Research Article

Green–Poisson Reconstruction and Hypersurface Recursion for Dirac Laplacians

Simone Farinelli[1]

1. Independent researcher

We study reconstruction of sections of Hermitian Dirac bundles through finite chains of hypersurfaces. For the invertible full Dirichlet realization of the Dirac Laplacian, the Green–Poisson formula separates boundary data from an interior source. We establish a composition formula that retains the sources at every level and specifies the Banach spaces and assembly maps required for its use. The induced Clifford connection, the multiplicity of restricted modules, and the interface conditions are treated explicitly. A transmission formula explains the difference between partitioning one global Green integral and using independent Dirichlet inverses on the pieces. A cylinder and a product neighborhood of a complex hypersurface provide exact two-step examples with nonzero intermediate sources. We also give a global source-corrected lift on a normal disk bundle and explain the obstruction to choosing a real diameter in a nontrivial normal bundle. The general composition theorem is conditional on a realized chain; the product example has bounded transfers on its specified finite-dimensional spaces.

Correspondence: papers@team.qeios.com — Qeios will forward to the authors

1. Introduction

One motivation for this work is the Hodge conjecture, which relates the topology of a smooth complex projective variety to its algebraic subvarieties. For such a variety X of complex dimension n, its rational Hodge classes of degree $2p$, $0 \leq p \leq n$, form the space

$$\mathrm{Hdg}^p(X, \mathbb{Q}) = H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X),$$

where the intersection is taken inside $H^{2p}(X,\mathbb{C})$ and $H^{p,p}(X)$ is the corresponding Hodge summand. The conjecture asserts that each such class is a finite rational linear combination of cycle classes $\mathrm{cl}(Z)$ of closed algebraic subvarieties of complex codimension p; these subvarieties may be singular. Here $\mathrm{cl}(Z)$ denotes the Poincaré dual cohomology class of the cycle. We refer to Deligne [1], Section 1 for the precise statement and to Voisin [2] for the Hodge-theoretic background.

After a Kähler metric has been chosen, Hodge theory identifies these classes with their harmonic (p,p) representatives. This supplies an analytic starting point for the Dirac–Dolbeault approach proposed in [3]. With the fixed-p twist specified in Example 2.6, the square of $\sqrt{2}(\bar{\partial}+\bar{\partial}^{*})$ is the Hodge Laplacian. The proposed programme seeks to relate harmonic representatives and their pairings to geometric data through boundary representations and dimension-changing constructions. To implement that idea one must first determine exactly which analytic reconstruction identities survive restriction and iteration.

An elliptic boundary representation reconstructs an interior section from boundary values and an interior source. Repeating such a representation on successively lower-dimensional manifolds requires compatible induced Clifford modules, well-posed realizations of the operators, and assembly maps supplying complete boundary traces. Restriction need not preserve harmonicity. Accordingly, a recursive formula must retain the resulting interior sources and must account for the transmission data at artificial interfaces. Pairing two real steps is relevant to a proposed complex-dimensional construction, but its geometric realization and complex structure require their own verification.

This paper gives a self-contained analytic formulation of that reconstruction problem. The Green–Poisson representation in Theorem 4.2, with its regularized proof in Appendix A, is the starting point. Theorem 3.1 gives the compatible hypersurface Clifford connection. Theorem 5.2 proves the composition formula, including every interior Green term, for an admissible realized chain. Theorems 7.3 and 7.4, together with Proposition 7.5, distinguish a partition of one global Green integral from reconstruction with independent piecewise Dirichlet inverses. The cylinder and Proposition 6.4 give exact two-step examples; the latter ends on a complex hypersurface in a product neighborhood.

The analytic background is classical. Dirac bundles and the Bochner–Weitzenböck description of their squares are treated by Lawson and Michelsohn [4], Chapter II, Booß-Bavnbek and Wojciechowski [5], pp. 10–28, and Gilkey [6][7]. For elliptic realizations, traces, Poisson operators and Green operators we use Grubb's boundary calculus [8], Sections 1.3–1.6, its account in [9], Chapters 4 and 9–11, and Taylor [10]. First-order boundary projectors, including Calderón and Atiyah–Patodi–Singer projectors, concern a

different realization problem; see [5][11][12][13]. The present analysis uses the second-order full Dirichlet problem. The Green–Poisson formula, the hypersurface formula and the underlying elliptic estimates are classical. We organize their simultaneous use with explicit source, module and interface compatibility and examine the consequences in exact models.

The work is organized as a two-paper programme. The present article, Paper A1, develops the analytic reconstruction framework. Paper B1, under development, concerns the further nonlinear existence and admissibility questions, together with the identification of geometric cycle classes required by the proposed Hodge application. All analytic assertions used in A1 are proved here or taken from the cited published literature. The composition theorem assumes a realized chain; it does not construct one on an arbitrary manifold. The complex product example has finite-dimensional transfer spaces. Unrestricted Sobolev continuation and realization of a prescribed Hodge class remain additional requirements of the programme.

Sections 2–4 establish the one-stage analytic framework. Section 5 proves the composition theorem, and Section 6 gives its examples. Sections 7 and 8 treat interfaces and fixed-domain differentiation. Section 9 addresses nontrivial normal bundles, Section 10 records the geometric scope of the results, and Section 11 concludes the paper.

2. Dirac data and elliptic realizations

All manifolds and bundle coefficients are smooth. A boundary is smooth and has no corners unless explicitly stated otherwise. Sobolev spaces are the L^2 spaces of sections, defined using a finite atlas and smooth bundle trivializations; the equivalent norm obtained from a smooth connection may also be used. We write Tr for the value trace and H^1_0 for the closure of compactly supported interior sections in H^1. Our Hermitian products are linear in the first argument.

Definition 2.1 (Dirac bundle and Dirichlet realization)**.** Let (M,g) be a compact Riemannian manifold with boundary, and let (V,h) be a finite-rank Hermitian bundle. A Dirac bundle consists of a Hermitian connection ∇ and a real-linear Clifford action $c : TM \to \mathrm{End}(V)$ satisfying

$$c(v)c(w) + c(w)c(v) = -2g(v,w)\mathrm{id}, \qquad c(v)^* = -c(v),$$

$$\nabla_X(c(Z)u) = c(\nabla^M_X Z)u + c(Z)\nabla_X u.$$

Its Dirac operator and Dirac Laplacian are

$$Q = \sum_{a=1}^{\dim M} c(e_a)\nabla_{e_a}, \qquad P = Q^2,$$

where (e_a) is any local orthonormal frame. We use the inward unit normal ν. The full Dirichlet realization of the *second-order* expression P on $L^2(M,V)$ has domain

$$D(P_D) = H^2(M,V) \cap H^1_0(M,V).$$

Whenever its inverse is used we assume

$$0 \notin \sigma(P_D). \tag{2.1}$$

This realization is not defined as the square of a first-order operator with an unspecified boundary projector.

Lemma 2.2 (Green identities). *For smooth sections u, v,*

$$\int_M \big(\langle Qu, v\rangle - \langle u, Qv\rangle\big) = -\int_{\partial M} \langle c(\nu)u, v\rangle, \tag{2.2}$$

$$\int_M \big(\langle Pu, v\rangle - \langle u, Pv\rangle\big) = -\int_{\partial M} \big(\langle c(\nu)Qu, v\rangle + \langle c(\nu)u, Qv\rangle\big). \tag{2.3}$$

The measures are the Riemannian measures on the indicated manifolds. For $u \in H^2 \cap H^1_0$ one has $(P_D u, u)_{L^2} = \|Qu\|^2_{L^2}$.

Proof. Apply the divergence theorem to the vector field whose pairing with a real tangent vector X is $\langle c(X)u, v\rangle$. The compatibility of ∇, the skew-adjointness of $c(X)$, and the outward normal $-\nu$ give (2.2). Apply that identity to (Qu, v) and (u, Qv) and add to obtain (2.3). Applying (2.2) to (Qu, u) with $\operatorname{Tr} u = 0$ gives the energy identity. Approximation and the Sobolev trace theorem extend it to the stated domain. These are the standard bundle Green identities; compare [8], Section 1.3 and [5]. $\square$

Proposition 2.3 (Dirichlet solution operators). *The operator P_D is self-adjoint, nonnegative, elliptic, and has compact resolvent. Under (2.1), let $\mathscr{G} = P_D^{-1}$. For every real $s \geq 2$ there are bounded operators*

$$\mathscr{G} : H^{s-2}(M,V) \longrightarrow H^s(M,V) \cap H^1_0(M,V),$$

$$\mathscr{P} : H^{s-1/2}(\partial M, V|_{\partial M}) \longrightarrow H^s(M,V)$$

such that $P\mathscr{G}f = f$, $P\mathscr{P}g = 0$, and $\operatorname{Tr}\mathscr{P}g = g$. The map

$$(P, \operatorname{Tr}) : H^s(M,V) \longrightarrow H^{s-2}(M,V) \oplus H^{s-1/2}(\partial M, V|_{\partial M})$$

is an isomorphism with inverse $(f,g) \mapsto \mathscr{G}f + \mathscr{P}g$. In particular,

$$\|\mathscr{G}f\|_{H^s} + \|\mathscr{P}g\|_{H^s} \leq C_s\big(\|f\|_{H^{s-2}} + \|g\|_{H^{s-1/2}}\big). \tag{2.4}$$

