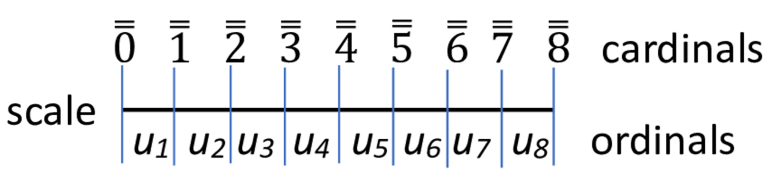

Figure 1. A measurement scale and a finite set

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Fig. 1 identifies that a cardinal is the limit of one or more ordinals. The cardinal $\bar{\bar{8}}$ remains the set designation as Cantor indicates.

3.1. A cardinal is the limit of a set

When the numerical value and unit of a quantity are defined as a cardinal and an ordinal, the cardinal does not have a property that determines finite set membership. Therefore a cardinal is not a member of a set, even when it designates a set. The cardinal is the limit of a finite set, as the upper limit of the ordinals u_1 - u_8 is the cardinal $\bar{\bar{8}}$. It is infinitesimally larger than the largest ordinal (u_8) of the set, as a limit must be.

Three paradoxes in Cantor's set theory have been identified. They disappear when cardinals are limits, not set members.

1. B. Russell identified that using a cardinal to define a set allows the logical possibility that a set is a member of itself, which creates a paradox.[14] When a cardinal is a limit, not a set member, this paradox is removed.
2. The C. Burali-Forti paradox identifies the cardinal of a set of all ordinal numbers as a set, which creates a paradox similar to the B. Russell paradox.[15] When the cardinal of the set of all ordinal numbers is a limit, not a set member, this paradox is removed.
3. Cantor asserts, but it has not been proven, there is no set of all cardinalities.[16] When ordinals are identified as finite set members and cardinals are identified only as set limits, his assertion appears to be true.

3.2. A transfinite class of cardinals is a superposition

Cantor recognized the difference between finite (empirical or theoretical) sets and transfinite sets: "C. Every finite aggregate E is such that it is equivalent to none of its parts... D. Every transfinite aggregate T is such that it has parts T_1 which are equivalent to it."[12] Cantor is stating that ordinals are finite in empirical sets and go to zero as a limit (i.e., without a limit) in transfinite sets, allowing the cardinals T and T_1 to become equivalent. These transfinite cardinals: T, T_1,... are proposed as a set theoretic QM superposition.

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4. Definition of a unit standard

A measurement apparatus scale is divided into units each with the same property in common with a *unit* standard. A unit standard establishes the limit of the measurement apparatus units by a secondary measurement process termed calibration in metrology (International Vocabulary of Metrology, VIM[17]).

Based upon the limit of a set as proposed in Section 3, a unit standard is defined as:

- A *unit standard* (e.g., the BIPM *units*: meter, kilogram, second)[18] independently identifies a property and defines the limit(s) of one or more independent sets of ordinals representing that property.

Multiple BIPM unit standards are often designated as one property. Examples: the position property may include units of time (seconds), the velocity property includes units of length (meters) divided by units of time (seconds) or the momentum property includes units of mass (kilograms) times units of velocity.

4.1. Unit standards in current measurement theory and empirical practice

In the widely applied representational measurement theory, calibration (VIM 2.39) to a unit standard is considered only empirical.[9] Then accuracy and precision are also considered empirical. QM measurements are based upon representational measurement theory.

In metrology the measurand (VIM 2.3) in Fig. 2 represents one or more quantities each with a numerical value (x) and a unit (u) which is not ordered. That is, all measurement instrument units are assumed to be equal to the unit standard after calibration, i.e., no unit order is required. This is the same as representational measurement theory.

In metrology, calibration may change both accuracy (VIM 2.13) and precision (VIM 2.15) as they are not recognized as independent variations of the numerical value and the unit, respectively. In this paper, except as used in the previous sentence, accuracy and precision are defined as variations of the numerical value and the unit, respectively.

Metrology measurement practice requires calibration of the measurement instrument to a unit standard. From Fig. 2, a measurement instrument mean unit ($\frac{1}{n}$) is much larger than a calibration state ($\frac{1}{m}$). Then a common metrology calibration process calibrates the mean unit of a set of measurement instrument units to be $\frac{1}{n}$ within $\pm\frac{1}{m}$ of the unit standard. As is discussed below, this calibration process becomes misleading when n and m are both small, e.g., QM measurements.

