

Research Article

Gravity as a Cross-Term Effective Gradient: Source-Position Dependence and Spherical-Symmetry Breaking in an A_5 -Seeded Finite Medium

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This paper studies a restricted precursor to gravitational modeling: where a two-body direction sign can first be carried in a finite, static, symmetric, linear-response, quadratic medium with no primitive direction. The main result is a distance-kernel localization theorem. For a vertex- and distance-transitive medium with equivariant single-source response u_a and equivariant symmetric Q , $B_Q(a, b) = \langle u_a, Qu_b \rangle$ depends only on the separation $d(a, b)$. Hence the self-energy is source-position independent, while the nontrivial two-source dependence is confined to a cross-distance kernel $h_Q(d)$. Within this restricted grammar and without external witnesses or higher-order/dissipative terms, the cross-distance kernel is the natural carrier of the two-body source-pair dependence; this scope statement is not advanced as an independent uniqueness theorem. The physical reading is made through the source-coupled Green variational form, for which the on-shell interaction has the attractive sign. Here, *readout* means an admissible finite extraction of effective observables, not an arbitrary external projection. The force channel is the coarse readout of an energy-lowering update tendency on the active substrate, not a passive measurement of a primitive force; the cross-distance kernel supplies the source-pair carrier whose separation gradient feeds it. As a minimal finite demonstration, the icosahedral A_5 seed yields a zero-mode-removed Green profile that verifies the attractive ordering. The same pairwise-kernel grammar also gives a finite N -source frustration hierarchy on the icosahedron. The paper does not derive Newton's constant, the inverse-square law, the equivalence principle, the continuum limit, general relativity, or quantum gravity; these are registered as open bridges.

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1. Introduction: Where to place the gravitational force?

1.1. Problem setting

Gravity is normally described as a force acting between two objects. In Newtonian mechanics^[1],

$$F = G \frac{Mm}{r^2}$$

is given as an empirical law, and in general relativity^{[2][3][4][5]} it is reinterpreted as the curvature of spacetime geometry. These two descriptions are completely different frameworks both technically and empirically, but they share one feature in common: in both cases, directionality is already incorporated at the stage of the effective equations. In Newtonian mechanics, directionality appears as a force in the direction connecting the two objects; in general relativity, it appears as geodesic deviation / free-fall direction determined by a given metric, on the readout side of the theory.

Neither, however, answers the more fundamental question:

Why do two objects have a direction that draws them toward each other?

This paper addresses this question not in terms of the primitive nature of force but reformulates it from a *cross-term structure on a finite medium*.

Furthermore, as a natural stress test for this two-source carrier claim, we will minimally address the multi-source case. If the directional carrier is truly pairwise and cross-term, the difficulty should arise not as a new primitive N -body force at the same secondary medium level but as a consistency problem between pairwise cross kernels. We confirm this point as a finite verification example on the A_5 icosahedron seed.

Reading frame: substrate, update tendency, force-channel readout.

The finite A_5 medium is treated as an active substrate, not a passive symbol table. Sources deform the substrate; deformation clouds of distinct sources overlap; the overlap generates a source-pair-dependent update tendency; the macroscopic force channel is the coarse readout—in the sense fixed in §2.3, an admissible finite extraction—of this tendency. The present paper addresses only the first half of this chain, namely the carrier of the two-body direction sign, and reserves Newton-scaling, second-order motion, and the continuum / GR limit for downstream extensions.

The field generated by a single object is uniform in all directions. If only source a exists, the field u_a created around it has shell equivalence centered at a and does not select any particular direction. Similarly, if only source b exists, u_b also has shell equivalence centered at b .

The problem arises when two omnidirectionally uniform fields overlap on the same medium. The two fields themselves do not have direction, but their product

$$\Phi_{ab}(x) := u_a(x)u_b(x)$$

has a bicenter structure. Φ_{ab} is neither spherically symmetric about a nor about b ; only the axial symmetry about the $a-b$ axis remains.

Thus, the direction of gravity does not lie within a single field. *The direction of gravity arises only at the intersection of two independent fields.*

1.2. The central proposition of this paper

The central proposition of this paper is not to treat gravity as a primitive force. Rather, it is to separate the following structure within a finite medium:

In a vertex-transitive and distance-transitive finite linear quadratic medium, the source-pair bilinear quantity falls to the cross-distance kernel.

A standalone source field

$$u_a(x) = G(d(x, a))$$

is one-center shell-uniform and has no preferred direction. The same applies to u_b . Furthermore, if the substrate is vertex- and distance-transitive and the single-source response and \mathcal{Q} are both consistent with graph automorphism, then the source-pair bilinear quantity

$$B_{\mathcal{Q}}(a, b) = \langle u_a, \mathcal{Q}u_b \rangle$$

is a function of the source separation alone:

$$B_{\mathcal{Q}}(a, b) = h_{\mathcal{Q}}(d(a, b)).$$

In particular, the self-energy

$$E_{\mathcal{Q}}(u_a) = \frac{1}{2}h_{\mathcal{Q}}(0)$$

does not depend on the absolute position of the source. Therefore, the self term cannot be responsible for the separation or direction of the source pair (a, b) .

Adopting the linear-response grammar

$$u_{a+b} = u_a + u_b$$

and the quadratic medium energy

$$E_{\mathcal{Q}}(u) = \frac{1}{2}\langle u, \mathcal{Q}u \rangle,$$

the energy excess due to the coexistence of two bodies is

$$I_{\mathcal{Q}}(a, b) = E_{\mathcal{Q}}(u_a + u_b) - E_{\mathcal{Q}}(u_a) - E_{\mathcal{Q}}(u_b) = h_{\mathcal{Q}}(d(a, b)).$$

The important point here is not the polarization identity itself, it is that finite symmetry compresses the source-pair dependence into a distance kernel and that kernel is then carried out by the cross term.

In addition, the pointwise overlap diagnostic of two fields,

$$\Phi_{ab}(x) = u_a(x)u_b(x),$$

has the two-center axis $a-b$. Hence, omnidirectionality belongs to the primary quantity of a single cloud, while the directional axis first emerges in the secondary quantity born from the encounter of two clouds.

To summarize the main thread:

one-center shell-uniformity \rightarrow distance-kernel localization \rightarrow self-energy independence
 \rightarrow cross-distance-kernel carrier \rightarrow attractive sign via Green reading.

1.2.1. Central M -layer theorem: distance-kernel localization

Here we explicitly state the algebraic observation that underpins the central proposition. We first divide the discussion into four levels.

Level	Content	In-scope status
Direction axis	Emergence of two-center axis $a-b$	Main subject of this paper
Direction sign	On that axis, which side (proximate/distal) is energy-lowering?	Main subject of this paper
Force law	Is the magnitude of $1/r^2$ type?	Downstream bridge
Dynamical motion	Does motion of the form $x = F$ actually occur?	Downstream bridge

Table 1. Four levels of two-body gravitational claim and their treatment in this paper.

Therefore, this paper does not derive full Newtonian gravity. It deals with where the carriers of the two-body direction axis and direction sign first appear in a finite, static, symmetric, linear-response, quadratic medium grammar.

The central M -layer theorem of this paper is not the polarization identity, nor the scope declaration of the first-occurrence grammar. The central point is the following *distance-kernel localization statement*.

Let Γ be a finite graph and assume its automorphism group acts vertex-transitively and distance-transitively. Let Q be a symmetric operator commuting with the graph automorphism action. Let u_a be a single source field centered at source a , constructed equivariantly from a fixed base source; that is, for any automorphism g ,

$$u_{ga} = gu_a.$$

Define the source-pair bilinear quantity by

$$B_Q(a, b) = \langle u_a, Qu_b \rangle.$$

Then $B_Q(a, b)$ is invariant under the simultaneous action of the graph automorphism group:

$$B_Q(ga, gb) = B_Q(a, b).$$

By distance-transitivity, two ordered source pairs with the same separation belong to the same orbit. Therefore, there exists a unique function h_Q on the distance shell such that

$$B_Q(a, b) = h_Q(d(a, b))$$

This is the central M -layer theorem of this paper, the *distance-kernel localization theorem*.

The consequences are immediate. First, the self-energy term is independent of the source location:

$$E_Q(u_a) = \frac{1}{2}B_Q(a, a) = \frac{1}{2}h_Q(0).$$

Hence the self-energy term cannot handle source-position dependence.

Second, in the linear quadratic expansion

$$E_Q(u_a + u_b) - E_Q(u_a) - E_Q(u_b),$$

the source-placement dependence is localized to the cross-distance kernel

$$h_Q(d(a, b)).$$

The importance of the cross term is not because the polarization identity is new. It is because finite symmetry pushes source-pair dependence into the cross-distance kernel.

Furthermore, the paper declares the following as part of its first-occurrence grammar:

In a static, endogenous, linear-response, quadratic grammar without a primitive direction or external witness, the natural carrier containing both sources is the cross-distance kernel.

This is a *scope declaration*: it specifies which carriers are inside the grammar of this paper, not a uniqueness theorem about all physical mechanisms. A self term contains only one source. An external term requires an external witness/background direction and is non-endogenous. A higher-order term lies outside the

quadratic model. A dynamic/dissipative term requires wake, memory, velocity, and time-reversal breaking and lies outside the static quadratic model. A nonlinear-response term discards $u_{a+b} = u_a + u_b$ and therefore lies outside the first-occurrence grammar of this paper.

The remaining carrier within the declared grammar is therefore the cross-distance kernel. The corresponding scope statement is recorded in machine-checkable form as `cross_unique_minimal_endogenous_direction_sign` in the companion Lean file `MinimalDirectionSign.lean`; the file makes the carrier list explicit and is referenced from Appendix B as a Lean infrastructure note rather than as a main-text M -claim.

In particular, the geometric description of GR, nonlinear frameworks, vertex descriptions of quantum fields, elastic/Eshelby-type continua, and wake/memory/dissipative dynamics are outside the scope of this paper and are treated as Open Bridges G_s, G_7, G_8 or as dynamic extensions in other papers.

Comparison with existing quantum gravity/emergent gravity programs.

This paper does not present a full theory competing with GR, string theory [6], LQG [7], asymptotic safety [8], causal sets [9], causal dynamical triangulations [10], AdS/CFT [11], or emergent gravity programs. Its niche is narrower: it asks where, given no primitive forces, a two-body direction-signed carrier first appears in a finite, static, quadratic medium grammar.

The relationship to existing programs may be summarized as in Table 2.

Program	Definition of gravity	Difference from this paper
GR	Curvature / free fall as a relation between stress-energy and geometry	The directional sign is already included in the effective geometry. This paper examines the preceding finite-medium carrier.
Jacobson-type thermodynamic gravity [12]	Reading Einstein's equations from horizon thermodynamics	Assumes horizon, entropy, and temperature. This paper deals with cross-carrier data in a source-pair finite quadratic medium.
Verlinde entropic gravity [13]	Gravity read as "entropy gradient / information gradient"	While the gradient concept is similar, this paper explicitly uses a finite substrate, one-center shell uniformity, invariant-group reduction, and a cross-term carrier.
Padmanabhan-type emergent gravity [14]	Reading gravitational dynamics from surface / bulk degrees of freedom and equipartition	The focus is thermodynamic / holographic counting. This paper focuses primarily on a finite-graph quadratic medium and source-pair interaction excess.
AdS/CFT / entanglement gravity [11][15]	Geometry as a readout of entanglement / duality	Powerful but relies on a specific duality structure. This paper extracts the carrier using a smaller static quadratic grammar.