Proof. The Bochner–Weitzenböck formula writes $P = \nabla^*\nabla + \mathscr{R}$ with a smooth Hermitian endomorphism $\mathscr{R}$; see [4], Chapter II, [5], pp. 26–28, and [6][7]. On H^1_0, the form $\|Qu\|^2_{L^2}$, after adding a sufficiently large multiple of $\|u\|^2_{L^2}$, is equivalent to the H^1 norm. Its self-adjoint realization has domain $H^2 \cap H^1_0$ by smooth Dirichlet boundary regularity. The compact embedding $H^1_0 \hookrightarrow L^2$ gives compact resolvent; Lemma 2.2 gives nonnegativity. These facts and the estimate

$$\|u\|_{H^s} \le C_s\big(\|Pu\|_{H^{s-2}} + \|u\|_{L^2}\big), \qquad \operatorname{Tr} u = 0,$$

are the classical elliptic Dirichlet theory [9], Chapters 4 and 9–11; see also [8], Sections 1.4–1.6 and [10]. The estimate holds on integer scales and extends to real $s \ge 2$ by the standard elliptic Sobolev regularity theorem. Assumption (2.1) bounds the L^2 inverse and hence gives the asserted bound for $\mathscr{G}$. Choose a bounded right inverse E_s of trace. Set

$$\mathscr{P} g = E_s g - \mathscr{G}(P E_s g).$$

This has the required equation and trace and satisfies (2.4). If two extensions are chosen, their difference belongs to the zero-trace domain and is removed by $\mathscr{G}P$; thus $\mathscr{P}$ is independent of E_s. Uniqueness follows from (2.1), proving the isomorphism. □

Proposition 2.4 (An invertibility certificate)**.** *Suppose every connected component of M meets its boundary. Assume Q extends as a smooth Dirac-type operator across that boundary and the extension has weak unique continuation: a distributional solution of $Qv = 0$ which vanishes on a nonempty open set vanishes on its connected component. Then $\ker P_D = 0$, and the self-adjoint elliptic Dirichlet realization P_D is invertible.*

Proof. Elliptic regularity makes a kernel element u smooth. The Green identity and $\operatorname{Tr} u = 0$ give $(P_D u, u) = \|Qu\|^2$, hence $Qu = 0$. Extend u by zero across the boundary. Its zero trace ensures that its first distributional derivatives have no boundary delta term, so the extension solves $Qv = 0$ and vanishes on an exterior collar. Unique continuation gives $u = 0$ on every component. The self-adjoint elliptic Dirichlet operator has compact resolvent; absence of zero from its kernel now excludes zero from its spectrum. □

Remark 2.5 (First-order boundary conditions)**.** The continuation hypothesis in Proposition 2.4 is an explicit sufficient condition for (2.1). Unique continuation for Dirac operators is treated in [5], pp. 43–49. Compact resolvent alone does not exclude a zero eigenspace. Full bundle Dirichlet data for Q^2 also differ from the absolute or relative de Rham conditions and from elliptic boundary projectors for Q; for these

other realizations see [11][12][13]. The Green functions for first-order local boundary problems in [14] do not replace the second-order inverse used here.

Example 2.6 (Dolbeault–Dirac data). On a Kähler manifold the Dolbeault–Dirac operator is

$$Q = \sqrt{2}(\bar\partial + \bar\partial^{*}), \qquad P = 2\Box_{\bar\partial}, \qquad \Box_{\bar\partial} = \bar\partial\bar\partial^{*} + \bar\partial^{*}\bar\partial.$$

For $(0,q)$-forms the bundle is $\Lambda^{0,*}T^{*}M$. For fixed p one instead uses $\Lambda^{p,0}T^{*}M \otimes \Lambda^{0,*}T^{*}M$, with its induced Hermitian connection; the untwisted $(0,*)$ bundle does not contain every (p,q)-form. The Kähler identities give $2\Box_{\bar\partial} = \Delta_d$ on ordinary complex forms. These constructions are described in [4], Chapter II, [6] and [15]. Every later identification with an intrinsic form bundle must respect its connection and grading.

3. Induced hypersurface Dirac bundles

Let $Y \subset M$ be a smooth two-sided hypersurface with unit normal ν and shape operator $A(X) = -\nabla^{M}_{X}\nu$. The construction uses real hypersurfaces; no complex structure on Y is assumed. Compatibility with the induced Clifford action requires the following connection correction.

Theorem 3.1 (Hypersurface Dirac bundle). *On $V|_{Y}$ define, for $X \in TY$,*

$$c_{Y}(X) = -c(\nu)c(X), \qquad \nabla^{Y}_{X} = \nabla^{V}_{X} + \frac{1}{2}c(\nu)c(AX). \tag{3.1}$$

Then $(V|_{Y}, h|_{Y}, \nabla^{Y}, c_{Y})$ is a Dirac bundle over Y. If $Q_{Y} = \sum_{a=1}^{\dim Y} c_{Y}(e_{a})\nabla^{Y}_{e_{a}}$, then, for an ambient smooth extension u,

$$Q_{Y}(u|_{Y}) = \left(-c(\nu)Qu - \nabla^{V}_{\nu}u + \frac{1}{2}\operatorname{tr}(A)u\right)\Big|_{Y}. \tag{3.2}$$

The right side is independent of the extension.

Proof. Clifford anticommutation gives $c_{Y}(X)^{2} = -|X|^{2}$ and $c_{Y}(X)^{*} = -c_{Y}(X)$. The correction in (3.1) is skew-adjoint. With $B_{X} = \frac{1}{2}c(\nu)c(AX)$, the Weingarten and Gauss formulas give

$$\nabla^{V}_{X}(c_{Y}(Z)) - c_{Y}(\nabla^{TY}_{X}Z) = c(AX)c(Z) + \langle AX, Z\rangle,$$

and $[B_{X}, c_{Y}(Z)]$ is the negative of this expression. Thus the Clifford connection identity holds. Expand Q_{Y} using (3.1); the zeroth-order sum is $-\frac{1}{2}\sum_{a} c(e_{a})c(Ae_{a}) = \frac{1}{2}\operatorname{tr}(A)$. The remaining tangential sum is $-c(\nu)Q - \nabla^{V}_{\nu}$, proving (3.2). □

This convention agrees with the spinorial Gauss formula in [16], Section 2, (6)–(9); the argument above applies directly to a general Clifford module. The resulting bundle is the *restricted* module. Identifying it with an intrinsic spinor, de Rham or Dolbeault module is an additional theorem, including any parity and multiplicity factors. In particular an odd real-dimensional hypersurface does not acquire a Dolbeault complex merely by restriction. For the de Rham bundle there is a vector-bundle splitting

$$(\Lambda^* T^* M)|_Y \simeq \Lambda^* T^* Y \oplus \nu^* \wedge \Lambda^* T^* Y;$$

it is not a rank-preserving identification with only one intrinsic exterior bundle.

3.1. Restricted modules and intrinsic multiplicities

Proposition 3.2 (Rank and projection tests)**.** *For a real hypersurface Y of an m-manifold, the restricted exterior bundle has rank 2^m, while $\Lambda^* T^* Y$ has rank 2^{m-1}. Tangential projection from the former to the latter is not an intertwiner for the action $c_Y(X) = -c(\nu)c(X)$ and the standard intrinsic de Rham action. For a complex hypersurface H of a complex n-manifold, $(\Lambda^{0,*} T^* X)|_H$ has rank 2^n, whereas $\Lambda^{0,*} T^* H$ has rank 2^{n-1}.*

Proof. The real orthogonal splitting is $\Lambda^* T^* M|_Y = \Lambda^* T^* Y \oplus \nu^* \wedge \Lambda^* T^* Y$. For a tangent unit vector X, the ambient de Rham action is $c(X) = X^* \wedge - \iota_X$. Thus $c_Y(X) 1 = -\nu^* \wedge X^*$ has zero tangential projection, whereas intrinsic multiplication sends 1 to X^*. This disproves the asserted intertwining on the whole restricted module. In the complex case the Hermitian tangential-normal splitting gives the graded vector-bundle decomposition

$$(\Lambda^{0,*} T^* X)|_H \simeq \Lambda^{0,*} T^* H \otimes \Lambda^{0,*}(N^*_{H/X}),$$

whose normal factor has rank two. This proves the rank statements; it is not an operator-identification theorem. □

Thus an intrinsic realization requires a specified summand, quotient or multiplicity identification, together with connection and operator compatibility. The real hypersurface formula is classical; compare [16][17].

