Research Article

Common Fixed Point Results for Fuzzy F-Contractive Mappings in a Dislocated Metric Space with Application

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This article presents a new approach to proving the existence and uniqueness of a common fixed point for fuzzy mappings that satisfy Ciric type F-contraction and Hardy-Roger type F-contraction in a complete dislocated metric space. These results are applied to multivalued mappings in dislocated metric spaces, and we have provided illustrative examples to demonstrate the power of our approach.

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Mathematics subject Classification: 54E50; 47H10; 54H25.

1. Introduction

The study and application of fixed point theory are critical to the advancement of Functional analysis, Mathematics, and the Sciences in general. In 1922, Banach [1] introduced a key result in metric fixed point theory, which has become known as the Banach contraction principle. This principle is a widely used tool for establishing the existence and uniqueness of solutions to a wide range of problems in Mathematics and Physical Sciences. Over the past few decades, the Banach contraction principle has been extended and generalized in many ways, with applications in a variety of areas, some of which can be found in [2][3][4][5][6][7][8].

Wardowski [9] introduced the concept of an F-contractive mapping on a metric space and proved a fixed point theorem for such a map on a complete metric space. Tomar and Sharma [10] employed the idea of F-contraction introduced by Wardowski to establish coincidence and common fixed point theorems for a pair of discontinuous, noncompatible self-maps in a noncomplete metric space.

Zadeh [11] first proposed the idea of fuzzy sets. Later, Weiss [12] introduced the concept of fuzzy mappings and proved various fixed point results. Building on this work, Heilpern [13] introduced the idea of fuzzy contraction mappings and proved a fixed point theorem for fuzzy contraction mappings that is a fuzzy analogue of Nadler's [14] fixed point theorem for multivalued mappings.

Shahzad et al. [15] introduced the notion of an F-contraction to establish some fixed point results for fuzzy mappings satisfying a new Ciric type rational F-contraction in complete dislocated metric spaces.

In light of the discussion above, we establish the existence and uniqueness of a common fixed point of fuzzy mappings satisfying Ciric type F-contraction and Hardy-Roger type F-contraction in a complete dislocated metric space. We also apply our main results to obtain a common fixed point result for multivalued mappings in dislocated metric spaces, and we provide some illustrative examples to demonstrate the applicability of our results.

Throughout this article, we will denote the set of real numbers by $\mathbb{R}$, the set of positive real numbers by $\mathbb{R}^+$, and the set of natural numbers by $\mathbb{N}$.

2. Preliminaries

In this section, we begin by introducing the notion of an F-contraction and providing some relevant definitions and examples.

Definition 2.1.[16][17] Let X be a nonempty set. A function $d: X \times X \to \mathbb{R}^+$. A pair (X, d) is called a distance space. If d satisfies the following conditions:

i. $d(x, y) = 0$ if $x = y$;

ii. $d(x, y) = d(y, x)$;

iii. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, a function $d: X \times X \to \mathbb{R}^+$ is called a dislocated metric on X. If d is a dislocated metric on X, then the pair (X, d) is said to be a dislocated metric space.

Definition 2.2.[11][18] Let $(X,\ d)$ be a metric space. A map $f: X \to X$ is an F-contraction if there exists $\tau > 0$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)) \ (2.1)$$

for all $x, y \in X$ with $fx \neq fy$, where $F: \mathbb{R}^+ \to \mathbb{R}$ is a function satisfying:

i. F is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

ii. For each sequence $\left\{\alpha_n\right\}_{n \in \mathbb{N}}$ of positive numbers, $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F\left(\alpha_n\right) = -\infty$;

iii. There exists $k \in (0, 1)$ such that $\lim_{n \to 0^+} \alpha^k F(\alpha) = 0$.

We denote F, the family of all functions $F: \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions (i)-(iii). Every F-contraction is a contractive map, i.e.,

$$d(fx, fy) < d(x, y)$$

for all $x, y \in X, fx \neq fy$ and hence is necessarily continuous.

Definition 2.3.[19] Let $(X,\ d)$ be a metric space and $f, g: X \to X$. A pair of self-maps f and g have a coincidence point at $x \in X$ if $fx = gx$. Further, a point $x \in X$ is a common fixed point of f and g if $fx = gx = x$.

Definition 2.4.[20] A fuzzy set in X is a function whose domain is X and whose range is the interval $[0, 1]$. The set of all fuzzy sets in X is denoted by $F(X)$. Given a fuzzy set A and a point x in X, the value $A(x)$ is called the degree of membership of x in A. The α-level set of a fuzzy set A is denoted by $[A]_\alpha$, and is defined as follows:

$[A]_\alpha = \{x: A(x) \geq \alpha\}$ where $\alpha \in (0, 1]$, $[A]_0 = \overline{\{x: A(x) > 0\}}$.

Definition 2.5.[21][22] Let X be a nonempty set and Y be a metric space. A mapping T is called a fuzzy mapping if it is a mapping from X into $F(Y)$, the set of all fuzzy sets on Y. The membership function of a fuzzy mapping T, denoted $T(x)(y)$, is the degree to which y is a member of $T(x)$. That is, $T(x)(y)$ is the degree of membership of y in the fuzzy set $T(x)$. For simplicity, we use the notation $[Tx]_\alpha$ to refer to the α-level set of $T(x)$, instead of $[T(x)]_\alpha$.

Definition 2.6.[18] A point $x \in X$ is called a fuzzy fixed point of a fuzzy mapping $T: X \to F(X)$ if there exists $\alpha \in (0, 1]$ such that $x \in [Tx]_\alpha$.

Definition 2.7.[16] Let (X, d_l) be a dislocated metric space.

(i) A sequence $\{x_n\}$ in (X, d_l) is called a Cauchy sequence if, given $\varepsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that, for all $n, m \ge n_0$, we have $d_l(x_m, x_n) < \varepsilon$ or $\lim_{n,m\to\infty} d_l(x_m, x_n) = 0$.

(ii) A sequence $\{x_n\}$ dislocated converges (for short d_l-converges) to x if $\lim_{n\to\infty} d_l(x_n, x) = 0$. In this case x is called a d_l-limit of $\{x_n\}$.

Definition 2.8.[16] Let K be a nonempty subset of a dislocated metric space X, and let $x \in X$.

An element $y_0 \in K$ is called a best approximation in K if

$d_l(x, K) = d_l(x, y_0)$, where $d_l(x, K) = \inf_{y \in K} y \in K\, d_l(x, y)$.

If each $x \in X$ has at least one best approximation in K, then K is called a proximal set.

Denote by $P(X)$ the set of all proximal subsets of X.

Definition 2.9.[16] The function $H_{d_l}: P(X) \times P(X) \to \mathbb{R}^+$, defined by

$$H_{d_l}(A, B) = max\left\{ \sup_{a\in A} d_l(a, B),\ \sup_{b\in B} d_l(A, b) \right\},$$

is called the dislocated Hausdorff metric on $P(X)$.