Table 2. Relationship between this paper and existing emergent / quantum gravity programs.

The originality of this paper is not that it "completely derives gravity." Its originality lies in the combination of the following five points:

finite substrate+one-center shell-uniformity + two-center invariance reduction
+ quadratic interaction excess + Lean-audited dependency map.

This combination provides a structural candidate where the attractive direction is not primitive and the carrier of the two-body direction sign is localized to a cross-distance kernel on a finite quadratic medium. This is not an alternative completing theory of mainstream programs but a smaller explanatory module.

Relationship with existing graph Green / discrete potential theory.

This paper does not claim that the Green's function of the graph Laplacian, electric potentials on finite graphs, discrete potential theory [16][17], network interpretations [18][19], spectral graph theory [20][21], or continuum-side Riesz / Dirichlet-type quadratic energies [22][23] are novel. These are standard objects belonging to existing discrete and continuum potential theory, including the recent discrete-differential-geometric formulations [24][25].

The difference in this paper does not lie in the existing Green / quadratic-energy machinery itself but in the foundational question it uses. Specifically, this paper prohibits a primitive force direction, external witnesses, higher-order corrections, and dynamic memory / dissipation, and identifies, within a *static quadratic finite-medium grammar*, the source-pair-dependent carrier responsible for the two-body direction sign as the cross-distance kernel.

Therefore, this paper does not replace existing potential theory. Rather, it reinterprets the standard quadratic structure as a finite-medium grammar for de-axiomatizing the primitive direction of Newtonian forces.

1.3. Role of this paper: an attractive-direction carrier preceding a Newton-type readout

This paper is not a paper that derives Newtonian gravity as a whole. It deals with a narrower structural question that lies before reaching a Newtonian readout:

In a finite quadratic medium without a primitive force, on which carrier does the two-body direction sign first localize?

To progress to Newtonian gravity, at least three independent layers are necessary:

- **Attractive-direction layer.** A layer that yields an attractive sign with a two-body directional axis and the proximate side as the lower-energy side.
- **Newton-scaling layer.** A layer that connects this channel to a $1/r^2$ -type scaling.
- **Dynamical-motion layer.** A layer that connects the static energy ordering to a dynamic readout of the form $\ddot{x} = F$.

This paper deals only with the first layer. In other words, this paper does not deal with “Newton’s laws” but with “the first carrier of an attractive direction sign preceding a Newtonian readout.”

The subsequent Newton-scaling layer, dynamical-motion layer, quasi-static identification, and continuum / GR limit are not derived in this paper. These are retained as downstream Newton-core extensions or Open Bridges.

The natural ordering of the papers is therefore

→ downstream Newton-core extension: scaling / motion / continuum bridges.

Read as a self-contained first-stage paper in this order, the arguments of this paper do not presuppose the establishment of the subsequent extensions.

The reformulation thesis adopted in this paper—“the quantum-gravity problem is misposed from the moment spacetime geometry and quantum wave ontology are considered as primitive continua, and both should be reformulated as finite readouts from a common finite substrate”—is the broader motivation.

This paper establishes not the entire thesis but rather its gravity-side branch as the *attractive-direction carrier* (explicit non-claims are summarized in §11).

We clarify here an important epistemic limitation regarding the reformulation thesis. Since the reformulation thesis places a finite substrate foundationally, it follows the structural feature that “infinity does not occupy a position as a foundational object.” However,

This paper does not claim that empirically observed UV divergence does not exist within this framework.

More weakly and more precisely,

This paper shifts the problem to a model class where infinity does not occupy a position as a foundational object.

This is a structural commitment. Whether the effective theory after taking the continuum limit (Open Bridge G_5) has any new divergent structure at the UV scale is a separate matter and outside the scope of this paper. This is an empirical question that can only be addressed after resolving Open Bridges G_5 and G_7 .

1.4. Low-resolution words to avoid

This paper does not use the following terms as central terms of its theory. While these terms function as intuitive descriptions, they would obscure the structural arguments presented here.

Low-resolution word	Substitution in this paper
“Level” (gravitational potential)	quadratic cross-term distance dependence
“Maintaining symmetry”	single-source shell uniformity
“Symmetry breaking”	reduction from one-center shell uniformity to two-center axial overlap
“Stabilization”	direction of decrease in interaction-energy readout
“Low cost / energetically favorable”	effective-gradient direction read off the cross term
“Gravitational”	separation-gradient readout of the overlap cross term
“Frictional” (linear friction in the narrow sense)	<i>resistance</i> (a general term for dissipative forces that are time-reversal antisymmetric)

Table 3. Low-resolution words and their substitutions in this paper.

In particular, the substitution of the term “symmetry breaking” is the methodological center of this paper. Both standalone fields u_a and u_b are one-center shell-uniform and invariant under the action of the respective stabilizer groups of a or b . The overlap product $u_a u_b$ cannot simultaneously possess shell uniformity with respect to each of the two centers; what remains is axial symmetry about the a - b axis.

This is not a negative statement of the form “the symmetry that the individual field possessed is lost due to overlap” but rather the positive statement

Structural reduction from one-center shell equivalence to two-center axial structure.

This paper introduces this understanding uniformly through the substitution rules described above.

2. Epistemic contract: four-tiered structure ($M/B/P/E$)

To support the logical structure of this paper, we declare in advance what kind of evidence each claim relies on. We call this the *epistemic contract*.

We adopt a four-tiered structure ($M/B/P/E$). The important point is to explicitly separate *bridge / ansatz* as an independent B -layer. The methodological issue addressed by this contract is to determine to what extent the development of standard quadratic forms should be treated as mathematical bookkeeping and to what extent it should be elevated to a content-claim unique to this paper. The content that could conventionally be called B_{cross} is decomposed into (i) a preliminary bookkeeping based on the polarization identity, (ii) the distance-kernel localization theorem, and (iii) a scope declaration of the first-occurrence grammar (i.e., which carriers are admitted into the model class of this paper). The technical M -content of the paper is concentrated in (ii); (iii) is a scope statement, not a theorem, and is recorded as such. The connection rule to continuum quadratic cross terms is treated independently as Open Bridge G_3 .

This four-tiered structure is an epistemic contract for reading this paper as a self-contained first-stage paper. Within the same text, finite seed data, distance-kernel localization, Green-primary attractive signs, and downstream bridges are explicitly separated.

2.1. M layer: machine-verifiable mathematical facts and Lean candidates

The M layer consists of mathematical propositions that can be stated at the level of finite graphs, finite linear algebra, and discrete difference operators without requiring physical interpretation. We further divide the M layer into two parts.

Distinction	Meaning	Treatment in this paper
Verified M	Propositions audited in Lean 4 / Mathlib with <code>sorry = 0</code> , <code>axiom = 0</code>	Seed graph, Green profile, Green ordering (reproduced in this paper)
Candidate M	Mathematically clear, Lean-compatible propositions, but in this paper registered as skeletons / roadmap	Invariant-group reduction of overlap density, cross-profile ordering, finite-difference gradient

Table 4. Distinction between verified M and candidate M .

The distinction is drawn to avoid overemphasizing the scope of Lean-audited solutions. The key point of this paper is not to confuse M -layer propositions with B/P -layer bridges, while not assuming that all M -layer candidates are currently machine-verified.

Verified M : finite seed data and Lean audit used in this paper.

The finite data of the icosahedron seed graph used in this paper are reproduced in Sections §3 and §7 of the main text and registered as appendices: Lean audit / finite rational calculations. The verified M -facts required in this paper are as follows.

Structural data of the seed graph Γ_0 of the regular icosahedron:

$$|V| = 12, |E| = 30, |F| = 20.$$

Shell counts from a base vertex:

$$(N_0, N_1, N_2, N_3) = (1, 5, 5, 1).$$

Zero-mode-removed Green profile:

$$G_0 = \frac{7}{36}, G_1 = \frac{1}{90}, G_2 = -\frac{7}{180}, G_3 = -\frac{1}{18}.$$

Strict Green ordering:

$$G_1 > G_2 > G_3.$$

N -body finite frustration thresholds on the regular icosahedron seed:

$$v_1 = 3, v_2 = 6, v_3 = 12.$$

Correspondingly, three sources can saturate all pair nearest-neighbor requirements; four sources force at least one non-neighboring pair; and seven sources force an antipodal pair. This finite verification example is registered as a build-OK verified audit in `IcosaNBodyFrustrationAudit.lean`.

These are treated as finite seed calculations in this paper. To the extent necessary for the claims of this paper, we do not rely on any other paper.

Direct M: algebraic identities formalizable directly within this paper.

The minimal algebraic kernel of this paper is the quadratic energy identity for finite scalar fields $u, v: V \rightarrow \mathbb{R}$:

$$E(u + v) = E(u) + E(v) + 2Cross(u, v), Cross(u, v) = \sum_{x \in V} u(x)v(x).$$

This is a pure algebraic identity in a finite inner-product space and is included in the companion Lean infrastructure of this program.

Candidate M: Lean implementation candidates newly registered in this paper.

The following are the candidate M -layer items to be newly registered in this paper:

- One-center shell-uniformity of the isolated field $u_a(x) = G(d(x, a))$.
- The overlap density $\Phi_{ab}(x) = u_a(x)u_b(x)$ is determined as a function of the distance pair $(d(x, a), d(x, b))$.
- For labeled source pairs, Φ_{ab} is invariant under $Stab(a, b)$, and for unlabeled (homogeneous) sources, this extends to $Stab(\{a, b\})$.
- The total cross term $C(d_{ab}) = \sum_x u_a(x)u_b(x)$ depends only on d_{ab} , by the distance-transitivity of the icosahedron graph.
- For finite computation in the mass quadratic reading, the ordering $C(1) > C(2) > C(3)$ holds.
- The finite-difference gradient $\Delta_{d_{ab}} C$ contains non-trivial information only along the a - b axis.
- In a static, symmetric, quadratic grammar without a primitive direction, the smallest direction-sign carrier is restricted to the cross term.

These are treated as *Lean implementation candidates* in this paper. The last item, the minimal direction-sign grammar, has already been implemented as a build-OK audit theorem in the companion Lean file `MinimalDirectionSign.lean`. The full skeleton / dependency map for completing the remaining items is maintained in the broader Lean infrastructure of this program; unresolved bridges are not hidden by `sorry`.

A key feature of the M layer is that all of its statements are complete mathematical propositions and require no physical interpretation. For example, the statement that “the invariant group of Φ_{ab} reduces to $Stab(a, b)$ for labeled source pairs” is a pure proposition concerning finite group actions and requires no interpretive language such as “symmetry breaking.”

2.2. M_{pol} and the scope declaration

We split what was previously called B_{cross} into one M -layer lemma and one scope declaration:

M_{pol} (M-layer lemma): In a linear-response, second-order medium grammar, the interaction excess is decomposed into cross terms by the polarization identity.

Scope declaration: The model class of this paper is the static, symmetric, endogenous, linear-response, quadratic grammar, in which the carriers admitted into the source-pair-dependent reading are restricted to the cross-distance kernel.

The first item, M_{pol} , is not a novelty of this paper; it is the standard bilinear expansion of a symmetric quadratic form. The energy excess from the coexistence of two bodies in the same medium is defined by

$$I_Q(a, b) := E_Q(u_a + u_b) - E_Q(u_a) - E_Q(u_b),$$

and adopting the *linear-response grammar*

$$u_{a+b} = u_a + u_b$$

yields algebraically

$$I_Q(a, b) = \langle u_a, Qu_b \rangle.$$

This is a bookkeeping lemma to avoid “assuming cross-term interactions a priori” and is not the central theorem of this paper.