Lemma 3.3 (Flat tangential squares)**.** *In a flat trivial Dirac bundle with constant Clifford matrices, successive coordinate hyperplanes inherit constant Clifford actions by Theorem 3.1. At each level the induced operator has square equal to minus the sum of the remaining coordinate second derivatives, acting componentwise on the full restricted bundle.*

Proof. Every coordinate hyperplane is totally geodesic, so the connection correction vanishes. Its induced Clifford matrices are constant and satisfy $c_a c_b + c_b c_a = -2\delta_{ab}$. Squaring $\sum c_a \partial_a$ cancels all mixed derivatives and yields $-\sum \partial_a^2$. Induction proves the claim at every coordinate level. □

This supplies an operator check in the flat model, not a canonical reduction to one intrinsic Dolbeault bundle. The rank does not fall merely because the base dimension falls.

4. Green–Poisson reconstruction

Definition 4.1 (Right-acting Dirichlet Green kernel).

Under (2.1), let $\mathscr{G} = P_D^{-1}$. For an interior source point $x \in M^\circ$, its right-acting kernel is

$$G_x(y) \in \mathrm{Hom}(V_x, V_y), \qquad y \in M \setminus \{x\}.$$

It is a section of $\pi_2^* V \otimes \pi_1^* V^*$ over $(M \times M) \setminus \Delta$, with $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$. It is smooth off the diagonal and understood distributionally across it. For every $\xi \in V_x$,

$$P_y(G_x(\cdot)\xi) = \delta_x \xi, \qquad \mathrm{Tr}_{y \in \partial M}(G_x(y)\xi) = 0. \tag{4.1}$$

Here $\delta_x \xi$ is the V-valued point mass at x; in particular, for a smooth test section ϕ with zero boundary trace,

$$\int_M \langle G_x(y)\xi, P\phi(y)\rangle \, d\mathrm{vol}_g(y) = \langle \xi, \phi(x)\rangle. \tag{4.2}$$

The identity on the diagonal in (4.1) is a distribution; there is no canonical off-diagonal identity map from V_x to V_y. The adjoint kernel $G_x(y)^* : V_y \to V_x$ is the alternative convention used for the integral operator. The kernel of $\mathscr{G}$ acts by $\mathscr{G}f(y) = \int_M G_x(y) f(x)\, d\mathrm{vol}_g(x)$. Self-adjointness gives $G_x(y)^* = G_y(x)$ off the diagonal.

Define the right-acting boundary kernel and its fibre adjoint by

$$A_x(y) = -c(\nu_y) Q_y G_x(y) : V_x \longrightarrow V_y, \qquad K(x,y) = A_x(y)^* : V_y \longrightarrow V_x.$$

The adjoint is essential because the operator acting on boundary values must map to the evaluation fibre V_x. All kernel compositions below use these displayed fibre directions. The distributional and off-diagonal descriptions follow from the elliptic Green construction; see [8][9][18].

Theorem 4.2 (Poisson representation). *For an interior point $x \in M^\circ$ and a smooth section u on M satisfying $Pu = 0$,*

$$\langle \xi, u(x) \rangle = \int_{\partial M} \langle A_x(y)\xi, u(y) \rangle \, d\mathrm{vol}_{\partial M}(y), \tag{4.3}$$

$$u(x) = \int_{\partial M} K(x,y) u(y) \, d\mathrm{vol}_{\partial M}(y). \tag{4.4}$$

Boundary evaluation of this kernel formula is understood only through its trace or limiting interpretation. Let $\mathscr{P}f$ denote the solution of $Pu = 0$, $\mathrm{Tr} u = f$. For smooth f it is smooth up to the boundary. For $s \geq 2$ it extends boundedly from $H^{s-1/2}(\partial M)$ to $H^s(M)$, with trace f.

Proof. Appendix A proves (4.3) by regularizing the point source before applying Green's identity. Taking the fibre adjoint and using nondegeneracy of h gives (4.4). Proposition 2.3 supplies the Poisson operator and its Sobolev bound. This is the conormal Poisson representation of the second-order elliptic Dirichlet problem; compare [8][9]. □

Proposition 4.3 (Boundary trace of the Poisson representation). *Let $f \in C^\infty(\partial M, V|_{\partial M})$ and $u = \mathscr{P}f$. Then $\mathrm{Tr} u = f$. In particular, for $z \in \partial M$,*

$$\lim_{x \to z,\ x \in M^\circ} \int_{\partial M} K(x,y) f(y) \, d\mathrm{vol}_{\partial M}(y) = f(z),$$

with the limit interpreted through the smooth Dirichlet solution; at finite Sobolev regularity the assertion is a trace identity, not a pointwise limit. No separate boundary jump term is added to the interior Poisson formula.

Proof. The representation theorem identifies the displayed integral with $\mathscr{P}f$ at interior points. The construction $\mathscr{P}f = Ef - \mathscr{G}(PEf)$ has trace f, and smooth Dirichlet regularity extends it to the boundary. The assertion follows from its continuity. For Sobolev data use the trace theorem and the stated boundedness of $\mathscr{P}$. □

Corollary 4.4 (Integrated Green–Poisson identity). *For smooth ψ and every smooth P-harmonic u, let*

$$\zeta_\psi(y) := \int_M G_x(y) \psi(x) \, d\mathrm{vol}_g(x).$$

Then

$$\int_M \langle \psi(x), u(x) \rangle \, d\mathrm{vol}_g(x) = -\int_{\partial M} \langle c(\nu_y) Q_y \zeta_\psi(y), u(y) \rangle \, d\mathrm{vol}_{\partial M}(y). \tag{4.5}$$

Proof.

The kernel convention gives $\zeta_\psi = \mathscr{G}\psi$. It is smooth, has zero boundary trace, and satisfies $P\zeta_\psi = \psi$. Apply (2.3) to (ζ_ψ, u); its second boundary term and its second volume term vanish. This proves (4.5) without an interchange involving the singular kernel. □

Proposition 4.5 (Source-corrected identity). *Every smooth section u satisfies*

$$u = \mathscr{P}\mathrm{Tr}u + \mathscr{G}(Pu). \tag{4.6}$$

Proof. Subtract the right side from u. The difference has zero trace and zero image under P, hence vanishes by (2.1). □

Example 4.6 (A boundary-only formula needs harmonicity)**.** Choose a nonzero $u \in C_c^\infty(M^\circ, V)$. Both u and all of its derivatives vanish on ∂M, so any formula involving only its boundary data has zero right-hand side and cannot reproduce u. Indeed Pu cannot vanish under the invertible Dirichlet realization: (4.6) gives $u = \mathscr{G}(Pu)$. Thus the Poisson boundary formula applies to $Pu = 0$, whereas an arbitrary section requires the interior Green source. This elementary test also rules out removing the volume term by a mere exchange of the two kernel variables.

Remark 4.7. For a section v vanishing on the boundary its tangential covariant derivatives vanish there. Thus $Qv = c(\nu)\nabla_\nu v$ and $-c(\nu)Qv = \nabla_\nu v$. This identifies the conormal Poisson formula. A Dirichlet Poisson trace is not a Calderón projector for the first-order equation $Qu = 0$; the latter is discussed in [5], pp. 67–104 and [11].

4.1. The Green corrector and its residual

Proposition 4.8 (Parametrix correction)**.** *The identity $G = E - H$ is valid if $E_x\xi$ is an exact fundamental section and H solves the homogeneous Dirichlet extension of its boundary values. A local parametrix need not be exact. More generally, if*

$$P(E_x\xi) = \delta_x\xi + r_x\xi$$

with $r_x\xi$ smooth and $E_x\xi$ smooth near the boundary, then

$$G_x\xi = E_x\xi - \mathscr{P}\mathrm{Tr}(E_x\xi) - \mathscr{G}(r_x\xi). \tag{4.7}$$

Proof. The right side has zero trace and its image under P is $\delta_x\xi$; uniqueness identifies it with the Dirichlet kernel. □

This is a consistency identity after the inverse has been established. On a closed component a global exact fundamental kernel cannot simply be assumed in the presence of zero modes. These distinctions matter for decomposition independence.

5. Realized dimension-lowering chains

We now separate geometric realization from the composition calculation.

Definition 5.1 (Admissible realized chain)**.** For $j = 0, \dots, r-1$, let (M_j, V_j, P_j) have the Dirichlet Poisson and Green operators $\mathscr{P}_j, \mathscr{G}_j$ above. Let M_r carry terminal data and choose their Banach space $\mathscr{X}_r$. For $0 \le j < r$, specify Banach spaces $\mathscr{X}_j \subset H^{s_j}(M_j, V_j)$, $s_j \ge 2$, source spaces $\mathscr{Y}_j \subset H^{s_j-2}(M_j, V_j)$, and bounded maps

$$I_j : \mathscr{X}_{j+1} \longrightarrow H^{s_j - 1/2}(\partial M_j, V_j).$$

Require $E_j := \mathscr{P}_j I_j : \mathscr{X}_{j+1} \to \mathscr{X}_j$ and $\mathscr{G}_j : \mathscr{Y}_j \to \mathscr{X}_j$ to be bounded. A compatible tuple has $\operatorname{Tr}_j u_j = I_j u_{j+1}$ and $h_j := P_j u_j \in \mathscr{Y}_j$. Norms on the subspaces are specified as part of the data. Terminal spaces can be finite-dimensional, including data on a zero-dimensional manifold.