Lemma 2.10.[17] Let A and B be nonempty proximal subsets of a dislocated metric space (X, d_l). If $a \in A$, then

$d_l(a, B) \le H_{d_l}(A, B)$.

Lemma 2.11.[23] Let (X, d_l) be a dislocated metric space. Let $(P(X), H_{d_l})$ be a dislocated Hausdorff metric space. Then, for all $A, B \in P(X)$ and for each $a \in A$, there exists $b_a \in B$ satisfying

$d_l(a, B) = d_l(a, b_a)$,

then

$H_{d_l}(A, B) \ge d_l(a, b_a)$.

3. Main Results

In this section, we begin with the following theorem.

Theorem 3.1 Let (X, d_l) be a complete dislocated metric space with $A, B: X \to W(X)$ as two fuzzy mappings on X and (A, B) a pair of Ciric type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \{BA(x_n)\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \le F\left(M_l(x, y)\right), \quad (3.1)$$

where

 doi.org/10.32388/SV98CN.2

$$M_l(x,y) = \max\left\{\begin{array}{c} d_l(x,y),\, d_l\left(x, [Ax]_{\alpha(x)}\right),\, d_l\left(y, [By]_{\alpha(y)}\right), \\ \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(y, [By]_{\alpha(y)}\right)}{d_l(x,y) + d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)}, \\ \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right) d_l\left(y, [By]_{\alpha(y)}\right)}{d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)} \end{array}\right\}$$

Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.1) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Proof. Let $x_0 \in X$ be any arbitrary point in X. Let $x_1 \in \left[Ax_0\right]_{\alpha\left(x_0\right)}$ be an element such that $d_l\left(x_0, \left[Ax_0\right]_{\alpha\left(x_0\right)}\right) = d_l\left(x_0, x_1\right)$. Again, let $x_2 \in \left[Bx_1\right]_{\alpha\left(x_1\right)}$ be an element such that $d_l\left(x_1, \left[Bx_1\right]_{\alpha\left(x_1\right)}\right) = d_l\left(x_1, x_2\right)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{2n+1} \in \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}$ and $x_{2n+2} \in \left[Bx_{2n+1}\right]_{\alpha\left(x_{2n+1}\right)}$, for all $n \in \mathrm{N} \cup \{0\}$. Also, $d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) = d_l\left(x_{2n}, x_{2n+1}\right)$ and $d_l\left(x_{2n+1}, \left[Bx_{2n+1}\right]_{\alpha\left(x_{2n+1}\right)}\right) = d_l\left(x_{2n+1}, x_{2n+2}\right)$. Hence, we define the iteration by $\left\{BA\left(x_n\right)\right\}$. If $M_l(x,y) = 0$, obviously, $x = y$ is a common fixed point of A and B. Then the proof is complete. Let $M_l(x,y) > 0$ for all $x, y \in \left\{BA\left(x_n\right)\right\}$ with $x \neq y$ and by using (3.1) and Lemma 2.11, we obtain

$$F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) \leq F\left(H_{d_l}\left(\left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right)\right)$$

$$\leq F\left(M_l\left(x_{2i}, x_{2i+1}\right)\right) - \tau,$$

for all $i \in \mathrm{N} \cup \{0\}$.

$$M_l\left(x_{2i}, x_{2i+1}\right) = \max\left\{\begin{array}{c} d_l\left(x_{2i}, x_{2i+1}\right),\, d_l\left(x_{2i}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right),\, d_l\left(x_{2i+1}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right), \\ \dfrac{d_l\left(x_{2i}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right) d_l\left(x_{2i+1}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right)}{d_l\left(x_{2i}, x_{2i+1}\right) + d_l\left(x_{2i}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right) + d_l\left(x_{2i+1}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right)}, \\ \dfrac{d_l\left(x_{2i}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right) d_l\left(x_{2i}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right) + d_l\left(x_{2i+1}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right) d_l\left(x_{2i+1}, [By]_{\alpha\left(x_{2i+1}\right)}\right)}{d_l\left(x_{2i}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right) + d_l\left(x_{2i+1}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right)} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} d_l\left(x_{2i}, x_{2i+1}\right),\, d_l\left(x_{2i}, x_{2i+1}\right),\, d_l\left(x_{2i+1}, x_{2i+2}\right), \\ \dfrac{d_l\left(x_{2i}, x_{2i+1}\right) d_l\left(x_{2i+1}, x_{2i+2}\right)}{d_l\left(x_{2i}, x_{2i+1}\right) + d_l\left(x_{2i}, x_{2i+2}\right) + d_l\left(x_{2i+1}, x_{2i+1}\right)}, \\ \dfrac{d_l\left(x_{2i}, x_{2i+1}\right) d_l\left(x_{2i}, x_{2i+2}\right) + d_l\left(x_{2i+1}, x_{2i+1}\right) d_l\left(x_{2i+1}, x_{2i+2}\right)}{d_l\left(x_{2i}, x_{2i+2}\right) + d_l\left(x_{2i+1}, x_{2i+1}\right)} \end{array}\right\}$$

$$\leq \max\left\{d_l\left(x_{2i}, x_{2i+1}\right), d_l\left(x_{2i+1}, x_{2i+2}\right)\right\} \quad (3.3)$$

If there exists $i \in \mathrm{N} \cup \{0\}$ such that $\max\left\{d_l\left(x_{2i}, x_{2i+1}\right), d_l\left(x_{2i+1}, x_{2i+2}\right)\right\} = d_l\left(x_{2i+1}, x_{2i+2}\right)$, then (3.3) becomes

$$F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) \leq F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) - \tau, \quad (3.4)$$

which is a contradiction. Therefore, $\max\left\{d_l\left(x_{2i}, x_{2i+1}\right), d_l\left(x_{2i+1}, x_{2i+2}\right)\right\} = d_l\left(x_{2i}, x_{2i+1}\right)$ for all $i \in \mathrm{N} \cup \{0\}$. Hence, from (3.3), we get

$F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) \leq F\left(d_l\left(x_{2i}, x_{2i+1}\right)\right) - \tau$, for all $i \in \mathbb{N} \cup \{0\}$. (3.5)

Similarly, we get

$F\left(d_l\left(x_{2i}, x_{2i+1}\right)\right) \leq F\left(d_l\left(x_{2i-1}, x_{2i}\right)\right) - \tau$, for all $i \in \mathbb{N}$. (3.6)

Letting (3.6) in (3.5), we get

$$F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) \leq F\left(d_l\left(x_{2i-1}, x_{2i}\right)\right) - 2\tau$$

By continuing the same way, we have

$$F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) \leq F\left(d_l\left(x_0, x_1\right)\right) - (2i+1)\tau,$$

Similarly, we obtain

$$F\left(d_l\left(x_{2i}, x_{2i+1}\right)\right) \leq F\left(d_l\left(x_0, x_1\right)\right) - 2i\tau \text{ (3.8)}$$