The second item is a *scope declaration*, not a theorem. Within the model class declared here, a self term contains only a single source, an external term imports an external witness, a higher-order term lies outside the quadratic grammar, and a dynamic / dissipative term lies outside the static grammar. The carrier admitted by the model class is therefore the cross-distance kernel. This is a statement about which carriers the paper places inside its grammar, not a claim of physical uniqueness across all gravity-relevant frameworks.

This decomposition means that the old B_{cross} is not simply “promoted” to the M layer. Rather, it is decomposed into an elementary polarization lemma and a scope declaration. The part to be read as a physical gravity channel continues to belong to $P_{grav} / P_{energy \rightarrow force}$.

Linear-response firewall: first-occurrence grammar.

The linear-response condition

$$u_{a+b} = u_a + u_b$$

is not a harmless notational convenience. It is a substantive assumption that restricts the grammar of this paper. This assumption places nonlinear elasticity, mutual screening, history-dependent responses, and Eshelby-type inclusion back-reactions outside the scope of the theorems of this paper.

The restriction is intentional. This paper does not describe all gravitational phenomena; it addresses the first-occurrence question: where can the two-body direction sign first appear before introducing nonlinear responses, delays, dissipation, and continuous geometry?

The model class of this paper should be read as a static, linear-response, quadratic, first-occurrence grammar. Models that are nonlinear, history-dependent, or involve wake / memory are not rejected but are sent to Open Bridges / dynamic extensions as separate grammars.

Open Bridge G_5 : continuum quadratic-form bridge.

The rule connecting the finite cross term

$$\sum_{x \in V} u_a(x)u_b(x)$$

to the quadratic cross term on the continuum side is still treated as Open Bridge G_5 . The minimal candidate is the scalar diagnostic bridge

$$\sum_x u_a u_b \rightarrow \int_{\Omega} \phi_a \phi_b d\mu,$$

while the principal candidate for the gravity-energy reading is the Dirichlet/Green bridge

$$(u_a, Lu_b) \rightarrow \int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV,$$

or, alternatively, a source-coupled Green form. Elastic/Eshelby-type support requires additional field lifting and is therefore removed from the main text and left as a future bridge candidate.

References to subsequent bridges (B_1, B_4).

In addition, §8 registers downstream bridge candidates needed to connect the cross-distance kernel to the Newtonian $1/r^2$ law:

- B_1 : shell-area parametrization, $W_n = (n + 1)^2$;
- B_4 : shell-uniform flux ansatz, $W_{\neq n} = C$;
- B_2 : radius calibration, $r_n = a_0(n + 1)$.

These are not part of the main claim of this paper; their registration is meant to make explicit the following statement:

An inverse-square readout is a downstream bridge problem.

The methodological focus of this paper is not to highlight the algebraic existence of cross terms. Rather, the focus is on using the elementary polarization lemma to separate the source-pair-dependent term and showing that this term becomes the minimal carrier of the two-body direction sign within a static quadratic endogenous grammar.

2.3. P layer: physical interpretation principles and localization of explanatory load

The P layer consists of principles for physically interpreting the mathematical structure of the M layer. P -layer claims are not machine-verifiable: what is being verified is a physical reading, so they are interpretive contracts rather than propositions.

It is important that this paper does not eliminate the explanatory load. The approach is the following:

Instead of eliminating the explanatory load, we localize it across the M layer, model- M layer, P layer, and Open Bridge layer.

The M layer states the cross-term carrier and carrier structure within a static quadratic grammar. The physical load is concentrated in the explicit commitments of Table 5.

Commitment	Content	Status
Linear response	Take $u_{a+b} = u_a + u_b$ as the minimum two-body response model	Model choice
Quadratic variational energy	Restrict the medium energy to the form $\frac{1}{2}(u, Qu)$	Model choice
Source coupling	Couple sources as $-(\rho, u)$	Model choice
Q selection	Take the source-coupled Green/Laplacian as the primary reading of physical gravitational energy	Major P /model commitment
Energy-to-force readout	Read the direction in which the energy decreases as the force channel	P layer
Readout-to-force interface	Treat the force channel as the macroscopic readout of an energy-lowering update tendency, not as a primitive vector field on the substrate	P layer/downstream bridge boundary
Finite readout/extraction	What is observed is not the substrate field itself but an admissible finite extraction of effective observables	P layer

Table 5. Explicit physical commitments of this paper.

Among these, *linear response* is not merely a simple model choice. The model class of this paper relies on treating two-body coexistence as a minimal linear superposition $u_{a+b} = u_a + u_b$. When nonlinear inclusion, mutual screening, history-dependent relaxation, and wake/memory responses are allowed, the interaction excess does not generally reduce to a simple bilinear cross term. Therefore, the main claim is internal to the *static linear-response quadratic grammar*, not a universal statement in nonlinear medium theory.

The P -layer principles adopted in this paper are registered as follows.

- P_{med} (*medium-as-active*). The finite A_5 substrate carries the influence of sources as a deformation u on a finite graph—an *active construct*. Force is therefore a readout of substrate deformation, not a primitive action at a distance.
- $P_{energy \rightarrow force}$ (*energy-to-force readout*). The direction in which the medium energy decreases with respect to source separation is read as the observed attractive channel.
- $P_{force-channel}$ (*readout-to-force interface*). The force channel is not a primitive vector field on the substrate. It is the macroscopic readout of an energy-lowering update tendency of the finite active medium. The cross-distance kernel supplies the source-pair carrier whose distance gradient can feed such a readout; the acceleration law itself remains a downstream dynamical bridge.
- P_{read} (*finite readout/admissible extraction commitment*). The observable quantities are not the substrate state itself (each component of u) but finite readouts: admissible finite extractions of effective observables such as interaction energy U_{int} and effective force F^{eff} . In the general finite-medium grammar, this means that the observable is not an arbitrary projection; in the A_5 -seeded implementation, it is further constrained by substrate symmetry, equivariance, and the absence of unwitnessed directions or branches.

As a P -layer registration specific to this paper, we add the following principle.

- P_{axial} (*overlap-as-axial-localization-diagnostic*). The pointwise product $\Phi_{ab} = u_a u_b$ of two shell-uniform source fields is read *not* as the gravitational energy density of the two sources but first as a *diagnostic indicating the location of the two-center axial structure*. The principal reading of the gravitational energy sign is placed in the source-coupled Green energy.

A_5 -constrained finite readout.

The term “readout” is used as the program-level name for an admissible finite “extraction of effective observables from the substrate state. It is not an arbitrary observational projection imposed from outside the model. In the A_5 -seeded setting, such extraction is constrained by the substrate symmetry; in the absence of

an A_5 -invariant or A_5 -equivariant witness, the readout is not allowed to introduce an unwitnessed direction, branch, or source-pair distinction. This clarification preserves the generality of the distance-kernel localization lemma, which is not A_5 -specific, while fixing the physical reading adopted in this program as an A_5 -constrained finite readout.

Readout-to-force interface.

The term “readout” in this paper should therefore not be understood as a merely external observation of a pre-existing primitive force. The finite medium is treated as an active substrate whose source-induced deformation carries an energy-ordering structure. The cross-distance kernel supplies the first endogenous carrier of the two-body source-pair dependence; its separation gradient is then read as an energy-lowering update tendency of the medium. The macroscopic force channel is the coarse dynamical readout—that is, the admissible finite extraction into an effective force variable—of this update tendency. Thus the paper does not identify the cross term itself with a force law, nor does it derive the acceleration equation. It only localizes the carrier whose energy-ordering can feed such a force-channel readout once the downstream dynamical bridge is supplied.

No-bias grammar and witnesses.

In this paper, a *witness* is a structure that allows the readout/extraction map to introduce certain distinctions. An *external witness* refers to a background direction, boundary condition, fixed external record, or non-endogenous symmetry-breaking device that does not originate from the source pair itself. The first-occurrence grammar of this paper prohibits importing two-body direction signs from such external witnesses and asks only what the source pair’s own cross-distance kernel can do.

This no-external-witness grammar is consistent with a no-bias selector of type $P_{\zeta}^{A_5}$; unless an A_5 -invariant or A_5 -equivariant witness is present, the readout/extraction map does not arbitrarily introduce unproven directions, branches, or source-pair distinctions. However, $P_{\zeta}^{A_5}$ is not an additional assumption of the distance-kernel localization theorem of this paper. It is a meta- P principle reinforcing this paper’s grammatical choice not to introduce primitive directions or external witnesses.

This arrangement avoids the difficulty of having mass overlap and Laplacian/Green energy coexist as readings: Φ_{ab} is a local diagnostic indicating “where the one-center shell-uniformity reduces to a two-center axial structure,” while the energy sign of the gravity channel is read off the source-coupled on-shell energy in §4.4.

The claims of the P layer cannot be derived from the facts of the M layer or from the B -layer bridges. They are prerequisites for interpreting this paper as “addressing the carrier of the gravitational direction sign.”

We note that no-bias selectors of type $P_{\zeta}^{A_5}$ can be useful auxiliary principles in downstream dynamical extensions; however, they are not used in the main theorem of this paper. This paper is completed within the scope of shell-uniformity, distance-kernel localization, and the source-coupled Green reading for single fields.

2.4. E layer: empirical correspondence

The E layer describes the correspondence between the structure of this paper and what it corresponds to in the empirical world. The E layer is not the subject of mathematical verification but of future empirical research and ongoing studies.

The E -layer correspondences relevant to this paper are:

- The two-center axial cross-gradient corresponds to the “direction connecting the source pair” of empirical attraction.
- The continuum quadratic cross term may appear as an effective form such as a scalar-field Dirichlet energy or an elastic energy; determining which form corresponds to the finite cross term of this paper requires the use of bridge G_5 .
- Newton’s constant G , the equivalence principle, the $1/r^2$ scaling, and the GR limit are not established here and are retained as Open Bridges (§11).

The E -layer correspondence describes what the $M + B + P$ stack captures in the experienced world; it does not claim that the correspondence is complete.

2.5. Summary of the four-layer structure

Layer	Content	Examples in this paper	Verification
M	Mathematical facts and Lean implementation candidates	Green profile, $G_1 > G_2 > G_3$, polarization bookkeeping lemma, shell uniformity, overlap axiality	Verified / candidate, explicit in the Lean audit
model- M	Mathematics within the adopted finite quadratic medium model	Source-coupled on-shell energy $U_{int}^{om} = -(\rho_a, G\rho_b)$; static channel classification	Theorem after explicit model assumptions
P	Physical interpretation principles	$P_{mech} P_{grav} P_{resist} P_{axial}$	Interpretive contract (not subject to verification)
B / Open	Downstream bridge / ansatz	G_5 (continuum quadratic-form bridge), B_1, B_4 (inverse-square coarse-graining)	Explicit registration / Open Bridge
E	Empirical correspondence	Correspondence with Newton / GR / observed gravity	Empirical observation and future research

Table 6. Summary of the four-layer structure adopted in this paper.

The four-tiered Lean audit aspect is detailed in §10. In short, on the Lean side, the M -layer claims are separated as theorems, model- M as theorems with assumptions of the adopted secondary medium model, B / Open bridges as explicit structure fields, and P -layer claims as interpretive contracts not subject to verification.