The subspaces are essential when cutting a closed boundary into pieces. Piecewise H^s functions do not in general glue to H^s functions. Every trace-matching condition, including derivatives required at the chosen regularity, must be included. Moreover both the Poisson contribution and the Green contribution must separately satisfy the conditions needed for the next composition. Cancellation of incompatible interface terms only in their sum does not justify composing the separate summands.

Theorem 5.2 (Recursive reconstruction with source terms)**.** *For an admissible realized chain and compatible tuple,*

$$u_0 = E_0 \cdots E_{r-1} u_r + \sum_{j=0}^{r-1} E_0 \cdots E_{j-1} \mathscr{G}_j h_j. \tag{5.1}$$

The empty product is the identity. Every term belongs to $\mathscr{X}_0$. In the specified norms,

$$\begin{aligned} \|u_0\|_{\mathscr{X}_0} \le{}& \left(\prod_{i=0}^{r-1} \|E_i\| \right) \|u_r\|_{\mathscr{X}_r} \\ &+ \sum_{j=0}^{r-1} \left(\prod_{i=0}^{j-1} \|E_i\| \right) \|\mathscr{G}_j\| \|h_j\|_{\mathscr{Y}_j}. \end{aligned}$$

Grouping two successive steps gives

$$u_j = E_j E_{j+1} u_{j+2} + \mathscr{G}_j h_j + E_j \mathscr{G}_{j+1} h_{j+1}. \tag{5.2}$$

If every $h_j = 0$, only the terminal term in (5.1) remains.

Proof. The source-corrected identity gives $u_j = E_j u_{j+1} + \mathscr{G}_j h_j$. Successive substitution proves (5.1) and (5.2); every intermediate summand belongs to the required domain by Definition 5.1. The norm bound follows from boundedness and the triangle inequality. □

The corresponding kernels may be understood as distribution kernels of the bounded composed operators. Writing them as pointwise iterated integrals requires an additional integrability or regularization argument. Fubini is not justified by notation alone. At level j the measure is $d\mu_j = d\mathrm{vol}_{g_j}$ for the specified induced metric.

Proposition 5.3 (A sufficient Fubini criterion)**.** *Let $S_0, \dots, S_r$ be compact manifolds with finite measures and finite-rank Hermitian bundles. Suppose E_j acts by a measurable kernel $K_j(x_j, x_{j+1})$ and f is terminal data. If, for almost every x_0,*

$$\int_{S_1 \times \cdots \times S_r} \prod_{j=0}^{r-1} \|K_j(x_j, x_{j+1})\| \, \|f(x_r)\| \, d\mu_1 \cdots d\mu_r < \infty,$$

then the iterated kernel formula for $E_0 \cdots E_{r-1} f$ is absolutely convergent and independent of the order of integration, with the order of the fibre maps kept fixed, whenever the successive operators are represented by these integrals. In particular this holds for bounded kernels and integrable f.

Proof. The norm of the composed fibre map applied to f is bounded by the displayed scalar integrand. Tonelli applied to this bound and Fubini applied in finite bundle trivializations justify the successive integrations. A finite partition of unity makes the conclusion independent of trivialization. If all kernels are bounded, the integral is at most the product of their bounds, the intermediate volumes, and $\|f\|_{L^1}$. □

Remark 5.4. Interior Poisson kernels are smooth when the evaluation point stays away from its own boundary. This does not make every iterated boundary restriction bounded. At singular interfaces one must prove the displayed integrability or use bounded Sobolev operator composition, without identifying it with an unjustified pointwise integral.

5.1. One-step and paired-step bookkeeping

If $\dim_{\mathbb{R}} M_0 = m$, a realized real hypersurface chain has $\dim_{\mathbb{R}} M_j = m - j$. Pairing two steps lowers real dimension by two. If $m = 2n$, an even level has the dimension of a potential complex $(n - j/2)$-manifold, but complex structure requires a separate theorem. Odd levels are real intermediates. Intrinsic form degree, cycle codimension and recursion depth must not be denoted by the same index.

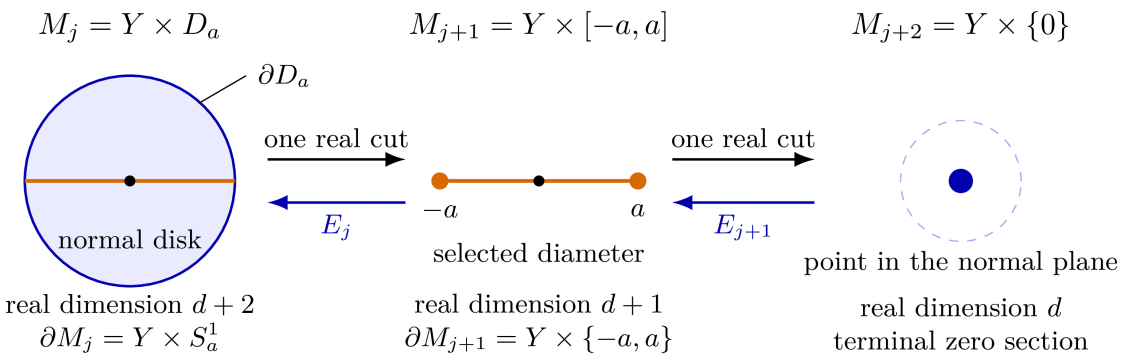


Figure 1. A paired real step, drawn in the normal plane of the product model. The common Y factor is suppressed: a point in the right panel represents the entire terminal manifold. Geometric cuts run to the right, whereas reconstruction runs to the left. At each arrow the source-corrected identity is $u_j = E_j u_{j+1} + \mathscr{G}_j h_j$. The selected diameter is an interior cut, not the outer circle boundary. Its assembly into outer boundary data must be specified, as it is in Proposition 6.4. Two real steps alone do not supply a complex structure.

Proposition 5.5 (Compatible refinement and propagated defects).

Let $E_i, \widetilde{E}_i : \mathscr{X}_{i+1} \to \mathscr{X}_i$ be bounded operators, $0 \le i < r$. Then

$$E_0 \cdots E_{r-1} - \widetilde{E}_0 \cdots \widetilde{E}_{r-1} = \sum_{j=0}^{r-1} E_0 \cdots E_{j-1}(E_j - \widetilde{E}_j)\widetilde{E}_{j+1} \cdots \widetilde{E}_{r-1}.$$

The norm of the difference is bounded by the sum of the products of the norms of these factors. In particular, if $\mathscr{Z}_j$ is a Banach space, $S_j : \mathscr{X}_{j+1} \to \mathscr{Z}_j$ and $R_j : \mathscr{Z}_j \to \mathscr{X}_j$ are bounded, and $R_j S_j = E_j$ on the admissible domain, replacing E_j by these two successive arrows preserves the operator product. In a source-corrected chain, the source terms and their transfers must also remain compatible.

Proof. Insert the mixed products in which the factors are changed one at a time and telescope. Submultiplicativity gives the estimate. Exact factorization makes every defect zero. For the source-corrected formula the same reasoning applies to each term, provided the source data and their maps are preserved. □

This is the precise conditional operator meaning of refinement invariance. Equality of pairings follows after applying any common continuous linear functional. It does not imply that independently cut Dirichlet Green operators agree: that requires the transmission data in Theorem 7.3. The half-derivative

boundary-to-interior gain cannot be iterated as a global Sobolev gain without the assembly maps and matching traces of Definition 5.1; this is a requirement of the realized-chain hypotheses.

6. Examples and compatibility tests

Example 6.1 (Restriction need not preserve harmonicity). For the scalar Laplacian in the plane, $u(x,y) = x^2 - y^2$ is harmonic. On the line $y = 0$ its restriction is x^2, whose one-dimensional Laplacian is nonzero. Thus ambient harmonicity does not supply $h_1 = 0$ in Theorem 5.2. This scalar example already rules out an unrestricted inference for all Dirac-bundle components.

Example 6.2 (One-step normalization). On $[0, L]$ with $Q = i\,d/dx$, $P = -d^2/dx^2$, the Dirichlet kernel is

$$G(x,y) = \frac{\min(x,y)(L - \max(x,y))}{L}.$$

Its inward derivatives at $y = 0, L$ are $1 - x/L$ and x/L. The Poisson formula therefore gives $u(x) = (1 - x/L) f(0) + (x/L) f(L)$, fixing the normal sign.

Example 6.3 (A nonconstant realized two-step chain). Let $M_0 = [0, L] \times S^1$ with the flat metric, $V_0 = \Lambda^* T^* M_0 \otimes \mathbb{C}$ and $Q_0 = d + d^*$. Its scalar sector has $P_0 = -\partial_t^2 - \partial_\theta^2$. Both boundary circles are totally geodesic. Their induced restricted Dirac bundles have the same fibre rank, and their squares act componentwise as $-\partial_\theta^2$ in the parallel trivialization. Cut each circle at $0, \pi$, obtaining four intervals M_1, and let M_2 consist of their endpoints with the shared values specified below.