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Following metrology measurement practice and recognizing that measurement result comparisons are fundamental, RMT, as shown in Fig. 2, includes a unit standard in a measurement system.

5. Review of RMT

Relative Measurement Theory defines a theoretical finite measurement system (Fig. 2) to include a measurand (one or more quantities), a measurement apparatus, a unit standard as defined above, and to be without any noise (external to the measurement system) or distortion (internal to the measurement system) including distortion due to the measurement apparatus affecting the measurand.

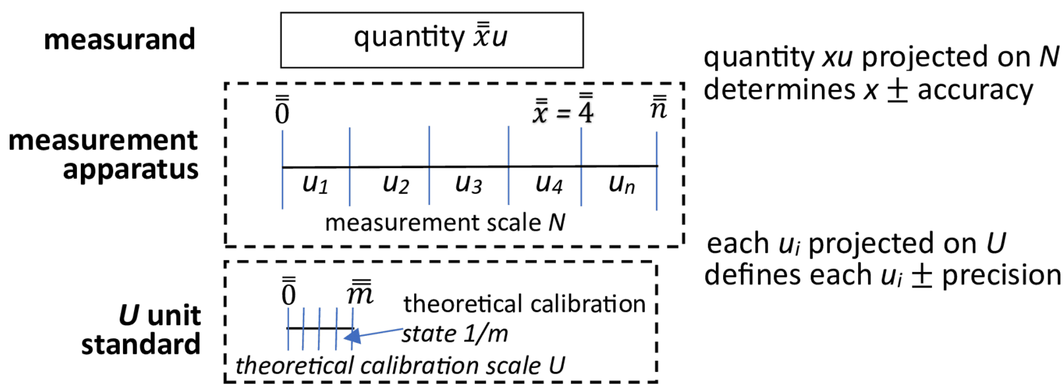

Figure 2. Theoretical measurement system

Applying set theory as shown in Fig. 1 to understand Fig. 2, the cardinal $\bar{\bar{4}}$ is the numerical value of a measurement result quantity which is also the limit of the sum of the ordinals u_1+u_2+u_3+u_4.

L. Euler identified, "It is not possible to determine or measure one quantity other than by assuming that another quantity of the same type is known and determining the ratio between the quantity being measured and that quantity."[19]

As example, a brick, similar to any physical entity (in metrology a measurand), has multiple properties expressed in units: three space dimensions, weight or mass (i.e., particle property), de Broglie wavelength (i.e., wave property), color, velocity, spin, etc. When a measurement apparatus measures an entity (e.g., the brick), any property (i.e., Euler's "known" type) is only "known" by being relative (i.e., calibrated) directly or indirectly at some time to a non-local BIPM standard representing that property or properties (expressed in units) and the unit's numerical value(s).

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In QM, the duality (wave or particle) of a measurand is not understood because QM (which does not formally treat units) does not recognize that any measurand's unit property or combination thereof is only known relatively (see ref 9), not absolutely.

Fig. 2 is formalized as:

$$\sigma(Q) = \sum_{i=1}^{i=x\pm a} u_i \pm 1/m \tag{2}$$

Equation (1), $Q = xu$, shows the current theoretical understanding of an empirical measurement is definitive (single measurement result). However, (2) identifies that a measurement result quantity's numerical value (x) has a fundamental accuracy ($\pm a$, see Section 5.1) and the unit (u_i) has a fundamental precision of $\pm\frac{1}{m}$. These forms of accuracy and precision are theoretical in a finite measurement process and always will establish a distribution of measurement results over enough measurements.

Maxwell's definition of an empirical measurement result (1) is not complete. Equation (2) is proposed as the formal representation of any one dimensional measurement apparatus relative to unit standard(s).[4]

Recognizing that a comparable measurement result quantity only exists relative to a unit standard, both the unit property and its precision relative to a unit standard are theoretical in a relative measurement system. A distribution of measurement results is not always caused by empirical effects, it is defined in (2), the proposed measurement theory.