2.6. Bridge / principles compact table

This subsection summarizes the M -layer theorems, P -layer principles, and Open Bridge / downstream bridge candidates appearing in this paper, together with their corresponding sections. It serves as a dependency map for reading this paper self-containedly.

$M / \text{model-}M$ layer theorems.

ID	Name	Content	Section	Source
M_{half}	Polarization bookkeeping lemma	In linear-response, second-order medium grammar: $I_{\mathcal{O}}(a, b) = (u_a, Qu_b)$	§4	Used as a standard lemma
M_{static}	Static channel classification	In a static, symmetric quadratic medium, conservative channels are classified as attractive, repulsive, or neutral; no resistive channels are produced	§6	New registration in this paper
M_{A_5}	A_5 finite channel calculation	$C(1) > C(2) > C(3)$ in mass quadratic; $G_1 > G_2 > G_3$ in Laplacian quadratic	§7	Finite seed calculation in this paper
M_{NB}	N -body finite frustration audit	Threshold statements $v_1 = 3, v_2 = 6, v_3 = 12$ and $N = 4, N = 7$	§7.6	IcosanBodyFrustrationAudit.lean build OK

Table 7. $M / \text{model-}M$ layer theorems registered in this paper.

B / Open-layer bridges.

ID	Name	Content	Section	Source
G_5	Continuum quadratic-form bridge	Connects the finite cross term to the quadratic cross term on the continuum side	§9	Open Bridge registered here
B_1	Shell-area parametrization	$W_n = (n + 1)^2$	§8	Downstream bridge candidate
B_4	Shell-uniform flux ansatz	$W_{j,n} = C$	§8	Downstream bridge candidate

Table 8. *B / Open-layer bridges registered in this paper.*

P-layer principles.

ID	Name	Content	Section	Source
P_{med}	Medium-as-active	The substrate is an active deformable structure	§3, §6	Adopted in this paper
$P_{grav} / P_{energy \rightarrow force}$	Gravity-channel readout	Read the energy-lowering direction as the physical gravity channel	§6, §7	Adopted and restricted in this paper
$P_{force-channel}$	Readout-to-force interface	Force is the macroscopic readout of an energy-lowering update tendency, not a primitive substrate vector	§2.3, §6	Adopted and restricted in this paper
P_{read}	Finite readout / admissible extraction	The directly observed quantity is an admissible finite extraction, constrained by A_5 symmetry in the program implementation	Throughout	Adopted in this paper
P_{axial}	Overlap-as-axial-localization-diagnostic	Read Φ_{ab} as a bicenter axial localization diagnostic	§5, §6	New registration in this paper
$P_{\zeta}^{A_5}$	No-bias grammar reinforcement	Auxiliary principle: do not read out distinctions of direction, branch, or source pair without a witness; not an additional assumption of the main theorem	§2.3, §6.4	M -backed meta- P auxiliary principle

Table 9. *P-layer principles registered in this paper.*

Notes on the table.

- The old B_{cross} is decomposed into a polarization bookkeeping lemma and a scope declaration of the first-occurrence grammar. The algebraic existence of cross terms itself is not the novelty of this paper.
- The conditional nature of this paper is mainly localized to $P_{med} / P_{grav} / P_{axial}$ and the subsequent bridges G_5, B_1 , and B_4 .
- B_2 (radius calibration), B_3 (force readout), B_5 (coupling calibration), B_{7a} (kinematical), and B_{7b} (dynamical) are not directly related to the main claim of this paper. They are retained as bridge candidates for the downstream Newton-core extension.

Localization of conditional content in this paper.

The core of the M -layer in this paper is the distance-kernel localization theorem ($M_{20} *$, §), and its corollary- M_{static} (static channel classification). The remaining non- M -layer is P_{med} (reading the finite graph as the active medium), P_{grav} (reading the energy-lowering direction as the physical gravity channel), P_{axial} (reading the local overlap as a bicenter axial localization diagnostic), G_5, B_1, B_4 (downstream bridges).

This summary serves as a reference supporting the local introductions of bridges in Sections 3–9.

3. Finite A_5 medium and the single source field

This section introduces the finite medium discussed in this paper and the single source field defined on it. The basic data are reproduced in this paper and used as the finite seed calculation in §7. This paper does not presuppose any other paper.

3.1. A_5 -seeded finite medium

As a seed of the finite medium, this paper considers the *icosahedron graph* Γ_0 . The regular icosahedron ^{[26], [27]} gives a polyhedral realization (the icosahedral rotation group) of A_5 ^[28].

The basic combinatorial data are

$$|V| = 12, |E| = 30, |F| = 20.$$

The graph-distance shell structure from a base vertex $a \in V$ is

$$(N_0, N_1, N_2, N_3) = (1, 5, 5, 1),$$

so the unique vertex at $N_3 = 1$ is the *antipodal vertex* of a (the opposite vertex of the regular icosahedron).

This seed graph is not a continuous space itself; it is a tool for obtaining a continuous space, distance, field, and force as subsequent readouts—a *finite substrate*.

The reasons for using the A_5 icosahedron seed are limited to three:

1. A_5 provides a specific finite graph Γ_0 as the rotation group of the icosahedron.
2. Γ_0 has a non-degenerate shell structure $(1, 5, 5, 1)$ with a diameter of 3.
3. The zero-mode-removed Laplacian Green profile shows the strict ordering $G_1 > G_2 > G_3$ and gives an attractive sign under source-coupled Green reading.

No further A_5 uniqueness claim is made in this paper. The connection to a broader A_5 program is the motivation for this paper, not a premise of the main theorem.

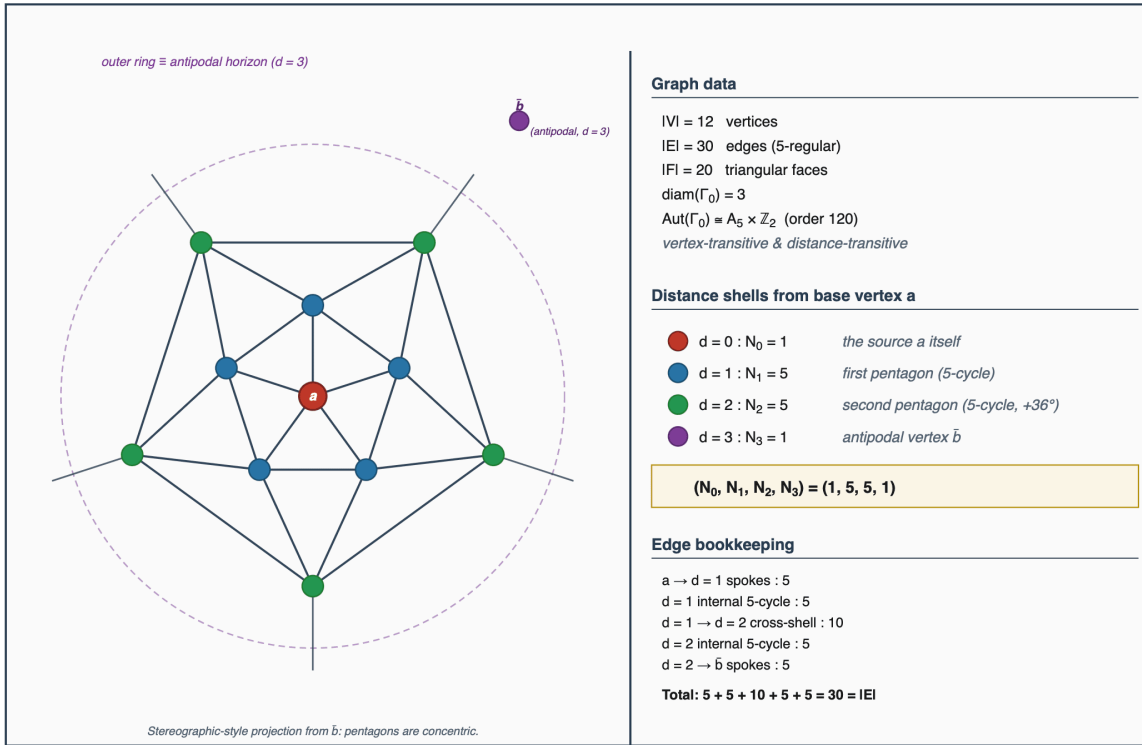


Figure 1. A_5 -seeded icosahedron graph Γ_0 and its distance-shell decomposition from base vertex a . **Left:** stereographic-style projection of the 12-vertex, 30-edge, 20-triangular-face icosahedron graph viewed from the antipodal vertex \bar{b} . The base vertex a (red, $d = 0$) is concentric with the first pentagon (blue, $d = 1$) and the second pentagon (green, $d = 2$, rotated by $+36^\circ$); the outermost shell consists of the antipodal vertex \bar{b} alone (purple, $d = 3$). **Right:** graph-structural data, distance-shell vertex counts $(N_0, N_1, N_2, N_3) = (1, 5, 5, 1)$, and edge bookkeeping $5 + 5 + 10 + 5 + 5 = 30$. The graph is vertex-transitive and distance-transitive, with automorphism group $\text{Aut}(\Gamma_0) \cong A_5 \times Z_2$ of order 120. This figure fixes the finite-seed data of the present paper and is used as the reference object for all M -layer computations of §§3 onward (Green profile, cross-distance kernel).

3.2. Green profile of a single source

The Laplacian $L = D - A$ (D : degree matrix; A : adjacency matrix) of Γ_0 has a one-dimensional zero mode $\ker L = \mathbb{R} \mathbf{1}$ spanned by the constant function $\mathbf{1} \in \mathbb{R}^V$. The zero-mode-removed Green equation is given for source $a \in V$ by

$$LG_a = \delta_a - \frac{1}{|V|} \mathbf{1}, \quad \sum_{x \in V} G_a(x) = 0,$$

which determines G_a uniquely. We define the deformation field on the finite medium when source a exists by

$$u_a(x) = G(d(x, a))$$

where $d(x, a)$ is the graph distance on Γ_0 and G is the zero-mode-removed Green profile.

By the high symmetry of the icosahedron, $G_a(x)$ depends only on the graph distance from a and not on the individual position of x , so it is a *distance-only* function. The Green value at each distance shell is

$$G_0 = \frac{7}{36}, G_1 = \frac{1}{90}, G_2 = -\frac{7}{180}, G_3 = -\frac{1}{18}.$$

These are used in this paper as finite rational calculations directly verifiable on the regular icosahedron graph. In particular, the strict ordering

$$G_1 > G_2 > G_3$$

holds and is referenced again in §7 (attractive-sign core) of this paper as the anchor for the attractive readout.

What matters for the discussion of this paper is not the specific Green numerical values but the distance-only structure itself: u_a depends only on graph distance. In §3.3 this is refined not by the phrase “directionless” but by invariance under the action of the graph automorphism group.

3.3. A single source does not have a preferred direction

The field u_a of a single source a takes equal values on each shell centered at a :

$$u_a(x) = G(d(x, a)) \implies u_a(x) = u_a(y) \text{ whenever } d(x, a) = d(y, a).$$

There is therefore no preferred direction in a single field.

What is important here is *not* to say “symmetry is preserved.” We instead use the following formulation.

Definition 1 (One-center shell-uniformity). A field $u: V \rightarrow \mathbb{R}$ on a finite graph Γ is one-center shell-uniform with center $a \in V$ when

$$d(x, a) = d(y, a) \implies u(x) = u(y)$$

holds for all $x, y \in V$.

Proposition 1 (Shell-uniformity of u_a). $u_a(x) = G(d(x, a))$ is one-center shell-uniform with center a .