By (3.7) and (3.8), we have

$$F\left(d_l\left(x_n, x_{n+1}\right)\right) \leq F\left(d_l\left(x_0, x_1\right)\right) - n\tau \text{ (3.9)}$$

On taking the limit as $n \to \infty$ in (3.9), we get

$$\lim_{n\to\infty} F\left(d_l\left(x_n, x_{n+1}\right)\right) = -\infty \text{ (3.10)}$$

Consider (3.10) and (F_2), we have

$$\lim_{n\to\infty} d_l\left(x_n, x_{n+1}\right) = 0. \text{ (3.11)}$$

From (3.10), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \left(\left(d_l\left(x_n, x_{n+1}\right)\right)^k F\left(d_l\left(x_n, x_{n+1}\right)\right)\right) = 0. \text{ (3.12)}$$

From (3.10), for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\left(d_l\left(x_n, x_{n+1}\right)\right)^k F\left(d_l\left(x_n, x_{n+1}\right) - F\left(d_l\left(x_0, x_1\right)\right)\right) \leq -\left(d_l\left(x_n, x_{n+1}\right)\right)^k n\tau \leq 0. \text{ (3.13)}$$

Using (3.11), (3.12) and taking the limit as $n \to \infty$ in (3.13), we get

$$\lim_{n\to\infty} \left(n\left(d_l\left(x_n, x_{n+1}\right)\right)^k\right) = 0. \text{ (3.14)}$$

Then, there exists $n_1 \in \mathbb{N}$ such that $n\left(d_l\left(x_n, x_{n+1}\right)\right)^k \leq 1$ for all $n \geq n_1$, that is,

$d_l\left(x_n, x_{n+1}\right) \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_1$. (3.15)

For all $m > n > n_1$, by using (3.15) and the triangle inequality, we get

$$\begin{aligned} d_l\left(x_n, x_m\right) &\leq d_l\left(x_n, x_{n+1}\right) + d_l\left(x_{n+1}, x_{n+2}\right) + \ldots + d_l\left(x_{m-1}, x_m\right) \\ &= \sum_{i=n}^{m-1} d_l\left(x_i, x_{i+1}\right) \leq \sum_{i=n}^{\infty} d_l\left(x_i, x_{i+1}\right) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, taking the limit as $n \to \infty$, we get

$$\lim_{m,n\to\infty} d_l\left(x_n, x_m\right) = 0. \text{ (3.17)}$$

This proves that $\left\{BA\left(x_n\right)\right\}$ is a Cauchy sequence in $\left(X,\ d_l\right)$.

Since $\left(X,\ d_l\right)$ is a complete dislocated metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty}\left\{BA\left(x_n\right)\right\} = x^*$. That is,

$$\lim_{n\to\infty} d_l\left(x_n, x^*\right) = 0. \quad (3.18)$$

By Lemma 2.11, we have

$$\tau + F\left(d_l\left(x_{2n+1}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)\right) \le \tau + F\left(H_{d_l}\left(\left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)\right), \quad (3.19)$$

Contractive condition (3.1) also holds for x^*, then we have

$$\tau + F\left(d_l\left(x_{2n+1}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)\right) \le F\left(M_l\left(x_{2n}, x^*\right)\right), \quad (3.20)$$

where

$$M_l\left(x_{2n}, x^*\right) = \max\left\{\begin{array}{c} d_l\left(x_{2n}, x^*\right), d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right), d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right), \\ \dfrac{d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, x^*\right) + d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right)}, \\ \dfrac{d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right)}, \end{array}\right\}$$

$$= max\left\{\begin{array}{c} d_l\left(x_{2n}, x^*\right), d_l\left(x_{2n}, x_{2n+1}\right), d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right), \\ \dfrac{d_l\left(x_{2n}, x_{2n+1}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, x^*\right) + d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right)}, \\ \dfrac{d_l\left(x_{2n}, x_{2n+1}\right) d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right)} \end{array}\right\} \quad (3.21)$$

Using (3.18) and taking the limit as $n \to \infty$ in (3.21), we get

$$\lim_{n\to\infty} M_l\left(x_{2n}, x^*\right) = d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right). \quad (3.22)$$

Since F is strictly increasing, then (3.20) implies

$$d_l\left(x_{2n+1}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) < M_l\left(x_{2n}, x^*\right). \quad (3.23)$$

Again, using (3.22) and taking the limit as $n \to \infty$ in (3.23), we get

$$d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) < d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) \quad (3.24)$$

a contradiction. So $d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) = 0$ or $x^* \in \left[Bx^*\right]_{\alpha\left(x^*\right)}$. Similarly, by using (3.18) and Lemma 2.11 and

$$\tau + F\left(d_l\left(x_{2n+2}, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right)\right) \le \tau + F\left(H_{d_l}\left(\left[Bx_{2n+1}\right]_{\alpha\left(x_{2n+1}\right)}, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right)\right), \quad (3.25)$$

we can also show from (3.25) that $d_l\left(x^*, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right) = 0$ or $x^* \in \left[Ax^*\right]_{\alpha\left(x^*\right)}$. Hence, A and B have a common fixed point x^* in X. Now,

$$d_l\left(x^*, x^*\right) \le d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(\left[Bx^*\right]_{\alpha\left(x^*\right)}, x^*\right) \le 0$$

Also,

$$d_l\left(x^*, x^*\right) \le d_l\left(x^*, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(\left[Ax^*\right]_{\alpha\left(x^*\right)}, x^*\right) \le 0.$$

Thus, $d_l\left(x^*, x^*\right) = 0$.

Theorem 3.2. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A : X \to W(X)$ be fuzzy mappings on X satisfying Ciric type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{A\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [Ay]_{\alpha(y)}\right)\right) \le F\left(M_l(x, y)\right), \quad (3.26)$$

where

$$M_l(x, y) = \max \left\{ \begin{array}{c} d_l(x, y), d_l\left(x, [Ax]_{\alpha(x)}\right), d_l\left(y, [Ay]_{\alpha(y)}\right), \\ \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(y, [Ay]_{\alpha(y)}\right)}{d_l(x, y) + d_l\left(x, [Ay]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)}, \\ \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(x, [Ay]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right) d_l\left(y, [Ay]_{\alpha(y)}\right)}{d_l\left(x, [Ay]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)} \end{array} \right\}$$