Use admissible scalar traces which extend to smooth, even, π-periodic functions on each circle. At finite Sobolev regularity use these symmetry subspaces of the global periodic Sobolev spaces, with their periodic Sobolev norms, and restrict them to the intervals. This definition includes all interface compatibility, including at critical trace exponents. The terminal values are constant on the endpoints belonging to each original circle; E_1 reconstructs these constants on the intervals. For sources use the even, π-periodic zero-mean subspace, with the same copies on the intervals. Its Dirichlet Green image glues: on the Fourier generators

$$\mathscr{G}_1(\cos(q\theta)) = \frac{\cos(q\theta) - 1}{q^2}, \qquad q = 2m,\ m \geq 1.$$

Fourier expansion and the Dirichlet estimates extend this boundedly to the stated source spaces. Thus both summands satisfy the interface requirements, and I_0 is the bounded assembly map to the two boundary circles. More precisely, let $\mathscr{X}_1$ be the paired global even, π-periodic $H^{5/2}$ spaces restricted to the four intervals, with the transported global norms. Let $\mathscr{X}_0$ be their P_0-harmonic Poisson extensions

in $H^3(M_0, V_0)$ and give it the H^3 norm. Take $\mathscr{Y}_0 = 0$, and for $\mathscr{Y}_1$ use the paired even, π-periodic, zero-mean $H^{1/2}$ source spaces restricted to those intervals. Together with the two-dimensional terminal space, these choices give $s_0 = 3$ and $s_1 = 5/2$. Proposition 2.3 and the displayed Fourier formula verify every boundedness condition in Definition 5.1.

For $q = 2m$, set $u_0(t, \theta) = e^{qt}\cos(q\theta)$. It is P_0-harmonic. On the intervals the restricted data have $h_1 = q^2 u_1$, while $h_0 = 0$. The endpoint values are 1 on the lower circle and e^{qL} on the upper circle. Consequently

$$E_0 E_1 u_2 = b(t), \qquad E_0 \mathscr{G}_1 h_1 = e^{qt}\cos(q\theta) - b(t), \quad b(t) = 1 + \frac{e^{qL} - 1}{L}t.$$

Equation (5.2) reconstructs u_0 exactly. Any finite linear combination of these modes gives another example, so this is not confined to constant sections. It demonstrates two real dimension drops and the necessity of source terms. It does not construct a complex hypersurface or prove the Hodge-specific geometric recursion.

6.1. A projective product-neighborhood test

The next construction puts the second cut on a genuine complex hypersurface. It is a deliberately narrow geometric model, with all transfers specified; it does not identify the first-order restricted module with an intrinsic Dolbeault module.

Proposition 6.4 (Two real cuts ending on a complex hypersurface). *Let Y be a smooth connected projective Kähler manifold and $X = Y \times \mathbf{P}^1$. Fix an affine coordinate $z = x + iy$ near 0, a closed coordinate disk $D_a = \{|z| \le a\}$, and the product metric $g_Y + dx^2 + dy^2$ on $M_0 = Y \times D_a$. Put*

$$M_1 = Y \times [-a, a] \quad (y = 0), \qquad M_2 = Y \times \{0\} = H.$$

On the full ambient Dolbeault–Dirac module $V_0 = \Lambda^{0,*}T^*X|_{M_0}$ take $Q_0 = \sqrt{2}(\overline{\partial} + \overline{\partial}^{*})$ and $P_0 = Q_0^2$. At each cut use the restricted ambient bundle with the corrected induced Clifford action and connection of Theorem 3.1; denote the successive squares by P_1, P_2. In the parallel product trivialization their differential expressions are

$$P_0 = P_Y - \partial_x^2 - \partial_y^2, \qquad P_1 = P_Y - \partial_x^2, \qquad P_2 = P_Y, \quad P_Y = 2\Box_{\overline{\partial},Y}.$$

All components of the full Dirichlet realizations of P_0 and P_1 have zero kernel. For $q \ge 0$, let $\mathscr{K}_q = \ker\left(P_Y|_{\Omega^{0,q}(Y)}\right)$, with the normal exterior factor of degree zero, and put

$$b(x) = \cosh a + \frac{\sinh a}{a}x, \qquad c(x) = e^x - b(x).$$

On $\mathscr{X}_2 = \mathscr{K}_q$ and $\mathscr{X}_1 = \mathscr{K}_q \otimes \operatorname{span}\{b, c\}$ define the bounded finite-dimensional boundary map $I_1(s) = (e^{-a}s, e^{a}s)$, so that $E_1 s = b(x)s$. Define the even harmonic extension on $\mathscr{X}_1$ by

$$\mathscr{H}(bs) = b(x)s, \qquad \mathscr{H}(cs) = (e^{x}\cos y - b(x))s.$$

With $\mathscr{X}_0 = \mathscr{H}\mathscr{X}_1$, source spaces $\mathscr{Y}_0 = \{0\}$ and $\mathscr{Y}_1 = \{-e^{x}s : s \in \mathscr{K}_q\}$, set $I_0(v) = \operatorname{Tr}_{\partial M_0}\mathscr{H}(v)$, and let E_0 be the full Dirichlet Poisson extension of this trace. Then $E_0 v = \mathscr{H}(v)$ and, for $s \in \mathscr{K}_q$,

$$u_0 = e^{x}\cos y\, s, \quad u_1 = e^{x}s, \quad u_2 = s, \qquad h_0 = P_0 u_0 = 0, \quad h_1 = P_1 u_1 = -e^{x}s,$$

$$u_0 = E_0 E_1 u_2 + E_0 \mathscr{G}_1 h_1, \qquad \mathscr{G}_1 h_1 = c(x)s \neq 0 \quad (s \neq 0). \tag{6.1}$$

The maps and identity are bounded for the indicated finite-dimensional spaces with any fixed Sobolev norms, including H^{s} for $s \geq 2$.

Proof. The two cuts are totally geodesic and their normal fields are parallel, so the induced connection has no correction term in these coordinates. The Clifford anticommutation relations give the displayed squares; the product splitting of the Dolbeault Laplacian gives P_Y on the tangential factor. Integration by parts shows the Dirichlet energy on M_i equals $\|Q_i v\|^2$. Alternatively, separation into the nonnegative spectrum of P_Y and the strictly positive Dirichlet spectrum of the disk or interval proves that both second-order Dirichlet kernels vanish.

The function b is affine, with $b(\pm a) = e^{\pm a}$, and c vanishes at both endpoints. Since $P_Y s = 0$, one has $P_1(cs) = -e^{x}s$. Thus the Dirichlet Green inverse sends $-e^{x}s$ to cs, while the Poisson extension of $I_1 s$ is bs. The two displayed extensions by $\mathscr{H}$ are P_0-harmonic and restrict at $y = 0$ to bs and cs; uniqueness of the disk Dirichlet problem proves $E_0 = \mathscr{H}$ on this space. Adding them yields $e^{x}\cos y\, s$. Finite dimensionality gives all asserted fixed-norm bounds. The end hypersurface H is complex because $z = 0$. □

Remark 6.5 (Normal multiplicity and the precise limit)**.** At H, the full restricted ambient exterior module is $\Lambda^{0,*}T^{*}Y \otimes \Lambda^{0,*}\mathbb{C}$, which has two normal exterior degrees. Projection to normal degree zero commutes with P_2 in this product, and its square-level action is P_Y. This does not identify the induced *first-order* Dirac operator with the intrinsic Dolbeault operator, nor establish a general Hodge-cohomology descent. The flat product metric is used only on the displayed coordinate neighborhood; no globally flat metric on $\mathbf{P}^1$ is asserted.

The finite-dimensional map $\mathscr{H}$ is not a bounded even harmonic continuation on the full interval Sobolev scale. For high-frequency $\sin(kx)$, the even continuation $\sin(kx)\cosh(ky)$ grows exponentially

at fixed $y \neq 0$, while its interval Sobolev norms grow polynomially. Such a map cannot be substituted without qualification for a uniform moving-domain transfer. For nontrivial normal bundles a single global product coordinate is also absent.

Remark 6.6 (The role of the product hypothesis)**.** The selected diameter in Proposition 6.4 uses a trivial normal bundle. Proposition 9.1 gives a concrete obstruction to making that selection globally. The invariant disk lift in Proposition 9.3 avoids a selected diameter but provides only one source-corrected stage.

7. Artificial interfaces and decomposition independence

The differential expression on each piece does not determine the global inverse. Both value and conormal matching enter the realization; see [8], Sections 1.3–1.6 and [9], Chapters 10–11. The following elementary example isolates the interface contribution.

Example 7.1 (Independent Dirichlet inverses)**.** Let $P = -d^2/dx^2$ on $[-1,1]$ with zero data at ± 1. For source 1, the global solution is

$$u_{\mathrm{global}}(x) = \frac{1 - x^2}{2}.$$

Solve separately on $[-1,0]$ and $[0,1]$, also imposing zero data at the artificial interface. The assembled solution is

$$u_{\mathrm{split}}(x) = \frac{|x| - x^2}{2}.$$

It has derivative jump 1 at 0, so distributionally

$$P u_{\mathrm{split}} = 1 - \delta_0 \neq 1.$$

Thus the sum of the two Dirichlet Green solutions does not equal the global Green solution; their values at 0 already differ. The operator is the square of $Q = i\,d/dx$, so the example lies within the Dirac-bundle setting. It shows why an interface correction is necessary.