This follows directly from the definition.

Refinement via graph automorphisms.

Let $\text{Aut}(\Gamma_0)$ be the graph automorphism group [29], [30] of Γ_0 , and write the subgroup fixing vertex a as

$$\text{Stab}(\{a\}) = \{g \in \text{Aut}(\Gamma_0) : g(a) = a\}.$$

For $g \in \text{Stab}(\{a\})$, graph distance is preserved ($d(g(x), a) = d(x, a)$), so

$$u_a(g(x)) = G(d(g(x), a)) = G(d(x, a)) = u_a(x).$$

Hence u_a is invariant under the action of $\text{Stab}(\{a\})$. This is the precise meaning of “a single field has no preferred direction.” The general framework of this refinement, in particular the role played by distance-regularity [31], [32] and association schemes [33], and the corresponding linear-representation language [34], [35], are standard.

Similarly, for source b .

The standalone field with source b ,

$$u_b(x) = G(d(x, b)),$$

is invariant under the action of $\text{Stab}(\{b\})$, and u_b alone has no preferred direction.

Note: this refers to “degeneracy of the invariant group”, not “symmetry breaking”.

We emphasize here that u_a and u_b each have separate invariant groups $\text{Stab}(\{a\})$ and $\text{Stab}(\{b\})$. The fact that their overlap product $\Phi_{ab} = u_a u_b$ is not in general invariant under either of these groups—and is instead invariant under $\text{Stab}(\{a, b\})$ which fixes the source pair $\{a, b\}$ —arises because Φ_{ab} has a bicenter axial structure. The detailed invariant-group content is refined further in §5; in this section we restrict ourselves to the properties of the single field.

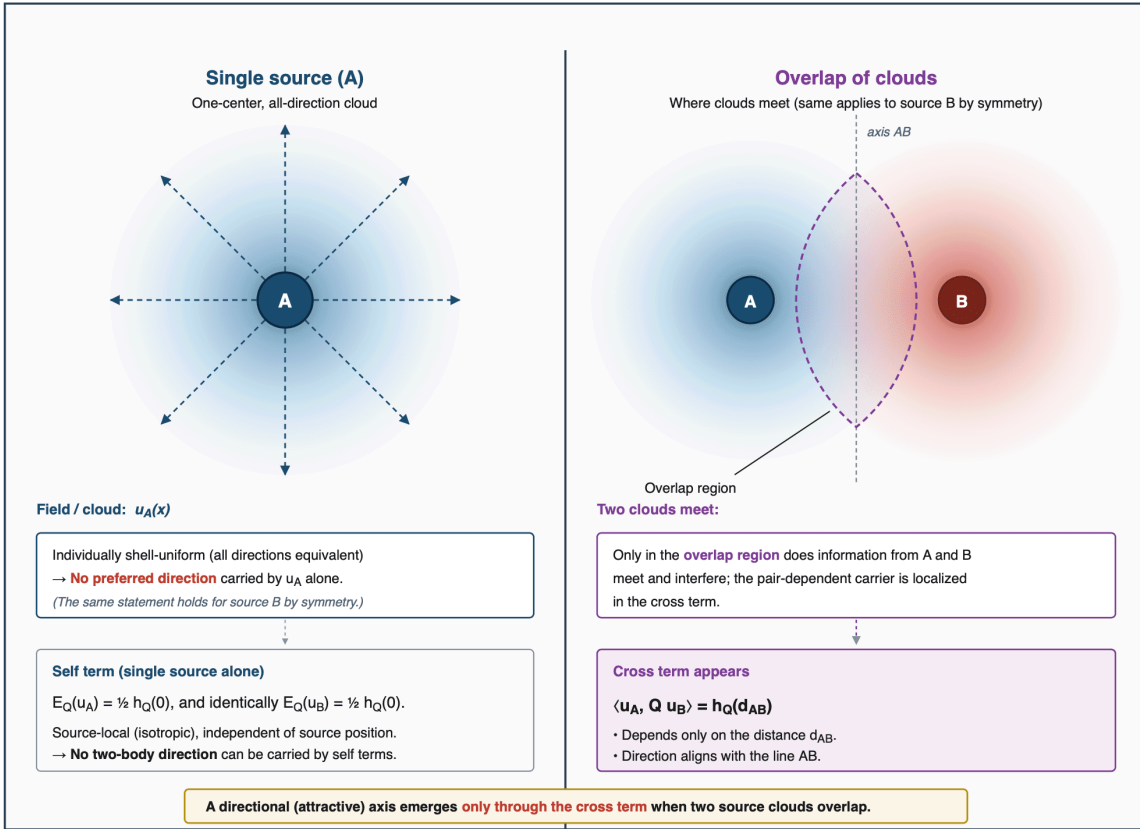


Figure 2. Cross-term origin of the attractive direction. **Left:** the structure of the single-source field $u_A(x)$. The one-center cloud centered on the source is shell-uniform in all directions and carries no preferred direction; the same statement holds for source B by symmetry. The self-energy $E_Q(u_A) = \frac{1}{2} h_Q(0)$ (and identically for u_B) is source-local and source-position-independent, hence cannot carry two-body directional information. **Right:** two source clouds A and B coexisting on the same medium. Only in the overlap region (purple dashed) do information from A and B meet and interfere, and the pair-dependent carrier $\langle u_A, Q u_B \rangle = h_Q(d_{AB})$ is localized in the cross term. This bilinear quantity depends only on the source separation d_{AB} , and its directional content aligns with the axis AB joining the two sources. Directionality therefore arises not within a single field but only in the cross term where two fields overlap. The figure visualizes the central reframing of §§1.1–1.2 of this paper.

4. Quadratic medium energy and distance-kernel localization: from cross term to distance kernel

In this section we determine the M -layer proposition that underlies the central proposition of this paper (§1.2.1). We demote the polarization identity to a bookkeeping lemma and then formulate the main M -layer claim, the *distance-kernel localization theorem*.

4.1. General secondary medium energy

For a scalar deformation field $u: V \rightarrow \mathbb{R}$ on a finite set V , consider the general symmetric quadratic form

$$E_Q(u) = \frac{1}{2} \langle u, Qu \rangle$$

where Q is a symmetric operator. In the mass quadratic, $Q = I$; in the Laplacian quadratic, $Q = L$. For two fields u, v , the Q -bilinear term is defined by

$$B_Q(u, v) = \langle u, Qv \rangle.$$

If Q is symmetric, then

$$E_Q(u+v) = E_Q(u) + E_Q(v) + B_Q(u, v).$$

This is standard polarization bookkeeping, not the novelty of this paper.

4.2. Polarization as bookkeeping, not novelty

Let u_a and u_b be the individual responses produced by two sources a and b , and adopt

$$u_{a+b} = u_a + u_b$$

as the minimal model of linear response. This assumption is part of the grammar of this paper and is a strong constraint excluding nonlinear responses, screening, hysteresis, and wake responses.

The increase in medium energy due to the coexistence of two bodies on the same medium is defined as the *interaction excess*,

$$I_Q(a, b) := E_Q(u_a + u_b) - E_Q(u_a) - E_Q(u_b).$$

By the standard polarization identity for symmetric quadratic forms,

$$I_Q(a, b) = \langle u_a, Qu_b \rangle.$$

This proposition is not an algebraic novelty of this paper. Its role here is to separate self-energy and source-pair bilinear terms as bookkeeping. The actual M -layer claim is the distance-kernel localization theorem of the next subsection.

4.3. Equivariant single-source fields

The standalone source field treated in this paper is constructed in accordance with the symmetry of the graph: for a graph automorphism g , moving source a to ga moves the field in the same way,

$$u_{ga} = gu_a.$$

On the icosahedron seed, $u_a(x) = G(d(x, a))$ is defined, so this equivariance follows from preservation of graph distance. We further assume that Q commutes with the graph automorphism action,

$$Q(gu) = g(Qu).$$

This condition stipulates that the medium operator does not implicitly violate the symmetry of the substrate. The Laplacian L and the identity I are typical examples satisfying this condition.

4.4. Distance-kernel localization theorem

Theorem 2 (Distance-kernel localization). *Let Γ be a finite graph and assume that its automorphism group acts vertex-transitively and distance-transitively on $V(\Gamma)$. Let Q be a symmetric operator on \mathbb{R}^V commuting with the automorphism action, and let $\{u_a\}_{a \in V}$ be a family of single-source fields satisfying the equivariance condition*

$$u_{ga} = gu_a.$$

Then the source-pair bilinear quantity

$$B_Q(a, b) = \langle u_a, Qu_b \rangle$$

depends only on the graph distance $d(a, b)$. Equivalently, there exists a unique function

$$h_Q: \{0, 1, \dots, \text{diam}(\Gamma)\} \rightarrow \mathbb{R}$$

such that

$$B_Q(a, b) = h_Q(d(a, b)).$$

Proof. For any automorphism g ,

$$\begin{aligned} B_Q(ga, gb) &= \langle u_{ga}, Qu_{gb} \rangle \\ &= \langle gu_a, Qgu_b \rangle \\ &= \langle gu_a, gQu_b \rangle \\ &= \langle u_a, Qu_b \rangle. \end{aligned}$$

Hence B_Q is constant on each orbit of ordered source pairs. By distance-transitivity, ordered pair orbits are classified by the distance shell. Therefore, $B_Q(a, b)$ is a function of $d(a, b)$ only. \square

This theorem unifies the former M20/M12 lemmas into one. The central M -kernel of this paper is not the existence of cross-terms themselves, but the point at which finite symmetry forces the source-pair bilinear quantity to fall to a distance kernel.

4.5. Self-energy is source-position-independent

Corollary 3 (Self-energy independence). *Under the hypotheses of Theorem 2, the self-energy*

$$E_Q(u_a) = \frac{1}{2} \langle u_a, Qu_a \rangle = \frac{1}{2} h_Q(0)$$

is independent of the source position $a \in V$.

Proof. Setting $a = b$ in Theorem 2 gives $B_Q(a, a) = h_Q(0)$, which depends only on the separation 0. Halving yields the claim. \square

Hence, the self-energy can describe the cost of a single source's deformation but cannot carry information about the separation of two sources.

4.6. Interaction excess is a cross-distance kernel

Corollary 4 (Cross-distance kernel). *Under the hypotheses of Theorem 2 and the linear-response grammar*

$$u_{a+b} = u_a + u_b,$$

the secondary interaction excess

$$I_Q(a, b) := E_Q(u_a + u_b) - E_Q(u_a) - E_Q(u_b)$$

satisfies

$$I_Q(a, b) = h_Q(d(a, b)).$$

Proof. By bilinearity and symmetry of $(\cdot, Q \cdot)$, $I_Q(a, b) = \langle u_a, Qu_b \rangle$. Theorem 2 then gives the claim. \square

The interaction excess is therefore not merely a cross-term; it is a *cross-distance kernel*. This is the precise M -layer proposition underpinning the physical interpretation of this paper.

4.6.1. Multi-source decomposition: pairwise cross-kernel sum

Corollary 5 (Pairwise N -body decomposition). *Under the hypotheses of Theorem 2, let $a_1, \dots, a_N \in V$ be pairwise distinct ($a_i \neq a_j$ for $i \neq j$), with total field*

$$u_{a_1 + \dots + a_N} = u_{a_1} + \dots + u_{a_N}.$$

Then the placement-dependent part of the interaction excess decomposes as

$$I_Q^{(N)}(a_1, \dots, a_N) = \sum_{1 \leq i < j \leq N} h_Q(d(a_i, a_j)).$$

Proof. Apply Corollary 4 to each ordered pair (a_i, a_j) with $i < j$, and sum. \square

This is not a proposition introducing a new primitive N -body carrier. The right-hand side is the distance-kernel localization theorem applied $\binom{N}{2}$ times and is closed within the same first-occurrence grammar. The multi-source hardness at this level, therefore, does not appear as an irreducible N -body force but as a constrained compatibility problem: whether pairwise distance kernels can be simultaneously realized on a finite graph.