Then, $\left\{A\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.26) holds for x^*, then A has a fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Proof Let $x_0 \in X$ be any arbitrary point in X. Let $x_1 \in \left[Ax_0\right]_{\alpha\left(x_0\right)}$ be an element such that $d_l\left(x_0, \left[Ax_0\right]_{\alpha\left(x_0\right)}\right) = d_l\left(x_0, x_1\right)$. Again, let $x_2 \in \left[Ax_1\right]_{\alpha\left(x_1\right)}$ be an element such that $d_l\left(x_1, \left[Ax_1\right]_{\alpha\left(x_1\right)}\right) = d_l\left(x_1, x_2\right)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{2n+1} \in \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}$ and $x_{2n+2} \in \left[Ax_{2n+1}\right]_{\alpha\left(x_{2n+1}\right)}$, for all $n \in \mathbb{N} \cup \{0\}$. Also, $d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) = d_l\left(x_{2n}, x_{2n+1}\right)$ and $d_l\left(x_{2n+1}, \left[Ax_{2n+1}\right]_{\alpha\left(x_{2n+1}\right)}\right) = d_l\left(x_{2n+1}, x_{2n+2}\right)$. Hence, we define the iteration by $\left\{A\left(x_n\right)\right\}$. If $M_l(x, y) = 0$, obviously, $x = y$ is a fixed point of A. Then the proof is complete. Let $M_l(x, y) > 0$ for all $x, y \in \left\{A\left(x_n\right)\right\}$ with $x \ne y$ and by using (3.26) and Lemma 2.11, then the results follow from Theorem 3.1.

Example 3.3. Let $X = \{0, 1, 2\}$ and $d_l(x, y) = x + y$ be a complete dislocated metric space defined by a pair of fuzzy mappings $A, B : X \to W(X)$ as follows:

$$A(x)(t) = \begin{cases} \alpha & if\, \frac{x}{6} \le t < \frac{x}{4} \\ \frac{\alpha}{2} & if\, \frac{x}{4} \le t \le \frac{x}{2} \\ \frac{\alpha}{4} & if\, \frac{x}{2} < t < x \\ 0 & if\, x \le t \le \infty \end{cases}$$

and

$$B(x)(t)=\begin{cases}\beta & if\ \frac{x}{8}\le t<\frac{x}{6}\\ \frac{\beta}{4} & if\ \frac{x}{6}\le t\le\frac{x}{4}\\ \frac{\beta}{6} & if\ \frac{x}{4}<t<x\\ 0 & if\ x\le t\le\infty\end{cases}$$

Define the function $F:\mathrm{R}^+\to\mathrm{R}$ by

$F(x)=\ln(x)$ for all $x\in\mathrm{R}^+$ and $F\in\Delta_F$.

Consider,

$[Ax]_{\frac{\alpha}{2}}=\left[\frac{x}{6},\frac{x}{2}\right]$ and $[By]_{\frac{\beta}{4}}=\left[\frac{y}{8},\frac{y}{4}\right]$

For $x\in X$, we define the sequence $\left\{BA\left(x_n\right)\right\}=\left\{1,\frac{1}{6},\frac{1}{48},\ldots\right\}$ generated by $x_0=1$ in X. We have

$$H_{d_l}\left([Ax]_{\frac{\alpha}{2}},[By]_{\frac{\beta}{4}}\right)=\max\left\{\sup_{a\in Sx}d_l\left(a,[By]_{\frac{\beta}{4}}\right),\sup_{b\in Tx}d_l\left([Ax]_{\frac{\alpha}{2}},b\right),\right\}$$

$$=max\left\{\sup_{a\in Sx}d_l\left(a,\left[\frac{y}{8},\frac{y}{4}\right]\right),\sup_{b\in Tx}d_l\left(\left[\frac{x}{6},\frac{x}{2}\right],b\right)\right\}$$

$$=max\left\{\sup_{a\in Sx}d_l\left(\frac{x}{6},\frac{y}{8}\right),\sup_{b\in Tx}d_l\left(\frac{x}{6},\frac{y}{4}\right)\right\}$$

$$=max\left\{\frac{x}{6}+\frac{y}{8},\frac{x}{6}+\frac{y}{4}\right\}$$

where

$$M_l(x,y)=\max\left\{\begin{array}{c}d_l(x,y),d_l\left(x,\left[\frac{x}{6},\frac{x}{2}\right]\right),d_l\left(y,\left[\frac{y}{8},\frac{y}{4}\right]\right),\\ \frac{d_l\left(x,\left[\frac{x}{6},\frac{x}{2}\right]\right)d_l\left(y,\left[\frac{y}{8},\frac{y}{4}\right]\right)}{d_l(x,y)+d_l\left(x,\left[\frac{y}{8},\frac{y}{4}\right]\right)+d_l\left(y,\left[\frac{x}{6},\frac{x}{2}\right]\right)},\\ \frac{d_l\left(x,\left[\frac{x}{6},\frac{x}{2}\right]\right)d_l\left(x,\left[\frac{y}{8},\frac{y}{4}\right]\right)+d_l\left(y,\left[\frac{x}{6},\frac{x}{2}\right]\right)d_l\left(y,\left[\frac{y}{8},\frac{y}{4}\right]\right)}{d_l\left(x,\left[\frac{y}{8},\frac{y}{4}\right]\right)+d_l\left(y,\left[\frac{x}{6},\frac{x}{2}\right]\right)}\end{array}\right\}$$

$$=max\left\{\begin{array}{c}d_l(x,y),d_l\left(x,\frac{x}{6}\right),d_l\left(y,\frac{y}{8}\right),\\ \frac{d_l\left(x,\frac{x}{6}\right)d_l\left(y,\frac{y}{8}\right)}{d_l(x,y)+d_l\left(x,\frac{y}{8}\right)+d_l\left(y,\frac{x}{6}\right)},\\ \frac{d_l\left(x,\frac{x}{6}\right)d_l\left(x,\frac{y}{8}\right)+d_l\left(y,\frac{x}{6}\right)d_l\left(y,\frac{y}{8}\right)}{d_l\left(x,\frac{y}{8}\right)+d_l\left(y,\frac{x}{6}\right)}\end{array}\right\}$$

$$=x+y$$

Case 1. Suppose $\max\left\{\frac{x}{6}+\frac{y}{8}, \frac{x}{6}+\frac{y}{4}\right\} = \frac{x}{6}+\frac{y}{8}$, and $\tau = \ln\left(\frac{8}{3}\right)$, we get

$$16x + 12y \leq 36x + 36y$$

$$\frac{8}{3}\left(\frac{x}{6}+\frac{y}{8}\right) \leq x + y$$

$$\ln\left(\frac{8}{3}\right) + \ln\left(\frac{x}{6}+\frac{y}{8}\right) \leq \ln(x+y).$$

That is,

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq F\left(M_l(x,y)\right)$$

Case 2. Again, suppose $\max\left\{\frac{x}{6}+\frac{y}{8}, \frac{x}{6}+\frac{y}{4}\right\} = \frac{x}{6}+\frac{y}{4}$, and $\tau = \ln\left(\frac{8}{3}\right)$, we get

$$16x + 24y \leq 36x + 36y$$

$$\frac{8}{3}\left(\frac{x}{6}+\frac{y}{4}\right) \leq x + y$$

$$\ln\left(\frac{8}{3}\right) + \ln\left(\frac{x}{6}+\frac{y}{4}\right) \leq \ln(x+y).$$