For a general second-order problem, a piecewise solution with matching values also needs matching conormal flux to avoid interface sources. For a homogeneous global problem, uniqueness proves equality only after the global PDE, all transmission conditions and the same external boundary data have been established. A piecewise corrector cannot be declared globally harmonic solely because the cut has measure zero.

Definition 7.2 (Transmission-compatible piecewise section). Let a smooth interior hypersurface Σ split M into two smooth pieces M_+ and M_-, with no corners at the external boundary. For $P = \nabla^*\nabla + \mathscr{R}$ with smooth Hermitian coefficients, a pair $u_\pm \in H^2(M_\pm, V)$ is transmission-compatible if its value traces agree in $H^{3/2}(\Sigma, V|_\Sigma)$ and

$$\nabla_{n_+} u_+ + \nabla_{n_-} u_- = 0 \quad \text{on } \Sigma,$$

where $n_\pm$ are the outward normals of the pieces. The flux equality is in $H^{1/2}(\Sigma, V|_\Sigma)$.

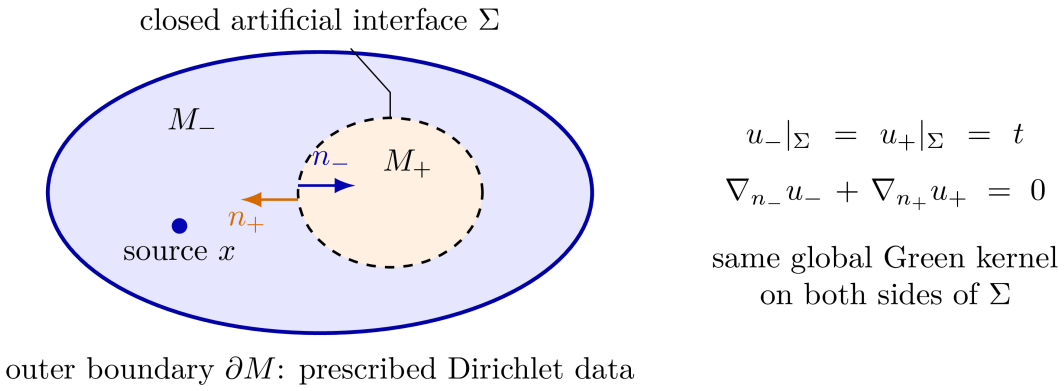


Figure 2. Transmission across a closed artificial cut disjoint from the external boundary, drawn schematically in two dimensions. The outward normals of the two pieces are opposite. The interface carries a common value and matching flux, rather than an extra homogeneous Dirichlet condition. For independently inverted piecewise Dirichlet problems, the missing trace t is restored by the two Poisson corrections in (7.2).

Theorem 7.3 (Transmission reconstruction). *Assume $0 \notin \sigma(P_D)$. If a transmission-compatible pair satisfies $Pu_\pm = f|_{M_\pm}$ for $f \in L^2(M, V)$ and has external trace $g \in H^{3/2}(\partial M, V)$, then its assembled section is the unique global H^2 solution and equals*

$$u = \mathscr{P} g + \mathscr{G} f.$$

Consequently any two such decompositions reconstruct the same section when f and g agree.

Proof. Equality of the value traces assembles the pair into an H^1 section. Integrating by parts on both pieces against a smooth global test section leaves an interface term given by the sum of the two outward normal derivatives. It vanishes by the flux condition. Thus $Pu = f$ distributionally on M, with the stated external trace. Elliptic interior and Dirichlet boundary regularity give $u \in H^2$ [9][10]. Subtract $\mathscr{P} g + \mathscr{G} f$ and use the zero-trace kernel assumption. The same uniqueness proves the last assertion. □

Theorem 7.4 (Cut invariance of the global Green integral)**.** *Suppose $\partial M \neq \varnothing$ and $0 \notin \sigma(P_D)$, with P and $\mathscr{G} = P_D^{-1}$ as in Definition 2.1. Let a smooth closed interior hypersurface Σ separate M into smooth pieces M_1, M_2; assume it does not meet ∂M. For $f \in L^2(M, V)$ put $u = \mathscr{G} f$. Then, as an identity of $H^2(M, V)$ sections,*

$$u(y) = \int_{M_1} G_x(y) f(x)\, d\mathrm{vol}_g(x) + \int_{M_2} G_x(y) f(x)\, d\mathrm{vol}_g(x). \tag{7.1}$$

The integrals denote the global Green operator applied to the two zero-extended source restrictions. Their sum is independent of Σ. For every source point $x \notin \Sigma$, the restrictions in the evaluation variable y of the same global kernel G_x have equal value traces across Σ and opposite outward conormal traces there (away from x). No homogeneous boundary condition is imposed on Σ.

Proof. The disjoint source decomposition $f = \mathbf{1}_{M_1} f + \mathbf{1}_{M_2} f$ holds almost everywhere; applying the fixed linear operator $\mathscr{G}$ proves (7.1). The integral notation is interpreted distributionally where the kernel is singular and almost everywhere where pointwise integration is available. Off its source, $G_x(\cdot)$ solves the same elliptic equation across the artificial surface and is smooth there. Its two restrictions therefore have the same trace and covariant normal derivative. Since $n_2 = -n_1$ on Σ, their outward conormal terms cancel in the two Green identities. The same cancellation holds for $u \in H^2$ in the Sobolev trace sense. The outer Dirichlet condition is that of P_D; there is no new condition at the cut. $\square$

Proposition 7.5 (The interface correction for piecewise Dirichlet kernels)**.** *Under the assumptions of Theorem 7.4, assume in addition that the full Dirichlet realizations $P_{i,D}$ on M_i are invertible. Let $\mathscr{G}_i = P_{i,D}^{-1}$ and let $\mathscr{P}_i t$ solve $Pv = 0$ on M_i with trace t on Σ and zero trace on the outer boundary of M_i. For $f \in L^2(M, V)$ define $v_i = \mathscr{G}_i(f|_{M_i})$ and $t = \mathrm{Tr}_\Sigma(\mathscr{G} f)$. Then*

$$(\mathscr{G} f)|_{M_i} = v_i + \mathscr{P}_i t, \qquad i = 1, 2, \tag{7.2}$$

and t satisfies the interface equation in $H^{1/2}(\Sigma, V|_\Sigma)$

$$(\Lambda_1 + \Lambda_2) t = -\big(\nabla_{n_1} v_1 + \nabla_{n_2} v_2\big)|_\Sigma, \qquad \Lambda_i t := \nabla_{n_i}(\mathscr{P}_i t)|_\Sigma. \tag{7.3}$$

Conversely, a trace $t \in H^{3/2}(\Sigma, V|_\Sigma)$ solving (7.3) and defining H^2 solutions $v_i + \mathscr{P}_i t$ assembles these solutions into $\mathscr{G} f$. The assumption on t can be weakened using the corresponding weak conormal trace spaces.

Proof. The restriction of $u = \mathscr{G} f$ and v_i solve the same inhomogeneous equation on M_i. Their difference is P-harmonic, vanishes at the outer boundary, and has trace t on Σ, giving (7.2) by uniqueness of the local Dirichlet problem. The global H^2 section u has matching values and conormal derivatives; inserting (7.2) yields (7.3). Conversely the common trace t gives matching values, and (7.3) gives matching flux.

Theorem 7.3 then identifies the assembly with $\mathscr{G} f$. Thus independent Dirichlet convolutions alone generally omit the two Poisson corrections $\mathscr{P}_i t$. □

Remark 7.6 (Fundamental sections and artificial cuts)**.** A global fundamental section which solves the exact delta equation and has the prescribed outer Dirichlet trace is, by invertibility, the unique global Dirichlet Green section. Its restrictions have transmission traces at an artificial cut, rather than additional zero Dirichlet traces. If a local parametrix is used, its residual source must be retained as in Proposition 4.8. If an exact fundamental section lacks the outer boundary condition, its harmonic boundary correction must be added. Agreement of values at the cut without agreement of conormal derivatives does not yield a global solution. The measure-zero property of the cut justifies partitioning a fixed global integral; it does not identify inverses defined using different boundary conditions.

Remark 7.7. This is a theorem about a fixed global operator and fixed complete data. It is not invariance of arbitrary dimension-changing constructions. Corners, changing operators and singular strata require separate results. The preceding split-interval counterexample violates precisely the flux condition.

8. Fixed-domain parameter differentiation

The following elementary identities prepare the moving-geometry analysis but do not establish a Nash–Moser theorem.