5.1. Worst case accuracy

In any finite measurement system when a measurand is projected[20] onto a measurement apparatus scale, the minimum state of the scale (*1/m*) determines the variation between two contiguous pair of units on the scale. In the worst case (rare when $\frac{1}{n} \gg \frac{1}{m}$) the minimum calibration state is not small enough to determine how close the maximum or minimum measurand quantity is to which unit of a contiguous pair of units on a scale.

As there is both a maximum and a minimum of a one dimensional measurand quantity and the units of the measurement instrument scale are not treated as equal, the variation of a quantity at each end of the scale is independent of the other end and is added in this worst case. Then the measurement result quantity's worst case numerical value of accuracy is: $a = 2(\frac{1}{n})$. Where accuracy (a) is the variation (in $\frac{1}{n}$, the mean u_i) of the numerical value of a measurement result quantity.

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5.2. Theoretical calibration determines precision

Theoretical calibration defines each *u* of the measurand as a u_i in calibration states (*1/m*) relative to a unit standard. Since the calibration states (*1/m*) are defined as equal and a unit standard (*U*) is defined exactly, each $u_i = U \pm \frac{1}{m}$. Theoretical calibration with statistical summing describe how each of varying u_i impacts the mean unit precision. Theoretical calibration is more rigorous in theory, but is not practical when *n* and *m* are large. However it becomes necessary when *n* and *m* are small, i.e., QM measurements.

5.3. Statistical summing of the theoretical calibration states

The measurement system formalized in (2) is without any noise or distortion (i.e., theoretical) yet produces a normal distribution of measurement results, not a single result. One verification of (2): making multiple experimental measurements of a fixed measurement system (minimum noise and distortion) will also produce a normal distribution of measurement result quantities at high enough precision. Statistical summing effects then add to any experimental precision effects created by noise or distortion. Statistical summing explains how both theoretical and empirical normal distributions occur.

In (2) each measurement result quantity (*Q*) consists of a numerical value (e.g., *x*= 4 in Fig. 2) of u_i. This measurement result quantity is the sum of $u_1+u_2+u_3+u_4$. In any measurement process each u_i has a probability of the completely random ±*1/m* precision even when the entity, the measurement instrument and the standard are fixed. The two random states of *+1/m* and *-1/m* change each of the four u_i,. When the four changed u_i are statistically summed over a large enough number of measurements, a total of $2^4 = 16$ measurement result quantities occur which approximate a normal distribution. A set of equal ordinals (*u*) with a cardinal numerical value becomes an empirical distribution of measurement instrument results ($\sum u_i$) by statistical summing. In QM statistical summing has been described as wave function collapse or decoherence (ref 5).

The probability of the unit precision of a measurement instrument changing a measurement result's quantity is rare when $\frac{1}{n} \gg \frac{1}{m}$ and *n* is large. General metrology practice: *m* is at least 10 times greater than *n*. However, when both *n* and *m* are small the statistical sum (empirical) diverges from the cardinal limit (set theoretical) and the probability of a measurement result quantity changing is significant.[8]

This understanding that normal distributions are caused by calibration states, as well as by noise or distortion, had not been recognized before RMT. RMT answers the question, "Why are Normal Distributions Normal?". The ubiquitous occurrence of normal distributions in metrology measurements (without noise or distortion) verifies the statistical summing of u_i, the precision of each u_i, theoretical calibration, and eq. (2).

6. Uncertainty and precision

In current physics (except metrology), without the concept of measurement precision relative to a unit standard, the uncertainty of finite measurements is often determined between two conjugate variables. Conjugate variables[21] is the physics term applied when two different quantities change in unison (synchronized) as their common units change.

Heisenberg's uncertainty relation, $p_1 q_1 \sim h$, is an example of the synchronized precision of conjugate variables (ref 11). p_1 is the precision of p (momentum) = mass units times velocity (position units/time units) and q_1 is the precision of q (position numerical value and unit at a time in units). Holding the mass fixed, p_1 and q_1 vary inversely with the time unit applied (the wavelength of the light). This explains the interaction between the precision of these conjugate variables using the units of a measurement result quantity.