That is,

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq F\left(M_l(x,y)\right)$$

Then, we can see that all the hypotheses in Theorem 3.1 are satisfied. Hence, A and B have a common fixed point

We have the following corollaries for Ciric type fuzzy F-contraction:

Corollary 3.4. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A, B : X \to W(X)$ be two fuzzy mappings on X and (A, B) a pair of Ciric type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{BA\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq F\left(M_l(x,y)\right),$$

where

$$M_l(x,y) = \max\left\{\begin{array}{c} d_l\left(y, [By]_{\alpha(y)}\right), \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(y, [By]_{\alpha(y)}\right)}{d_l(x,y) + d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)}, \\ \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right) d_l\left(y, [By]_{\alpha(y)}\right)}{d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)} \end{array}\right\}$$

Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.28) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Corollary 3.5. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A, B : X \to W(X)$ be two fuzzy mappings on X and (A, B) a pair of Ciric type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{BA\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq F\left(M_l(x,y)\right),$$

where

$$M_l(x,y) = \max \left\{ \begin{array}{c} d_l(x,y), d_l\left(x,[Ax]_{\alpha(x)}\right), d_l\left(y,[By]_{\alpha(y)}\right), \\ \dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right) d_l\left(y,[By]_{\alpha(y)}\right)}{d_l(x,y) + d_l\left(x,[By]_{\alpha(y)}\right) + d_l\left(y,[Ax]_{\alpha(x)}\right)} \end{array} \right\}$$

Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.29) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Corollary 3.6. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A, B: X \to W(X)$ be two fuzzy mappings on X and (A, B) a pair of Ciric type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{BA\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \le F\left(M_l(x,y)\right),$$

where

$$M_l(x,y) = \max \left\{ \begin{array}{c} d_l(x,y), d_l\left(x,[Ax]_{\alpha(x)}\right), d_l\left(y,[By]_{\alpha(y)}\right), \\ \dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right) d_l\left(x,[By]_{\alpha(y)}\right) + d_l\left(y,[Ax]_{\alpha(x)}\right) d_l\left(y,[By]_{\alpha(y)}\right)}{d_l\left(x,[By]_{\alpha(y)}\right) + d_l\left(y,[Ax]_{\alpha(x)}\right)} \end{array} \right\}$$

Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.30) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Corollary 3.7. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A, B: X \to W(X)$ be two fuzzy mappings on X and (A, B) a pair of fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{BA\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \le F\left(d_l(x,y)\right), \quad (3.31)$$

Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.31) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Corollary 3.8. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A: X \to W(X)$ be fuzzy mappings on X satisfying fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{A\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [Ay]_{\alpha(y)}\right)\right) \le F\left(d_l(x,y)\right), \quad (3.32)$$

Then, $\left\{A\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.32) holds for x^*, then A has a fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Now, we consider Hardy-Rogers-type fuzzy F-contraction for a pair of mappings.

Theorem 3.8. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A, B: X \to W(X)$ be a pair of fuzzy mappings on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{AB\left(x_n\right)\right\}$, $x \ne y$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \le$$

$$F\left(\begin{array}{r} a_1 d_l(x,y) + a_2 d_l\left(x, [Ax]_{\alpha(x)}\right) + \\ a_3 d_l\left(y, [By]_{\alpha(y)}\right) + a_4 \dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right) d_l\left(y,[By]_{\alpha(y)}\right)}{d_l(x,y) + d_l\left(x,[By]_{\alpha(y)}\right) + d_l\left(y,[Ax]_{\alpha(x)}\right)} \\ + a_5 \dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right) d_l\left(x,[By]_{\alpha(y)}\right) + d_l\left(y,[Ax]_{\alpha(x)}\right) d_l\left(y,[By]_{\alpha(y)}\right)}{d_l\left(x,[By]_{\alpha(y)}\right) + d_l\left(y,[Ax]_{\alpha(x)}\right)} \end{array}\right)$$

and $a_1, a_2, a_3, a_4, a_5 > 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ and $a_3 \neq 1$. Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.32) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Proof. From the proof of Theorem 3.1, we see that $x_1 \in \left[Ax_0\right]_{\alpha\left(x_0\right)}$ and $x_2 \in \left[Bx_1\right]_{\alpha\left(x_1\right)}$, with (3.32) and Lemma 2.11, we obtain

$$\tau + F\left(d_l\left(x_1, x_2\right)\right) \leq \tau + F\left(d_l\left(x_1, \left[Bx_1\right]_{\alpha\left(x_1\right)}\right)\right)$$

$$\leq \tau + F\left(H_{d_l}\left(\left[Ax_0\right]_{\alpha\left(x_0\right)}, \left[Bx_1\right]_{\alpha\left(x_1\right)}\right)\right)$$

$$\leq F\left(\begin{array}{r} a_1 d_l\left(x_0, x_1\right) + a_2 d_l\left(x_0, \left[Ax_0\right]_{\alpha\left(x_0\right)}\right) + \\ a_3 d_l\left(x_1, \left[Bx_1\right]_{\alpha\left(x_1\right)}\right) + a_4 \dfrac{d_l\left(x_0,\left[Ax_0\right]_{\alpha\left(x_0\right)}\right) d_l\left(x_1,\left[Bx_1\right]_{\alpha\left(x_1\right)}\right)}{d_l\left(x_0,x_1\right) + d_l\left(x_0,\left[Bx_1\right]_{\alpha\left(x_1\right)}\right) + d_l\left(x_1,\left[Ax_0\right]_{\alpha\left(x_0\right)}\right)} \\ + a_5 \dfrac{d_l\left(x_0,\left[Ax_0\right]_{\alpha\left(x_0\right)}\right) d_l\left(x_0,\left[Bx_1\right]_{\alpha\left(x_1\right)}\right) + d_l\left(x_1,\left[Ax_0\right]_{\alpha\left(x_0\right)}\right) d_l\left(x_1,\left[Bx_1\right]_{\alpha\left(x_1\right)}\right)}{d_l\left(x_0,\left[Bx_1\right]_{\alpha\left(x_1\right)}\right) + d_l\left(x_1,\left[Ax_0\right]_{\alpha\left(x_0\right)}\right)} \end{array}\right)$$

$$= F\left(\begin{array}{r} a_1 d_l\left(x_0, x_1\right) + a_2 d_l\left(x_0, x_1\right) + \\ a_3 d_l\left(x_1, x_2\right) + a_4 \dfrac{d_l\left(x_0,x_1\right) d_l\left(x_1,x_2\right)}{d_l\left(x_0,x_1\right) + d_l\left(x_0,x_2\right) + d_l\left(x_1,x_1\right)} \\ + a_5 \dfrac{d_l\left(x_0,x_1\right) d_l\left(x_0,x_2\right) + d_l\left(x_1,x_1\right) d_l\left(x_1,x_2\right)}{d_l\left(x_0,x_2\right) + d_l\left(x_1,x_1\right)} \end{array}\right)$$

$$\leq F\left(\left(a_1 + a_2 + a_4 + a_5\right) d_l\left(x_0, x_1\right) + a_3 d_l\left(x_1, x_2\right)\right)$$