4.7. Hierarchical relationship between two readings: Green energy as primary, mass overlap as diagnostic

We do not treat the mass quadratic and the Laplacian / Green quadratic as equivalent physical theories. They are different quadratic forms, and in general

$$h_A(d) \neq h_L(d).$$

Source-coupled Green / Laplacian readings are treated as the primary readings for gravity-channel energy in this paper; mass overlap is treated as a localization diagnostic.

For $Q = I$,

$$h_A(d(a, b)) = B_A(a, b) = \sum_{x \in V} u_a(x)u_b(x)$$

is the sum over V of the pointwise overlap density $\Phi_{ab}(x) = u_a(x)u_b(x)$. Mass overlap is therefore used as a local diagnostic indicating where two one-center shell-uniform fields contract into a two-center axial structure.

The energy sign of the gravity channel is read from the source-coupled quadratic functional

$$\mathcal{F}_Q(u; \rho) = \frac{1}{2} \langle u, Qu \rangle - \langle \rho, u \rangle.$$

The equilibrium field satisfies $Qu_\rho = \rho$, and writing $G = Q^{-1}$ gives $u_\rho = G\rho$. The corresponding on-shell energy is

$$\mathcal{F}_Q(u_\rho; \rho) = -\frac{1}{2} \langle \rho, G\rho \rangle.$$

For two sources $\rho = \rho_a + \rho_b$, the two-body part is

$$U_{int}^{on}(\rho_a, \rho_b) = -\langle \rho_a, G\rho_b \rangle.$$

The Laplacian seed reading involves zero-mode removal, with the Green equation

$$Lu_b = \delta_b - \frac{1}{|V|} \mathbf{1},$$

and from the mean-zero property of u_a ,

$$h_L(d(a, b)) = \langle u_a, Lu_b \rangle = \left\langle u_a, \delta_b - \frac{1}{|V|} \mathbf{1} \right\rangle = u_a(b) = G(d(a, b)).$$

Therefore, the abstract distance kernel h_Q becomes the concrete Green profile G_d when $Q = L$.

The choice $Q = L$ is not a theorem that nature necessarily realizes; it is the model commitment of this paper. The Laplacian is, however, the smallest graph-local quadratic operator measuring relative deformation rather than absolute displacement and is consistent with the zero-mode-removed Green response and with the subsequent Dirichlet / Poisson continuum bridge.

$$\langle u, Lu \rangle \rightarrow \int |\nabla \phi|^2.$$

We therefore adopt $Q = L$ as the minimal gravity-channel model. This does not exclude other operators such as $L + m^2 I$, which belong to a different model class.

4.8. Summary of this section

The following M -layer structure has been obtained in this section:

$$\begin{aligned} & \Rightarrow \text{source-pair bilinear invariant} \Rightarrow \text{distance kernel} \\ & \Rightarrow \text{self term is constant} \Rightarrow \text{source-placement dependence is localized to the cross-distance kernel.} \end{aligned}$$

This is the central M -kernel of this paper. The scope declaration of the first-occurrence grammar (§6.4) is a separate, non-theorem statement specifying the surrounding model class, and the physical gravity reading belongs to the still-later P/E layer.

5. Overlap density: where does spherical-symmetry breaking occur?

This section is the central pivot of this paper. By combining the one-center shell-uniformity of the single field established in §3 and the cross-term structure of the secondary energy established in §4, we determine, via invariant-group theory, where the spherical-symmetry breaking is localized. The argument is that the breaking occurs not in the single field but only in the *product of two fields* (the overlap density).

5.1. Pointwise overlap density

The *local overlap density* of two source fields is defined by

$$\Phi_{ab}(x) = u_a(x)u_b(x).$$

By §3.2, each field is

$$u_a(x) = G(d(x, a)), u_b(x) = G(d(x, b)),$$

so the overlap density becomes

$$\Phi_{ab}(x) = G(d(x, a))G(d(x, b)).$$

It is uniquely determined as a function of the graph-distance pair $(d(x, a), d(x, b))$.

This is a function of the two variables $d(x, a)$ and $d(x, b)$. Therefore,

- the value cannot be determined from the shell $(d(x, a) = \text{const})$ around a alone;
- the value cannot be determined from the shell $(d(x, b) = \text{const})$ around b alone.

At each point x , the value of $\Phi_{ab}(x)$ is determined only after x has been characterized by *two* distance indices: its distance from a and its distance from b .

5.2. From one-center shell-uniformity to two-center axial structure

As established in §3.3, the standalone field u_a is invariant under the action of $Stab(\{a\})$, and u_b is invariant under the action of $Stab(\{b\})$. The question is whether the product $\Phi_{ab} = u_a u_b$ is still invariant under these groups.

Labeled and unlabeled source pairs.

We distinguish two kinds of invariant group, summarized in Table 10.

Source treatment	Notation	Meaning
Labeled source pair	$Stab(a, b) = \{g : g(a) = a, g(b) = b\}$	For sources of distinct masses, distinct couplings, or otherwise distinguishable labels
Unlabeled / homogeneous source pair	$Stab(\{a, b\}) = \{g : g(\{a, b\}) = \{a, b\}\}$	When the exchange $a \leftrightarrow b$ is admitted between sources of the same kind and strength

Table 10. Two kinds of invariant group for source pairs.

For the minimal claim of this paper, the labeled invariant group $Stab(a, b)$ suffices. We extend the discussion to $Stab(\{a, b\})$, which includes the exchange symmetry, only for homogeneous sources.

Proposition 6 (Invariant group of Φ_{ab}). *The overlap density $\Phi_{ab}(x) = u_a(x)u_b(x)$ has the following invariance under the action of $Aut(\Gamma_0)$.*

1. If $g \in Aut(\Gamma_0)$ satisfies both $g(a) = a$ and $g(b) = b$, then

$$\Phi_{ab}(g(x)) = \Phi_{ab}(x).$$

2. If $g \in Aut(\Gamma_0)$ satisfies $\{g(a), g(b)\} = \{a, b\}$ (preserving the source pair as a set), then

$$\Phi_{ab}(g(x)) = \Phi_{ab}(x).$$

3. For g with $g \in Stab(\{a\})$ but $g \notin Stab(\{b\})$, $\Phi_{ab}(g(x))$ is in general not equal to $\Phi_{ab}(x)$.

Proof. (1) Graph distance is invariant under graph automorphism: if $g(a) = a$ and $g(b) = b$, then $d(g(x), a) = d(x, a)$ and $d(g(x), b) = d(x, b)$. Hence $\Phi_{ab}(g(x)) = G(d(x, a))G(d(x, b)) = \Phi_{ab}(x)$.

(2) If $\{g(a), g(b)\} = \{a, b\}$, the claim follows from the symmetric role of a, b in the definition of Φ_{ab} (since $u_a u_b = u_b u_a$). Combining (1) and (2), Φ_{ab} is invariant under

$$Stab(\{a, b\}) = \{g \in Aut(\Gamma_0) : g(\{a, b\}) = \{a, b\}\}.$$

(3) If $g(a) = a$ and $g(b) \neq b$, then $d(g(x), a) = d(x, a)$ but $d(g(x), b) \neq d(x, b)$ may occur. Since G is non-constant ($G_0 \neq G_1 \neq G_2 \neq G_3$), there exists x with $\Phi_{ab}(g(x)) \neq \Phi_{ab}(x)$.

□

Corollary 7 (Reduction of invariant groups)

$$(\text{invariant group of } \Phi_{ab}) = Stab(\{a, b\}) \subseteq Stab(\{a\}), Stab(\{b\}).$$

This is the invariant-group-theoretic precise form of the *reduction from one-center shell uniformity to two-center axial structure*.

That is: u_a and u_b are individually invariant under large groups ($Stab(\{a\})$ and $Stab(\{b\})$ respectively), but their product retains as its invariant group only the *common part* of the two. This is an axial structure: an automorphism that fixes the source pair $\{a, b\}$ acts on the graph by preserving the a - b axis (rotations around the axis remain, as elements of $Stab(\{a, b\})$) but cannot reorient the axis itself.

$$\text{spherical / shell-uniform (single-center)} \longrightarrow \text{axial / two-center}$$

This is a reduction of the invariant group and is the precise form of the phenomenon often loosely described as “symmetry breaking”.

5.3. The breaking is localized not in a single field but in the cross term

This is the most important claim of this paper.

Proposition 8 (Localization of the breaking). *Spherical-symmetry breaking does not occur on u_a alone or on u_b alone. Breaking occurs only in the product $\Phi_{ab} = u_a u_b$.*

Verification.

- For u_a : invariant under $Stab(\{a\})$ (§3.3). Hence rotational symmetry around source a is fully preserved.
- For u_b : similarly invariant under $Stab(\{b\})$, so rotational symmetry around source b is fully preserved.
- For Φ_{ab} : by Corollary 7, the invariant group reduces to $Stab(a, b)$ for labeled source pairs and to $Stab(\{a, b\})$ for homogeneous pairs. The one-center shell uniformity is broken; only the axial structure remains.

The phenomenon of “spherical-symmetry breaking” is therefore not a property of the source fields themselves but a derived phenomenon that first appears in the *algebraic composition* of two source fields, namely $u_a u_b$.

Conclusion: the origin of the gravitational direction.

The direction of gravity does not arise from “the object initially pulling toward the object”. Rather, a preferred direction—the a - b axis—is generated in the overlap region of two clouds that are individually compressed equally in all directions:

The attractive direction arises not as a single source field but as an axial structure of the overlap density.

This is the invariant-group-theoretic content of the central proposition (§1.2). The boxed claim “omnidirectionality is a property of a single field, while directionality is a property of an overlap quadratic quantity” is refined as the invariant-group reduction of Corollary 7.

Remark: this is not specific to the substrate.

The discussion of §5.2 and §5.3 is not specific to the A_5 icosahedral seed. The general structure—a distance-only field on any vertex-transitive finite graph, plus the cross term—holds in the same way. The role of the A_5 seed is to anchor an attractive sign by ensuring that the Green values per distance shell satisfy the strict ordering $G_1 > G_2 > G_3$, as refined in §7. The invariant-group-theoretic content of this section itself is robust to substrate choice.

5.4. Directionality is a second-order quantity

Lemma 9 (Direction as a second-order quantity). *The single-source fields u_a and u_b are each one-center shell-uniform and possess no preferred direction. The secondary product*

$$\Phi_{ab}(x) = u_a(x)u_b(x)$$

depends on the bicenter pair $(d(x, a), d(x, b))$ and possesses an axial structure determined by $\{a, b\}$. Hence the directional content of the source pair is a second-order quantity in the source fields.

Proof. The first claim is Proposition 1. The second follows from Propositions 6 and 8. \square

Thus, the attractive direction extracted in this paper is not a direction hidden in any single field; it is a direction that appears as a quadratic quantity only when two omnidirectionally equal fields overlap on the same medium.