Proposition 8.1 (Variation of the inverse and Poisson operator)**.** *Pull a differentiable family of Dirichlet problems to fixed bundle and Sobolev spaces, with the trace operator fixed. Suppose $P_a : H^s \cap H^1_0 \to H^{s-2}$ is invertible near the parameter in question, and write $\mathscr{G}_a = P_{a,D}^{-1}$. If the coefficients vary differentiably in the operator norms used, then*

$$D\mathscr{G}_a[v] = -\mathscr{G}_a (DP_a[v]) \mathscr{G}_a, \tag{8.1}$$

$$D\mathscr{P}_a[v] f = -\mathscr{G}_a (DP_a[v]) \mathscr{P}_a f. \tag{8.2}$$

For a differentiable family $I_{j,a}$ and $E_{j,a} = \mathscr{P}_{j,a} I_{j,a}$,

$$DE_{j,a}[v] = -\mathscr{G}_{j,a} (DP_{j,a}[v]) \mathscr{P}_{j,a} I_{j,a} + \mathscr{P}_{j,a} (DI_{j,a}[v]).$$

Proof. On fixed Sobolev spaces the inverse map is differentiable, as follows from the resolvent identity. A common bounded trace extension E gives $\mathscr{P}_a = E - \mathscr{G}_a P_a E$, so its differentiability is justified as well. Differentiate $P_a \mathscr{G}_a = I$ on the fixed Dirichlet domain to obtain (8.1). Differentiate $P_a \mathscr{P}_a f = 0$ and $\operatorname{Tr} \mathscr{P}_a f = f$. The derivative has zero trace and source $-(DP_a[v]) \mathscr{P}_a f$, giving (8.2). The last assertion is the product rule. □

Proposition 8.2 (Compact parameter families). *Let K be a compact parameter set. Suppose, after fixed-domain identifications, $a \mapsto P_{a,D}$ is continuous as a family of invertible operators from $H^s \cap H^1_0$ to H^{s-2} for a fixed $s \geq 2$, and $a \mapsto P_a$ is continuous from H^s to H^{s-2}. Then $\mathscr{G}_a$ and $\mathscr{P}_a$ have uniformly bounded norms on the indicated Sobolev scales for $a \in K$.*

Proof. Inversion is continuous on the invertible bounded operators between two fixed Banach spaces: for P_b close to P_a, invert $I + \mathscr{G}_a(P_b - P_a)$ by a Neumann series. Thus $a \mapsto \mathscr{G}_a$ is norm-continuous. Compactness bounds its norm, and the formula $\mathscr{P}_a = E - \mathscr{G}_a P_a E$ with a common trace extension bounds the Poisson norm. For a geometric family one verifies the stated continuity and elliptic regularity from uniform coefficient and ellipticity bounds; see [8][9]. □

Remark 8.3. An open neighborhood in an infinite-dimensional parameter space need not be compact. Uniform bounds there, control of derivative loss, and smooth tame dependence require separate estimates. Propositions 8.1 and 8.2 do not supply those estimates.

9. Normal disk bundles and global realizations

The product model separates two real directions in a trivial complex normal line. This separation need not exist globally.

Proposition 9.1 (Obstruction to a selected normal diameter). *Let Y be a compact complex hypersurface of a complex manifold X, and let $N = N_{Y/X}$ be its complex normal line bundle. A cooriented real hypersurface $S \subset X$ containing Y determines a nowhere-zero real section of the underlying oriented rank-two bundle $N_{\mathbb{R}}$. Consequently $c_1(N) = 0$. If $Y = \mathbf{P}^{n-1} \subset \mathbf{P}^n$ is a hyperplane and $n \geq 2$, there is no such cooriented S. In this hyperplane case there is also no real line subbundle of $N_{\mathbb{R}}$, so an unoriented family of normal diameters is impossible.*

Proof. Choose a Hermitian metric on X. A global unit normal to the cooriented S, restricted to Y, is orthogonal to TY and is nonzero in $N_{\mathbb{R}}$. A nonzero real section v and its rotation Jv form an oriented frame; equivalently, v is a nonzero section of the complex line N. Thus N is trivial and $c_1(N) = 0$.

For a hyperplane the transverse affine coordinate changes by the transition functions of $\mathscr{O}(1)$, so $N \simeq \mathscr{O}(1)|_{\mathbf{P}^{n-1}}$. Its first Chern class restricts to the generator on any projective line and is nonzero; see [15][19] for these standard normal-bundle and Chern-class descriptions. This excludes cooriented S. Finally, every real line bundle over $\mathbf{P}^{n-1}$ is orientable, since $H^1(\mathbf{P}^{n-1}; \mathbb{Z}/2) = 0$, and an oriented real line

bundle is trivial. A real line subbundle of $N_{\mathbb{R}}$ would therefore have a nonzero section, giving the same contradiction. □

Remark 9.2. The general implication above concerns a *cooriented* hypersurface. Without coorientation a real line bundle need not have a global nonzero section. The stronger unoriented conclusion for a hyperplane uses the vanishing of its first cohomology group; it is not asserted for every base. This obstruction concerns the selected-diameter construction, not every possible invariant dimension-changing procedure.

The following one-stage construction needs no chosen normal diameter.

Proposition 9.3 (One-stage source-corrected disk lift)**.** *Let $L \to Y$ be a smooth Hermitian complex line bundle over compact Y, $M = D_a(L)$, and $V = \pi^* V_0$. Let a smooth Laplace-type P on V have an invertible full Dirichlet realization. For a smooth radial profile $\chi(r^2)$ equal to one near the zero section and nonzero at the outer boundary, set $Js = \chi(r^2)\pi^* s$, $b_s = \mathrm{Tr} Js$, $h_s = PJs$. For each integer $m \geq 2$, the trace lies in $H^{m-1/2}(\partial M, V|_{\partial M})$ when $s \in H^m(Y, V_0)$, and $h_s \in H^{m-2}(M, V)$. Then*

$$Js = \mathscr{P} b_s + \mathscr{G} h_s .$$

For every integer $m \geq 2$, $J : H^m(Y, V_0) \to H^m(M, V)$ and $s \mapsto h_s : H^m(Y, V_0) \to H^{m-2}(M, V)$ are bounded at fixed geometry. This is a global nonproduct one-stage lift, not a reverse restriction or a second real stage.

Proof. The squared fibre radius is smooth at the zero section. In finite bundle charts derivatives of Js through order m are bounded smooth coefficients times base derivatives of s through order m. Integration over compact fibres gives the first bound; applying P gives the second. The trace theorem supplies $b_s \in H^{m-1/2}(\partial M)$; the Poisson solution with this trace lies in $H^m(M)$ under the stated elliptic realization. The difference of the two sides has zero trace and zero P-image, hence vanishes by Dirichlet uniqueness. □

Remark 9.4 (Scope of the disk lift)**.** Proposition 9.3 is an extension-plus-Green identity at fixed geometry. Its boundary is the circle bundle $S_a(L)$, not a pair of globally selected normal directions. It proves neither a reverse restriction map nor a second ordinary Dirichlet stage at the zero section. A collapse of the circle fibres requires a separately specified operator and domain; it is not an ordinary boundary problem on a smooth manifold of one lower dimension.

10. Geometric scope

The reconstruction theorem has two distinct inputs: a realized family of geometric levels and the bounded maps of Definition 5.1. It proves a formula for data already satisfying these hypotheses. It does not produce the levels or the assembly maps from arbitrary terminal data. The cylinder gives a realized infinite-dimensional symmetry subspace; Proposition 6.4 gives an explicit finite-dimensional product example. The high-frequency continuation test following that proposition prevents identifying its finite-mode extension with an unrestricted Sobolev operator.

The induced module is a restricted module of unchanged fibre rank. Any reduction to an intrinsic de Rham or Dolbeault module must identify the summand or quotient and verify its connection, operator, and grading. Pairing two real dimension drops supplies the correct real dimension for a complex hypersurface but does not supply its complex structure. Nor does a formal iteration eliminate interior sources: those terms remain in (5.1) unless their vanishing has been proved.

For a moving geometric construction one must pull all bundles and domains to fixed reference spaces before using Proposition 8.1. Uniform bounds on a full parameter neighborhood, including all assembly and interface operators, are additional analytic assertions. The fixed-domain identities here do not imply a smooth tame inverse for a nonlinear problem.

Proposition 10.1 (Ambient boundaries and cohomological pairings). *Let $i : B \to X$ be a smooth compact oriented $(d+1)$-dimensional chain with boundary, and let $Z = i_*(\partial B)$ with its boundary orientation. For every smooth closed d-form α on X,*

$$\int_Z \alpha = 0.$$

In particular, a nonempty compact complex k-submanifold of a Kähler manifold cannot be such an ambient boundary.

Proof. Stokes' theorem gives $\int_Z \alpha = \int_B i^*(d\alpha) = 0$. For a complex k-submanifold and Kähler form κ, the form $\kappa^k/k!$ restricts to its positive volume form, so its integral is strictly positive. This contradicts the boundary identity. The current-theoretic version follows from the same boundary pairing; compare [19].

□

Thus analytic reconstruction alone does not provide the cohomology class of a cycle or preservation of a rational period lattice. Such claims require additional geometric and cohomological arguments. They are outside the theorems of this paper.

11. Conclusions

We have formulated Green–Poisson reconstruction for sections of Hermitian Dirac bundles using a specified, invertible full Dirichlet realization of the Dirac Laplacian. The regularized boundary proof makes the kernel convention and normal sign explicit. The induced hypersurface connection supplies compatible restricted Clifford modules, with their multiplicities retained.