Since F is strictly increasing, we have

$$d_l\left(x_1, x_2\right) < \left(a_1 + a_2 + a_4 + a_5\right) d_l\left(x_0, x_1\right) + a_3 d_l\left(x_1, x_2\right)$$

implies

$$d_l\left(x_1, x_2\right) < \left(\frac{a_1 + a_2 + a_4 + a_5}{1 - a_3}\right) d_l\left(x_0, x_1\right)$$

From $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ and $a_3 \neq 1$, we deduce $1 - a_3 > 0$ and so

$$d_l\left(x_1, x_2\right) < d_l\left(x_0, x_1\right)$$

Therefore,

$$F\left(d_l\left(x_1, x_2\right)\right) \le F\left(d_l\left(x_0, x_1\right)\right) - \tau.$$

Again, from the proof of Theorem 3.1, we see that $x_{2i+1} \in \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}$ and $x_{2i+2} \in \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}$, with (3.32) and Lemma 2.11, we obtain

$$\tau + F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) \le \tau + F\left(d_l\left(x_{2i+1}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right)\right)$$

$$\le \tau + F\left(H_{d_l}\left(\left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right)\right)$$

$$\le F\left(\begin{array}{r} a_1 d_l\left(x_{2i}, x_{2i+1}\right) + a_2 d_l\left(x_{2i}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right) + \\ a_3 d_l\left(x_{2i+1}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right) + a_4 \dfrac{d_l\left(x_{2i}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right) d_l\left(x_{2i+1}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right)}{d_l\left(x_{2i}, x_{2i+1}\right) + d_l\left(x_{2i}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right) + d_l\left(x_{2i+1}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right)} \\ + a_5 \dfrac{d_l\left(x_{2i}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right) d_l\left(x_{2i}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right) + d_l\left(x_{2i+1}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right) d_l\left(x_{2i+1}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right)}{d_l\left(x_{2i}, \left[Bx_{2i+1}\right]_{\alpha\left(x_{2i+1}\right)}\right) + d_l\left(x_{2i+1}, \left[Ax_{2i}\right]_{\alpha\left(x_{2i}\right)}\right)} \end{array}\right)$$

$$= F\left(\begin{array}{r} a_1 d_l\left(x_{2i}, x_{2i+1}\right) + a_2 d_l\left(x_{2i}, x_{2i+1}\right) + \\ a_3 d_l\left(x_{2i+1}, x_{2i+2}\right) + a_4 \dfrac{d_l\left(x_{2i}, x_{2i+1}\right) d_l\left(x_{2i+1}, x_{2i+2}\right)}{d_l\left(x_{2i}, x_{2i+1}\right) + d_l\left(x_{2i}, x_{2i+2}\right) + d_l\left(x_{2i+1}, x_{2i+1}\right)} \\ + a_5 \dfrac{d_l\left(x_{2i}, x_{2i+1}\right) d_l\left(x_{2i}, x_{2i+2}\right) + d_l\left(x_{2i+1}, x_{2i+1}\right) d_l\left(x_{2i+1}, x_{2i+2}\right)}{d_l\left(x_{2i}, x_{2i+2}\right) + d_l\left(x_{2i+1}, x_{2i+1}\right)} \end{array}\right)$$

$$\le F\left(\left(a_1 + a_2 + a_4 + a_5\right) d_l\left(x_{2i}, x_{2i+1}\right) + a_3 d_l\left(x_{2i+1}, x_{2i+2}\right)\right)$$

Since F is strictly increasing, we have

$$d_l\left(x_{2i+1}, x_{2i+2}\right) < \left(a_1 + a_2 + a_4 + a_5\right) d_l\left(x_{2i}, x_{2i+1}\right) + a_3 d_l\left(x_{2i+1}, x_{2i+2}\right)$$

implies

$$d_l\left(x_{2i+1}, x_{2i+2}\right) < \left(\frac{a_1 + a_2 + a_4 + a_5}{1 - a_3}\right) d_l\left(x_{2i}, x_{2i+1}\right)$$

From $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ and $a_3 \ne 1$, we deduce $1 - a_3 > 0$ and so

$$d_l\left(x_{2i+1}, x_{2i+2}\right) < d_l\left(x_{2i}, x_{2i+1}\right)$$

Hence,

$$F\left(d_l\left(x_{2i+1}, x_{2i+2}\right)\right) \le F\left(d_l\left(x_{2i}, x_{2i+1}\right)\right) - \tau.$$

Following the same arguments in Theorem 3.1, we have $\left\{BA\left(x_n\right)\right\} \to x^*$, that is,

$$\lim_{n \to \infty} d_l\left(x_n, x^*\right) = 0. \ (3.33)$$

By Lemma 2.11, we have

$$\tau + F\left(d_l\left(x_{2n+1}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)\right) \le \tau + F\left(H_{d_l}\left(\left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)\right),$$

using (3.32), we have

$$\tau + F\left(d_l\left(x_{2n+1}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)\right) \le$$

$$F\left(\begin{array}{r} a_1 d_l\left(x_{2n}, x^*\right) + a_2 d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) + \\ a_3 d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + a_4 \dfrac{d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, x^*\right) + d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right)} \\ +a_5 \dfrac{d_l\left(x_{2n}, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, \left[Ax_{2n}\right]_{\alpha\left(x_{2n}\right)}\right)} \end{array}\right)$$

$$\le F\left(\begin{array}{r} a_1 d_l\left(x_{2n}, x^*\right) + a_2 d_l\left(x_{2n}, x_{2n+1}\right) + \\ a_3 d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + a_4 \dfrac{d_l\left(x_{2n}, x_{2n+1}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, x^*\right) + d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right)} \\ +a_5 \dfrac{d_l\left(x_{2n}, x_{2n+1}\right) d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right)} \end{array}\right)$$

Since F is strictly increasing, we have

$$d_l\left(x_{2n+1}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) < a_1 d_l\left(x_{2n}, x^*\right) + a_2 d_l\left(x_{2n}, x_{2n+1}\right) + a_3 d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + a_4 \frac{d_l\left(x_{2n}, x_{2n+1}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, x^*\right) + d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right)}$$

$$+a_5 \frac{d_l\left(x_{2n}, x_{2n+1}\right) d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right) d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)}{d_l\left(x_{2n}, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(x^*, x_{2n+1}\right)}$$

Taking the limit as $n \to \infty$ in (3.33), we get

$$d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) < a_3 d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right)$$

a contradiction. So $d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) = 0$ or $x^* \in \left[Bx^*\right]_{\alpha\left(x^*\right)}$. Similarly, by using (3.32), (3.33) and Lemma 2.11 and