5.5. Overlap as the encounter between clouds

The field u_a produced by a single source a is an omnidirectionally compressed cloud centered on a . Likewise, u_b is an omnidirectionally compressed cloud centered on b . Individually, neither cloud has a preferred direction.

When two such clouds overlap on the same finite medium, however, the pointwise overlap

$$\Phi_{ab}(x) = u_a(x)u_b(x)$$

arises. This overlap preserves neither the shell uniformity centered on a nor the shell uniformity centered on b ; what remains is a two-center axial structure determined by the a - b axis.

The direction of gravity therefore does not exist within any single cloud. Directionality first appears in the secondary overlap born from the encounter of two omnidirectionally compressed clouds:

Omnidirectional compression is neutral as a first-order quantity, but the encounter of two clouds produces an axial direction as a second-order quantity.

6. From interaction excess to effective channel

As established in §5, the overlap density Φ_{ab} has a two-center axial structure. In §4 we confirmed, as a bookkeeping lemma, that the interaction excess due to two-body coexistence is decomposed into cross terms within a linear-response, quadratic grammar. In this section we collapse this into a scalar profile depending on the source separation d_{ab} and classify the static channels as attractive, repulsive, or neutral.

6.1. Total interaction excess

In the mass-quadratic reading, the total interaction excess is obtained by summing the local overlap density over all vertices,

$$I_M(a, b) = \sum_{x \in V} \Phi_{ab}(x) = \sum_{x \in V} u_a(x)u_b(x).$$

This is a scalar; however, it depends on the source separation

$$d_{ab} = d(a, b) \in \{0, 1, 2, 3\}.$$

Proposition 10 (d_{ab} -dependence of the total interaction excess). $I_M(a, b)$ is invariant under the action of $Aut(\Gamma_0)$ and depends, among the relative positions of the source pair (a, b) , only on d_{ab} . By distance-transitivity, it can be written as

$$I_M(a, b) = C(d_{ab}),$$

where $C: \{0, 1, 2, 3\} \rightarrow \mathbb{R}$ is a sequence depending only on the graph distance.

6.2. Scalar but directional: effective gradient via discrete differences

At first sight, $C(d_{ab})$ appears scalar and therefore directionless. However, when the position of source a is moved on Γ_0 , d_{ab} changes, and the value of the interaction excess changes accordingly.

On a finite graph, discrete differences replace continuous derivatives. The change when source a moves to an adjacent vertex a' is

$$\delta C = C(d_{a'b}) - C(d_{ab}),$$

with $d_{a'b} \in \{d_{ab} - 1, d_{ab}, d_{ab} + 1\}$.

In particular, the increment along a one-step change in source separation,

$$\Delta_d C = C(d + 1) - C(d),$$

provides the effective gradient in configuration space. This is not a primitive force in continuous space, but the difference of energy profiles over the finite source configurations.

We emphasize that $\Delta_d C$ carries non-trivial information only along the a - b axis. When source a moves perpendicularly to the a - b direction so that $d_{a'b} = d_{ab}$, the interaction excess does not change. Directionality is therefore a derived consequence of the two-center axial overlap.

6.3. Static channel classification: trichotomy by elimination

When the interaction excess or on-shell interaction energy $U_{int}(d)$ depends only on the source separation $d = d_{ab}$, the conservative channels arising from a static, symmetric, secondary medium are classified by the difference

$$\Delta U(d) = U_{int}(d + 1) - U_{int}(d).$$

Three cases arise (Table 11).

Condition	Channel	Meaning
$U_{int}(d + 1) > U_{int}(d)$	Attractive	Energy increases with separation; the proximate direction is energy-lowering.
$U_{int}(d + 1) < U_{int}(d)$	Repulsive	Energy decreases with separation; the distal direction is energy-lowering.
$U_{int}(d + 1) = U_{int}(d)$	Neutral	Independent of separation.

Table 11. Trichotomy of static conservative channels.

The conservative channels arising from a static symmetric quadratic medium therefore fall into three classes—attractive, repulsive, neutral—purely by the trichotomy of differences. This is an algebraic fact.

Resistive channels: structural exclusion.

Time-reversal-antisymmetric resistive channels do not appear in this static classification. Operationally, we adopt the following definition:

Resistive channel: a dissipative force that is antisymmetric under time reversal.

This includes linear friction ($F \sim -\gamma d$), quadratic resistance ($F \sim -\beta d |d|$), Coulomb friction ($F \sim -\mu N \text{sgn} d$), memory kernels ($F = -\int K(t-s)d(s)ds$), wake forces, retardation effects, Lindblad dissipation, and so on.

To produce such a resistive term, at least one of the following is required: time delay, wake kernel, memory, dissipation, or time-reversal breaking. The static, symmetric, linear-quadratic framework adopted here is, however, time-reversal symmetric (the hypothesis of §4.1). It therefore follows that:

A time-reversal-symmetric, static, symmetric, linear quadratic framework cannot structurally produce a time-reversal-antisymmetric resistive channel.

This is not a claim that resistive channels cannot physically exist. The claim is narrower: resistive channels depending on velocity, memory, retardation, or dissipation cannot appear inside a time-reversal-symmetric, static, symmetric, linear-quadratic grammar. The cross-term mechanism of this paper does not deal with resistive channels; they are deferred to a subsequent wake/retardation bridge. This exclusion proposition is registered in §10.1 as M22, as a grammar-scope audit rather than a physical no-go.

Force types outside the scope of this paper.

The framework of this paper does *not* address the following force types.

1. **Stochastic forces** (Brownian motion, Langevin force): treated as stochastic processes, outside the deterministic dynamical framework here.
2. **Velocity-dependent non-dissipative forces** (Lorentz, Coriolis): require vector potentials and rotations; though velocity-dependent and non-dissipative, they fall structurally outside the scalar-field framework of this paper. They are not resistive channels.
3. **Many-body interactions** (three or more bodies): outside the scope of the present two-body focus.

These are explicitly separated from this paper and are to be treated within different frameworks.

Local versus global classification.

The trichotomy is a *local classification at each d* . Global characters such as “short-range attractive / long-range repulsive” (Lennard-Jones type) are treated as combinations of local trichotomies. On the A_3 seed, the attractive ordering is verified at all $d \in \{1, 2, 3\}$ in §7 and is therefore globally attractive. Distance-dependent sign changes are admissible on more general substrates, but they too can be represented as local combinations of the three options.

Microscopic versus macroscopic level.

This paper excludes the resistive channel only at the *microscopic level*. The possibility that emergent dissipation arises from macroscopic coarse-graining (a typical example being Hamiltonian dynamics yielding Brownian motion via the fluctuation-dissipation theorem) is consistent with the microscopic claim of this paper. Such emergent dissipation belongs to the coarse-graining bridge and is treated in the context of Open Bridge G_3 .

In a static, symmetric, linear-quadratic framework, the channel at the microscopic level is restricted to three options: attractive / repulsive / neutral.

6.4. Scope of the first-occurrence grammar

§6.3 confirmed that static conservative channels fall into the three classes attractive / repulsive / neutral. Here we step further back and *declare the scope* of the first-occurrence grammar adopted in this paper, i.e., which carriers of the two-body source-pair dependence are placed inside the model class.

This subsection is not a theorem. The central M -layer theorem of this paper is M20*, the distance-kernel localization theorem (§4.4): in a vertex-transitive, distance-transitive, Q -equivariant finite quadratic medium, the source-pair bilinear quantity falls to a distance kernel, $B_Q(a, b) = h_Q(d(a, b))$. Self-energy independence and the cross-distance-kernel readout follow as corollaries. The scope declaration below specifies the surrounding model class of this M -content.

Scope declaration.

The model class of this paper is the static, endogenous, linear-response, quadratic grammar, with no primitive direction and no external witness. Within this class, the admissible source-pair-dependent carriers are restricted as follows.

Carrier	Example	Verdict	Reason
Self term	$E_{\mathcal{C}}(u_a), E_{\mathcal{C}}(u_b)$	Excluded	Contains only one source; cannot encode source-pair dependence.
External term	$\langle f, u_a \rangle$	Excluded	Imports a background witness; not endogenous to the source pair.
Cross term	$\langle u_a, Qu_b \rangle$	Admitted	Contains both sources, remains quadratic, requires no external witness.
Higher-order term	$u_a^2 u_b, u_a u_b^2, u_a^2 u_b^2$	Outside grammar	Lies outside the quadratic class.
Dynamic / dissipative	$d, \text{wake, memory, dissipation}$	Outside grammar	Requires velocity, memory, retardation, time-reversal breaking.
Nonlinear-response	Corrections from $u_{a+b} \neq u_a + u_b$	Outside grammar	Discards the linear-response axiom.

Table 12. Carriers admitted into the first-occurrence grammar declared in this paper.

This is a declaration of grammar, not a theorem of physical uniqueness. Self terms are excluded because they contain only one source. External terms are excluded because the first-occurrence question of this paper asks what the source pair can carry on its own, before background witnesses are admitted; admitting them would trivially import direction. Higher-order, dynamic, and nonlinear-response carriers are outside the model class declared here and are routed to other grammars (Open Bridges, dynamic extensions, nonlinear elasticity, wake / memory dynamics).

The remaining admissible carrier within the model class is therefore the cross-distance kernel $\langle u_a, Qu_b \rangle = h_{\mathcal{C}}(d(a, b))$, which is precisely the object the distance-kernel localization theorem describes. The work of identifying this carrier is the work of declaring the model class; the structural fact that this object depends only on $d(a, b)$ is the M -content (§4.4).

Question: Which candidate can carry the two-body direction sign in a static, linear-response, quadratic, endogenous grammar without primitive direction or external witness?

Candidate carrier	Reason for falling outside the grammar (filter)	Verdict
Self term <i>E_Q(u_A) alone</i>	Contains only one source ; no information about a second source. <i>Self-energy is source-position independent; cannot encode pair direction.</i>	X Disqualified
External background <i>imported direction</i>	Requires an external witness or background direction. <i>Not endogenous; direction is presupposed, not produced by the substrate.</i>	X Disqualified
Higher-order term <i>cubic, quartic, ...</i>	Lies outside the quadratic grammar of this paper. <i>May contain both sources, but only as higher-order corrections to the cross interaction.</i>	X Disqualified
Dynamic / dissipative <i>wake, memory, friction</i>	Requires wake, memory, velocity, or time-reversal breaking . <i>Lies outside the static grammar; routed to dynamical extensions / Open Bridge G_8.</i>	X Disqualified
Nonlinear response <i>screening, hysteresis</i>	Discards the linear-response axiom $u_{\mathbf{a}+\mathbf{b}} = u_{\mathbf{a}} + u_{\mathbf{b}}$. <i>Breaks the first-occurrence grammar; routed to nonlinear extensions.</i>	X Disqualified

only one candidate survives

Cross distance kernel $\langle u_A, Q u_B \rangle = h_Q(d_{AB})$	Static, endogenous, linear-response, quadratic; contains both sources; depends only on graph distance $d_{\{AB\}}$. <i>Direction aligns with the line AB; passes all five filters.</i>	✓ Surviving
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Epistemic status:
This is a **carrier-scope corollary**, not a physical uniqueness theorem. The claim is internal to the static linear-response quadratic endogenous grammar, not a universal statement about gravity in nonlinear / dynamical / continuum frameworks.