For an admissible realized chain, the recursive reconstruction formula separates the transferred terminal data from the source contributions generated at each intermediate level. The transmission results identify the common trace and matching conormal flux needed to reconstruct a global solution from pieces. They also explain why partitioning one fixed global Green integral and summing independent piecewise Dirichlet solutions are different operations. The cylinder and the complex product-neighborhood example verify these statements in explicit settings and exhibit nonzero intermediate sources.

The normal disk-bundle construction gives a global one-stage lift without choosing a normal diameter. The normal-bundle obstruction shows why the selected-diameter realization in the product example does not extend automatically to nontrivial normal geometry. Together, these results specify an analytic framework in which a proposed dimensional construction can be checked step by step.

For the intended Hodge application, the remaining work is to realize the required geometric chains and bounded transfers, establish the necessary uniform estimates for the nonlinear problem, and prove the claimed identification with prescribed rational Hodge classes. Those are additional tasks for the companion investigation. The present paper establishes the stated reconstruction results under their explicit hypotheses and supplies exact models and compatibility tests for that further work.

Appendix A. Regularized proof of the boundary representation

Fix $x \in M^\circ$ and $\xi \in V_x$. Take a small coordinate ball around x disjoint from ∂M, a smooth bundle trivialization on that ball, and a nonnegative scalar approximate identity with respect to $d\mathrm{vol}_g$. Transport ξ smoothly in this trivialization and multiply by that approximate identity to obtain $\rho_\epsilon \in C_c^\infty(M^\circ, V)$. Then

$$\int_M \langle \rho_\epsilon(y), u(y) \rangle \, d\mathrm{vol}_g(y) \longrightarrow \langle \xi, u(x) \rangle$$

for every smooth u. This is independent of the trivialization because all such smooth extensions have the same value at x.

Put $v_\epsilon = \mathscr{G}\rho_\epsilon$. Elliptic Dirichlet regularity makes it smooth up to the boundary, with $Pv_\epsilon = \rho_\epsilon$ and $\mathrm{Tr} v_\epsilon = 0$. For $Pu = 0$, Lemma 2.2 gives

$$\int_M \langle \rho_\epsilon, u\rangle = -\int_{\partial M} \langle c(\nu) Q v_\epsilon, u\rangle.$$

The term involving $c(\nu)v_\epsilon$ vanishes by the zero trace. The Schwartz kernel of the inverse is smooth away from its diagonal, including where one variable lies in the boundary and the other stays in a compact interior set; its differentiated boundary kernels have the same property [9], Chapters 10–11; see also [8]. Consequently, on a fixed boundary collar separated from the source,

$$v_\epsilon(\cdot) \longrightarrow G_x(\cdot)\xi$$

with every derivative needed in the boundary integral. Passing to the limit yields

$$\langle \xi, u(x)\rangle = -\int_{\partial M} \langle c(\nu_y) Q_y G_x(y)\xi, u(y)\rangle = \int_{\partial M} \langle A_x(y)\xi, u(y)\rangle.$$

This proves (4.3). The argument uses Green's identity only for the smooth v_ϵ, not directly for the singular section $G_x\xi$.

Appendix B. Closed components and harmonic projections

Proposition B.1 (Reduced Green operator on a closed manifold). *Let M be closed and let $P = Q^2$. Let Π be the L^2-orthogonal projection onto the finite-dimensional space $\ker P$. There is a bounded reduced inverse on each elliptic Sobolev scale,*

$$\mathscr{G}^{\mathrm{red}} = (P|_{(\ker P)^\perp})^{-1}(I - \Pi),$$

and every smooth section satisfies

$$u = \Pi u + \mathscr{G}^{\mathrm{red}} P u.$$

For a measurable partition $M = M_1 \cup M_2$ up to a null set and $f \in L^2$, one has

$$\mathscr{G}^{\mathrm{red}} f = \mathscr{G}^{\mathrm{red}}(\mathbf{1}_{M_1} f) + \mathscr{G}^{\mathrm{red}}(\mathbf{1}_{M_2} f).$$

The equation of this normalized solution is $Pu = f - \Pi f$.

Proof. The self-adjoint elliptic operator has discrete spectrum, finite kernel, and a positive spectral gap on $(\ker P)^{\perp}$. Inverting the nonzero eigenvalues gives $\mathscr{G}^{\mathrm{red}}$ on L^2; elliptic regularity gives its Sobolev bounds [6][10]. The spectral decomposition gives $\mathscr{G}^{\mathrm{red}} P u = (I - \Pi) u$. Linearity applied to the source partition proves the last identity. □

Remark B.2 (Missing harmonic data). A boundary integral over an empty boundary cannot recover a nonzero global harmonic section. The component Πu is independent data. If a closed global problem is reconstructed from piecewise Dirichlet problems, those problems must use the restrictions of $f - \Pi f$, rather than f, and retain the global normalization $u \perp \ker P$. A distributional identity $P_y E_x = \delta_x \mathrm{id}$ for all x cannot hold when the kernel is nonzero: pairing with a nonzero harmonic section contradicts the identity at a point where that section is nonzero. Shrinking a boundary does not remove these compatibility conditions or provide uniform bounds on singular kernels.

Statements and Declarations

Funding

No specific funding was received for this work.

Potential Competing Interests

No potential competing interests to declare.

Author Contributions

S.F. was the sole author and is responsible for all aspects of the manuscript.

Use of Generative AI

ChatGPT assisted with mathematical reconstruction, proof checking, references and manuscript preparation. The author is responsible for the mathematical statements and their verification.

References

1. ^*Deligne P (2006). "The Hodge Conjecture." In: Carlson J, Jaffe A, Wiles A, editors. The Millennium Prize Problems. Providence, RI, and Cambridge, MA: American Mathematical Society and Clay Mathematics Institute, p. 45–53. https://www.claymath.org/wp-content/uploads/2022/06/hodge.pdf.*

2. ^Voisin C (2002). Hodge Theory and Complex Algebraic Geometry I. Cambridge: Cambridge University Press. doi:10.1017/CBO9780511615344.
3. ^Farinelli S (2024). "The Dirac–Dolbeault Operator Approach to the Hodge Conjecture." Symmetry. **16**(10):1291. doi:10.3390/sym16101291.
4. a, b, cLawson HB Jr, Michelsohn ML (1989). Spin Geometry. Princeton, NJ: Princeton University Press.
5. a, b, c, d, e, fBooß-Bavnbek B, Wojciechowski KP (1993). Elliptic Boundary Problems for Dirac Operators. Boston: Birkhäuser. doi:10.1007/978-1-4612-0337-7.
6. a, b, c, dGilkey PB (1995). Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem. 2nd ed. Boca Raton, FL: CRC Press.
7. a, bGilkey PB (2004). Asymptotic Formulae in Spectral Geometry. Boca Raton, FL: Chapman & Hall/CRC. doi:10.1201/9780203490464.
8. a, b, c, d, e, f, g, hGrubb G (1996). Functional Calculus of Pseudodifferential Boundary Problems. 2nd ed. Boston: Birkhäuser. doi:10.1007/978-1-4612-0769-6.
9. a, b, c, d, e, f, g, hGrubb G (2009). Distributions and Operators. New York: Springer. doi:10.1007/978-0-387-84895-2.
10. a, b, c, dTaylor ME (1996). Partial Differential Equations II: Qualitative Studies of Linear Equations. New York: Springer.
11. a, b, cBär C, Ballmann W (2012). "Boundary Value Problems for Elliptic Differential Operators of First Order." In: Surveys in Differential Geometry. Vol. 17. Somerville, MA: International Press, p. 1–78. doi:10.4310/SDG.2012.v17.n1.a1. https://arxiv.org/abs/1101.1196.
12. a, bBär C, Ballmann W (2016). "Guide to Boundary Value Problems for Dirac-Type Operators." In: Arbeitstagung Bonn 2013. Birkhäuser, p. 43–80. doi:10.1007/978-3-319-43648-7_3. https://arxiv.org/abs/1307.3021.
13. a, bGilkey PB (1993). "On the Index of Geometrical Operators for Riemannian Manifolds with Boundary." Adv Math. **102**(2):129–183. doi:10.1006/aima.1993.1063.
14. ^Raulot S (2011). "Green Functions for the Dirac Operator under Local Boundary Conditions and Applications." Ann Glob Anal Geom. **39**(4):337–359. doi:10.1007/s10455-010-9236-y.
15. a, bHuybrechts D (2005). Complex Geometry: An Introduction. Berlin: Springer. doi:10.1007/b137952.
16. a, bHijazi O, Montiel S, Zhang X (2001). "Dirac Operator on Embedded Hypersurfaces." Math Res Lett. **8**(1–2):195–208. doi:10.4310/MRL.2001.v8.n2.a8.

17. ^Bureš J (1993). "Dirac Operators on Hypersurfaces." Commentat Math Univ Carolin. **34**(2):313–322. https://dml.cz/handle/10338.dmlcz/118583.

18. ^Seeley RT (1966). "Singular Integrals and Boundary Value Problems." Am J Math. **88**(4):781–809. doi:10.2307/2373078.

19. a, bBott R, Tu LW (1982). Differential Forms in Algebraic Topology. New York: Springer. doi:10.1007/978-1-4757-3951-0.

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.