$\tau + F\left(d_l\left(x_{2n+2}, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right)\right) \le \tau + F\left(H_{d_l}\left(\left[Bx_{2n+1}\right]_{\alpha\left(x_{2n+1}\right)}, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right)\right)$, (3.34)

also, we can show from (3.34) that $d_l\left(x^*, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right) = 0$ or $x^* \in \left[Ax^*\right]_{\alpha\left(x^*\right)}$. Hence, A and B have a common fixed point x^* in $\left(X,\ d_l\right)$. Now,

$$d_l\left(x^*, x^*\right) \le d_l\left(x^*, \left[Bx^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(\left[Bx^*\right]_{\alpha\left(x^*\right)}, x^*\right) \le 0$$

Also,

$$d_l\left(x^*, x^*\right) \le d_l\left(x^*, \left[Ax^*\right]_{\alpha\left(x^*\right)}\right) + d_l\left(\left[Ax^*\right]_{\alpha\left(x^*\right)}, x^*\right) \le 0.$$

Thus, $d_l\left(x^*, x^*\right) = 0.$

If we take $A = B$ in Theorem 3.8, we have the following Theorem.

Theorem 3.9. Let $\left(X,\ d_l\right)$ be a complete dislocated metric space with $A : X \to W(X)$ be fuzzy mappings on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{AA\left(x_n\right)\right\}$, $x \neq y$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [Ay]_{\alpha(y)}\right)\right) \leq$$

$$F\left(\begin{array}{r} a_1 d_l(x, y) + a_2 d_l\left(x, [Ax]_{\alpha(x)}\right) + \\ a_3 d_l\left(y, [Ay]_{\alpha(y)}\right) + a_4 \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(y, [Ay]_{\alpha(y)}\right)}{d_l(x, y) + d_l\left(x, [Ay]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)} \\ + a_5 \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(x, [Ay]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right) d_l\left(y, [Ay]_{\alpha(y)}\right)}{d_l\left(x, [Ay]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)} \end{array}\right)$$

and $a_1, a_2, a_3, a_4, a_5 > 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ and $a_3 \neq 1$. Then, $\left\{AA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.33) holds for x^*, then A has a fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0.$

If we take $a_1 = 0$ in Theorem 3.8, we have the following Corollary.

Corollary 3.8. Let $\left(X,\ d_l\right)$ be a complete dislocated metric space with $A, B : X \to W(X)$ be a pair of fuzzy mappings on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{AB\left(x_n\right)\right\}$, $x \neq y$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq$$

$$F\left(\begin{array}{r} a_1 d_l\left(x, [Ax]_{\alpha(x)}\right) + a_2 d_l\left(y, [By]_{\alpha(y)}\right) \\ + a_3 \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(y, [By]_{\alpha(y)}\right)}{d_l(x, y) + d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)} \\ + a_4 \dfrac{d_l\left(x, [Ax]_{\alpha(x)}\right) d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right) d_l\left(y, [By]_{\alpha(y)}\right)}{d_l\left(x, [By]_{\alpha(y)}\right) + d_l\left(y, [Ax]_{\alpha(x)}\right)} \end{array}\right)$$

and $a_1, a_2, a_3, a_4 > 0$ with $a_1 + a_2 + a_3 + a_4 = 1$ and $a_2 \neq 1$. Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.32) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0.$

If we take $a_1 = a_2 = 0$ in Theorem 3.8, we have the following Corollary.

Corollary 3.8. Let $\left(X,\ d_l\right)$ be a complete dislocated metric space with $A, B : X \to W(X)$ be a pair of fuzzy mappings on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{AB\left(x_n\right)\right\}$, $x \neq y$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq$$

$$F\left(\begin{array}{c} a_1 d_l\left(y,[By]_{\alpha(y)}\right)+a_2\dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right)d_l\left(y,[By]_{\alpha(y)}\right)}{d_l(x,y)+d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)} \\ +a_3\dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right)d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)d_l\left(y,[By]_{\alpha(y)}\right)}{d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)} \end{array}\right)$$

(3.32)

and $a_1, a_2, a_3 > 0$ with $a_1 + a_2 + a_3 = 1$ and $a_1 \neq 1$. Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.32) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

If we take $a_1 = a_2 = a_3 = 0$ in Theorem 3.8, we have the following Corollary.

Corollary 3.8. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A, B : X \to W(X)$ be a pair of fuzzy mappings on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{AB\left(x_n\right)\right\}$, $x \neq y$ and $\alpha(x) \in (0, 1]$satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq$$

$$F\left(\begin{array}{c} a_1\dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right)d_l\left(y,[By]_{\alpha(y)}\right)}{d_l(x,y)+d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)}+ \\ a_2\dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right)d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)d_l\left(y,[By]_{\alpha(y)}\right)}{d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)} \end{array}\right)$$

(3.32)

and $a_1, a_2 > 0$ with $a_1 + a_2 = 1$. Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.32) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

If we take $a_2 = a_3 = 0$ in Theorem 3.8, we have the following Corollary.

Corollary 3.8 Let $\left(X, d_l\right)$ be a complete dislocated metric space with $A, B : X \to W(X)$ be a pair of fuzzy mappings on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{AB\left(x_n\right)\right\}$, $x \neq y$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}\left([Ax]_{\alpha(x)}, [By]_{\alpha(y)}\right)\right) \leq$$

$$F\left(\begin{array}{c} a_1 d_l(x,y)+a_2\dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right)d_l\left(y,[By]_{\alpha(y)}\right)}{d_l(x,y)+d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)} \\ +a_3\dfrac{d_l\left(x,[Ax]_{\alpha(x)}\right)d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)d_l\left(y,[By]_{\alpha(y)}\right)}{d_l\left(x,[By]_{\alpha(y)}\right)+d_l\left(y,[Ax]_{\alpha(x)}\right)} \end{array}\right)$$

and $a_1, a_2, a_3 > 0$ with $a_1 + a_2 + a_3 = 1$. Then, $\left\{BA\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (3.32) holds for x^*, then A and B have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

4. Application

As an application of our work, we will now show how Theorem 3.1 and Theorem 3.8 can be used to prove the existence of common fixed points for multivalued mappings in a dislocated metric space. The following theorem follows directly from our previous results.