In theory, and without treating accuracy, (3) presents the same Heisenberg conjugate variables, not as a relationship between the uncertainty of two properties (above), but as two standard deviation (σ) distributions of the measurement results of the same properties (left side). The right hand side, applying (2), shows two distributions of measurement results. u_i has a superscript which indicates this property consists of multiple BIPM units, even though only the time units ($1/m$) are varied. $1/m$ = smallest wavelength of light (time unit) applied. Both the QM and metrology measurement result distributions are equal:

$$\sigma(p)\sigma(q) = \left(\sum_{i=1}^{i=p} u_i^p \pm 1/m\right)\left(\sum_{i=1}^{i=q} u_i^q \pm 1/m\right) \tag{3}$$

The standard deviations of p and q shown on the left side each equal the equivalent measurement result distribution on the right hand side of (3) established by two versions of equation (2). The equality shown in (3) is proven in RMT.[10] The different p or q of each measurement result are members of the distribution shown in (2), not an indication of non-commuting measurement results of conjugate variables. Equation (3) verifies this paper's proposal that QM and metrology measurements results in theory are equivalent.

7. Understanding measurement discrepancies and misunderstandings

The property of a quantity's unit (*u*), as well as the numerical value (*x*) and well ordered unit (u_i), with accuracy and precision respectively, define independent measurement result quantities. The property, numerical value, unit, accuracy and precision are all relative. *xu* is a notation appropriate for counting. Measurement as shown in (2) has accuracy and precision relative to a unit standard.

Measurements as shown in (2) is required by Tarski's undefinability theorem or Gödel's incompleteness theorems. These theories apply to any formal system (e.g., a measurement system), identifying that comparable measurement results and their accuracy and precision (Tarski) or measurement apparatus capable of comparable measurement results (Gödel) are relative. These requirements are verified by all BIPM based experimental measurements.

In metrology (VIM), the calibration state (*1/m*) is usually considered instrument resolution. The apparently instantaneous interactions of entangled particles has been shown to be an illusion created by not recognizing the effect of two calibration processes.[22] When a calibration process is recognized as part of a measurement, the imprecisely calibrated measurement system proposed by Schrödinger is no longer a paradox.[23]

In most QM measurement calculations (using QM Dirac notation) a quantity is a numerical value and the *unit* is defined to be unity (one). Unity *units* simplify a quantity to a numerical value. In QM experiments good metrology practice ensures the unit variation is not significant. However, in QM interferometer experiments when measurement result quantity units are not recognized, the property of a measurement quantity's unit (spin) appears paradoxical.[22]

When the number of units and calibration states is small (e.g., neutron spin experiments[24]), the different measurement result quantities are distributions, not non-commuting. Mathematics rigorously represents physical systems. Measurement results of conjugate variables that do not commute would violate this.

The non-local structure described by J. S. Bell is a unit standard which makes possible comparisons between independent measurement results.

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8. Conclusion

Maxwell's measurement theory assumed measurement results are definitive (single numerical value). Later Heisenberg's uncertainty made definitive measurement results impossible in theory, but metrology measurement theory did not change. Then QM superpositions, when measured, generated distributions. The dichotomy between metrology measurement results assumed to be definitive and QM superpositions known to cause measurement result distributions is confounded when calibration to a unit standard is considered empirical in metrology and units are not treated in QM.

This paper relates finite metrology to transfinite QM by defining measurement units as ordinals and recognizing transfinite cardinals as superpositions in CTSE. Then metrology becomes, not an experimental appendage to physics theory, but the foundation for all physics.

When metrology describes all measurements, the apparent discrepancies and misunderstandings between metrology and QM disciplines disappear. This resolves the issues explored by: Bell's inequality, Bohr's complementarity, Copenhagen interpretation, Einstein-Podolsky-Rosen, Everett's many-worlds, Heisenberg's uncertainty, Mermin's instantaneous interactions of entangled particles, Rovelli's relational QM, Schlosshauer's measurement problem and Zurek's envariance.

Footnotes

[1] *units* are italicized when perfect (or assumed perfect) in this paper.

[2] In this paper a *measurement instrument* is physical and has a unit scale. A *measurement apparatus* may be physical (scale) or theoretical (well ordered ordinals).

[3] Cantor's cardinal notation, with two over bars, is applied to identify cardinality.

[4] The notation shown in (2) is offered as an improvement over the notation used in RMT (5).

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