Theorem 4.1 Let $\left(X,\ d_l\right)$ be a complete dislocated metric space with $R, S: X \to W(X)$ be two multivalued fuzzy mappings on X and (R, S) a pair of Ciric type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{SR\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}(Rx, Sy)\right) \le F\left(M_l(x, y)\right), \ (4.1)$$

where

$$M_l(x, y) = \max\left\{\begin{array}{c} d_l(x, y), d_l(x, Rx), d_l(y, Sy), \\ \frac{d_l(x, Rx)d_l(y, Sy)}{d_l(x, y) + d_l(x, Sy) + d_l(y, Rx)}, \\ \frac{d_l(x, Rx)d_l(x, Sy) + d_l(y, Rx)d_l(y, Sy)}{d_l(x, Sy) + d_l(y, Rx)} \end{array}\right\}$$

Then, $\left\{SR\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (4.1) holds for x^*, then R and S have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Proof. Let $\alpha: X \to (0, 1]$ be an arbitrary mapping. Consider a fuzzy mapping $A, B: X \to W(X)$ defined by

$$(Ax)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx, \end{cases}$$

$$(Bx)(t) = \begin{cases} \alpha(x), & t \in Sx, \\ 0, & t \notin Sx. \end{cases}$$

We have that

$$[Ax]_{\alpha(x)} = \{t: Ax(t) \ge \alpha(x)\} = Rx,$$

and

$$[Bx]_{\alpha(x)} = \{t: Bx(t) \ge \alpha(x)\} = Sx.$$

Thus, condition (4.1) becomes condition (3.1) in Theorem 3.1. It implies that there exists $x^* \in [Ax]_{\alpha(x)} \cap [Bx]_{\alpha(x)} = Rx \cap Sx$.

Now, we consider Hardy-Rogers-type fuzzy F-contraction to a pair of mappings.

Theorem 4.2 Let $\left(X,\ d_l\right)$ be a complete dislocated metric space with $R, S: X \to P(X)$ be a pair of multivalued fuzzy mappings on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{SR\left(x_n\right)\right\}$, $x \ne y$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}(Rx, Sy)\right) \le$$

$$F\left(\begin{array}{c} a_1 d_l(x, y) + a_2 d_l(x, Rx) + \\ a_3 d_l(y, Sy) + a_4 \frac{d_l(x, Rx)d_l(y, Sy)}{d_l(x, y) + d_l(x, Sy) + d_l(y, Rx)} \\ + a_5 \frac{d_l(x, Rx)d_l(x, Sy) + d_l(y, Rx)d_l(y, Sy)}{d_l(x, Sy) + d_l(y, Rx)} \end{array}\right)$$

and $a_1, a_2, a_3, a_4, a_5 > 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ and $a_3 \ne 1$. Then, $\left\{SR\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (4.3) holds for x^*, then R and S have a common fixed point $x^* \in X$ and $d_l\left(x^*, x^*\right) = 0$.

Proof. Let $\alpha : X \to (0, 1]$ be an arbitrary mapping. Consider a fuzzy mapping $A, B : X \to W(X)$ defined by

$$(Ax)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx, \end{cases}$$

$$(Bx)(t) = \begin{cases} \alpha(x), & t \in Sx, \\ 0, & t \notin Sx. \end{cases}$$

We have that

$$[Ax]_{\alpha(x)} = \{t : Ax(t) \geq \alpha(x)\} = Rx,$$

and

$$[Bx]_{\alpha(x)} = \{t : Bx(t) \geq \alpha(x)\} = Sx.$$

Hence, condition (4.3) becomes condition (3.32) in Theorem 3.8. It implies that there exists $x^* \in [Ax]_{\alpha(x)} \cap [Bx]_{\alpha(x)} = Rx \cap Sx$.

Theorem 4.3. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $R : X \to P(X)$ be a multivalued fuzzy mapping on X satisfying Ciric type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{R\left(x_n\right)\right\}$ and $\alpha(x) \in (0, 1]$ satisfying the following conditions:

$$\tau + F\left(H_{d_l}(Rx, Ry)\right) \leq F\left(M_l(x, y)\right), \quad (4.4)$$

where

$$M_l(x, y) = \max \left\{ \begin{array}{c} d_l(x, y), d_l(x, Rx), d_l(y, Ry), \\ \frac{d_l(x, Rx) d_l(y, Ry)}{d_l(x, y) + d_l(x, Ry) + d_l(y, Rx)}, \\ \frac{d_l(x, Rx) d_l(x, Ry) + d_l(y, Rx) d_l(y, Ry)}{d_l(x, Ry) + d_l(y, Rx)} \end{array} \right\}$$

Then, $\left\{R\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (4.1) holds for x^*, then there exists $x^* \in X$ such that $x^* \in Rx^*$.

Proof. Let $\alpha : X \to (0, 1]$ be an arbitrary mapping. Consider a fuzzy mapping $A : X \to F(X)$ defined by

$$(Ax)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx. \end{cases}$$

We have that

$$[Ax]_{\alpha(x)} = \{t : Ax(t) \geq \alpha(x)\} = Rx.$$

Hence, condition (4.5) becomes condition (3.27) in Theorem 3.2. It implies that there exists $x^* \in X$ such that $x^* \in \left[Ax^*\right]_{\alpha\left(x^*\right)} = Rx^*$.

Corollary 4.3. Let $\left(X, d_l\right)$ be a complete dislocated metric space with $R : X \to P(X)$ be a multivalued fuzzy mapping on X satisfying Hardy-Rogers-type fuzzy F-contraction. Suppose there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $x, y \in \left\{R\left(x_n\right)\right\}, x \neq y$ and $\alpha(x) \in (0, 1]$satisfying the following conditions:

$$\tau + F\left(H_{d_l}(Rx, Ry)\right) \leq F\left(\begin{array}{c} a_1 d_l(x, y) + a_2 d_l(x, Rx) + \\ a_3 d_l(y, Ry) + a_4 \frac{d_l(x, Rx) d_l(y, Ry)}{d_l(x, y) + d_l(x, Ry) + d_l(y, Rx)} \\ + a_5 \frac{d_l(x, Rx) d_l(y, Ry)}{d_l(x, y) + d_l(x, Ry) + d_l(y, Rx)} \end{array}\right), \quad (4.6)$$

and $a_1, a_2, a_3, a_4 > 0$ with $a_1 + a_2 + a_3 + a_4 = 1$ and $a_3 \neq 1$. Then, $\left\{R\left(x_n\right)\right\} \to x^* \in X$. Moreover, if condition (4.6) holds for x^*, then there exists $x^* \in X$ such that $x^* \in Rx^*$.

5. Conclusion

In this paper, we have established the existence and uniqueness of common fixed points for fuzzy mappings that satisfy Ciric type F-contraction and Hardy-Roger type F-contraction in complete dislocated metric spaces. In addition, we have applied our main results to prove common fixed point theorems for multivalued mappings in dislocated metric spaces. To demonstrate the usefulness of our approach, we have provided several examples.

Statements and Declarations

Acknowledgments

The authors would like to express their deepest gratitude to all those who have contributed in some way to the completion of this work.

Author's contribution

All authors have contributed equally to the writing of this paper.

Conflict of interests

The authors have no conflicts of interest to declare.

Research involving human participants and/or animals

This article does not contain any studies with human participants or animals that were conducted by the authors.

Funding

Not applicable.

Data Availability

Not applicable.

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Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.