Within a static, endogenous, linear-response, quadratic grammar without primitive direction or external witness, the cross distance kernel is the unique allowed carrier of the two-body direction sign. [Lean: MinimalDirectionSign.lean · cross_unique_minimal_endogenous_direction_sign]

Figure 3. Scope of the first-occurrence grammar declared in this paper. Within a static, linear-response, quadratic, endogenous grammar without primitive direction or external witness, candidate carriers of the two-body source-pair dependence are listed: self terms (single source, source-position-independent), external-background terms (require an external witness; not endogenous), higher-order terms (outside the quadratic class), dynamic / dissipative terms (require wake / memory / time-reversal breaking; outside the static class), and nonlinear-response terms (discard the linear-response axiom $u_{\mathbf{a}+\mathbf{b}} = u_{\mathbf{a}} + u_{\mathbf{b}}$; outside the first-occurrence grammar). The admitted carrier is the cross-distance kernel $\langle u_A, Q u_B \rangle = h_Q(d_{AB})$. This figure is a scope declaration, not a physical uniqueness theorem; the corresponding machine-readable enumeration is recorded as `cross_unique_minimal_endogenous_direction_sign` in `MinimalDirectionSign.lean` (Appendix 10).

Lean infrastructure note.

The carrier list above is recorded in machine-readable form in the companion Lean file `MinimalDirectionSign.lean` as the inductive type `Carrier` and the associated theorem `cross_unique_minimal_endogenous_direction_sign`. The role of this Lean object is to make the scope declaration mechanically traceable: any future extension of the paper that introduces a non-cross carrier must update the type `Carrier` explicitly, preventing silent expansion of the model class. The Lean object is not asserted as a non-trivial mathematical theorem; its size and content match the size and content of the scope declaration above.

6.5. Gravity-channel readout

In the source-coupled on-shell reading of §4.7, the interaction energy under the Laplacian Green model is

$$U_{in}^{on}(d) = -\lambda G(d), \lambda > 0,$$

where λ is a dimensional coupling.

Under $P_{grav} / P_{energy \rightarrow force}$ the energy-lowering direction is read as the physical gravity channel. On a finite graph, this is not a continuous force but the configuration-space readout

$$F^{eff} \sim -\Delta_d U_{in}(d).$$

This formula is a force-channel readout, not a primitive force law. Here “readout” retains the sense fixed in §2.3: an admissible finite extraction of an effective observable, not an arbitrary projection. The cross-distance kernel gives the source-pair energy-ordering carrier; the finite difference of that ordering gives an update tendency along the two-center axis; the observed force variable is the coarse dynamical readout of that tendency. Passing from this static channel to an acceleration equation such as $\Delta^2 x = F$ requires the downstream dynamical bridge and is not derived in this paper.

The energy-lowering reading is not a claim that every variational stabilization should be called gravity; it is a restricted reading convention. When a source-pair-dependent static channel remains as a conservative channel along the two-center axis, the lower-energy direction is read as the static precursor of the attractive-force channel.

This paper therefore does not assert in general that gravity equals stabilization. It identifies specific static energy-ordering channels as precursors of gravitational attraction in finite media.

The remaining non- M -layer content here is the interpretation of differences in U_{int} as the direction of actual gravity. The cross-term localization of the source-position dependence is M / model- M , but the association with observed gravity belongs to P/E .

6.6. Minimal description of gravity

Compressing the structure of §3–§6, the description of the attractive channel in this paper is summarized as follows.

- (i) single source: $u_a(x) = G(d(x, a))$ (one-center shell-uniform);
- (ii) self-energy: $E_G(u_a)$ (source-position-independent);
- (iii) source-pair dependence: $I_G(a, b) = (u_a, Qu_b)$ (cross-term localization);
- (iv) axial diagnostic: $\Phi_{ab}(x) = u_a(x)u_b(x)$ (two-center axial);
- (v) static classification: $U_{int}(d+1) \gtrless U_{int}(d) \Rightarrow$ attractive / repulsive / neutral;
- (vi) force-channel readout: $F^{eff} \sim -\Delta_d U_{int}(d)$ (coarse readout of update tendency).

Among these, (i)–(v) are organized within the M / model- M layer in the finite quadratic medium model. The center is (ii)–(iii): distance-kernel localization causes the source-pair bilinear quantity to fall to $h_G(d)$, and the self-energy becomes a constant. The scope declaration of §6.4 specifies that, within the first-occurrence grammar adopted in this paper (no primitive direction, no external witness), the cross-distance kernel is the admitted carrier of the two-body source-pair dependence; this is a statement about the model class, not a uniqueness theorem. The physical force-channel readout in (vi) belongs to the P layer and remains distinct from the downstream derivation of a motion law.

7. Minimal finite demonstration of the attractive sign on the A_5 icosahedron seed

The distance-kernel localization / cross-distance-kernel carrier mechanism established in §5–§6 is a general structure independent of the substrate. In this section, we specialize to the A_5 icosahedron seed Γ_0 and confirm that an attractive sign is selected under source-coupled Green reading.

The limitations are stated up front. This section does not complete an approximate model of continuum gravity. The icosahedron graph has only four distance values $d = 0, 1, 2, 3$ and a strong antipodal structure $N_3 = 1$. What this section provides is therefore neither large-graph universality nor a continuum limit but a *minimal finite demonstration of a direction-sign mechanism*.

7.1. Source-coupled Green ordering: verified M -layer anchor

The zero-mode-removed Green profile introduced in §3.2 takes, on each distance shell, the values

$$G_0 = \frac{7}{36}, G_1 = \frac{1}{90}, G_2 = -\frac{7}{180}, G_3 = -\frac{1}{18}.$$

For the distance shells 1, 2, 3,

$$G_1 > G_2 > G_3.$$

This is the central M -layer fact of the finite seed calculation used in this paper, directly verifiable as a finite rational calculation on the regular icosahedron graph.

In the Laplacian Green reading, the corresponding distance kernel is $h_L(d) = (u_a, Lu_b)$. From §4.7, using the zero-mode-removed Green equation and the zero-mean property of u_a , we obtain $h_L(d(a, b)) = u_a(b) = G(d(a, b))$. The abstract distance kernel h_Q therefore becomes the concrete Green profile G_d when $Q = L$.

Under the source-coupled on-shell reading,

$$U_{int}^m(d) = -\lambda h_L(d) = -\lambda G_d, \lambda > 0.$$

By the Green ordering,

$$U_{int}(1) < U_{int}(2) < U_{int}(3),$$

i.e., the smaller the source separation, the lower the on-shell energy.

In this sense, the A_5 seed achieves an attractive sign under the source-coupled Green reading.

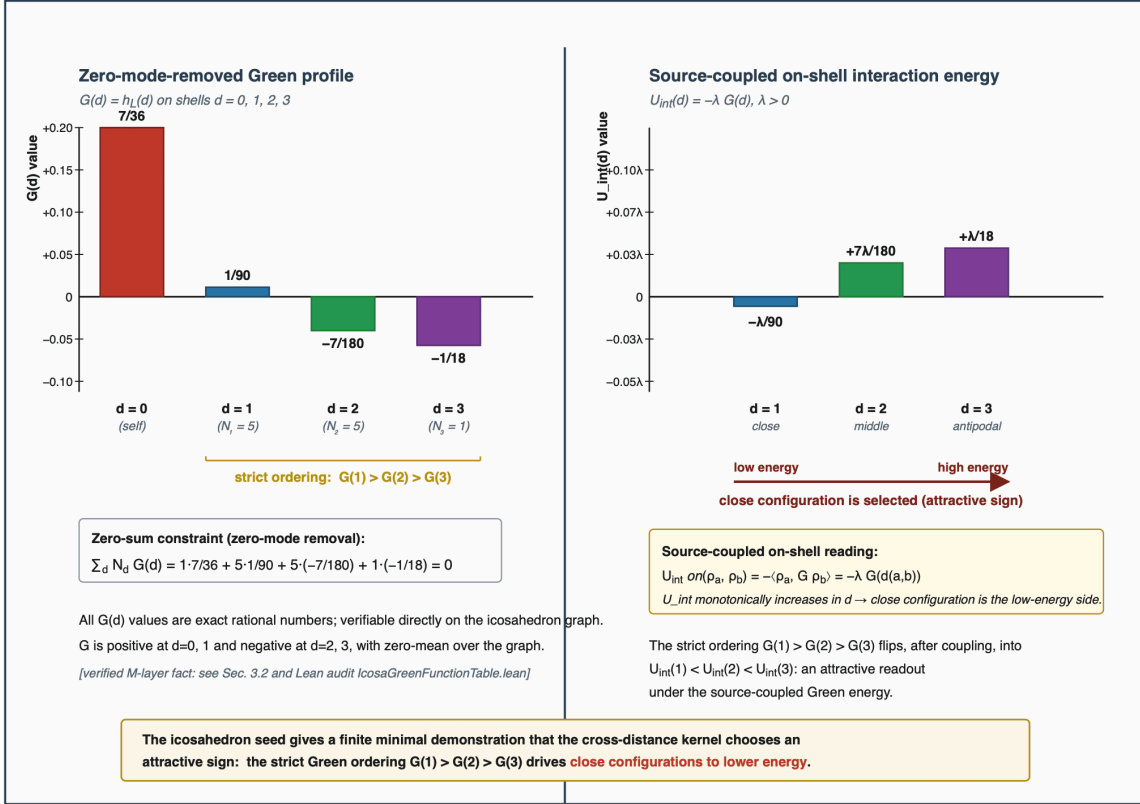


Figure 4. Zero-mode-removed Green profile and attractive sign on the icosahedron seed. **Left:** the Green profile $G(d) = h_L(d)$ on shells $d = 0, 1, 2, 3$, displayed as a bar chart with exact rational values $G_0 = 7/36$ (red), $G_1 = 1/90$ (blue), $G_2 = -7/180$ (green), and $G_3 = -1/18$ (purple). At $d = 1, 2, 3$ the strict ordering $G(1) > G(2) > G(3)$ holds, and the zero-sum constraint $\sum_d N_d G(d) = 0$ is satisfied. **Right:** the corresponding source-coupled on-shell interaction energy $U_{int}(d) = -\lambda G(d), \lambda > 0$. Sign reversal yields $U_{int}(1) < U_{int}(2) < U_{int}(3)$, so the close configuration ($d = 1$) is selected as the low-energy side and the antipodal configuration ($d = 3$) as the high-energy side. The cross-distance kernel therefore selects an attractive sign under the source-coupled Green-energy reading. The icosahedron seed thus provides a finite minimal demonstration that the cross-distance kernel chooses an attractive sign (§3.2, §7.1).

7.2. What the A_5 seed demonstrates

The A_5 seed does not establish the existence of a cross term: cross terms appear as bookkeeping in any linear-quadratic medium. Nor does it establish that A_5 is the unique generator of the attractive ordering; comparisons with general control graphs show that the attractive ordering is non-unique within the test class.

The positive content provided by the A_5 seed is more limited (Table 13).

Element	Content	Status
Non-trivial finite shell geometry	$(N_0, N_1, N_2, N_3) = (1, 5, 5, 1)$, diameter 3, antipodal layer	M anchor
Green attractive ordering	$G_1 > G_2 > G_3$	Verified M anchor
Source-coupled sign	$U_{int}^{on}(d) = -\lambda G_d$	Model- M / P commitment
Program compatibility	A_5 Hodge / Grad sector, $A_5 \rightarrow 2I \rightarrow E_8$ ladder, $\beta_0 = 11$ marker	Broader program anchor

Table 13. Positive content provided by the A_5 seed.

The A_5 seed is therefore not a unique generator but a compatibility anchor implementing the carrier mechanism on a finite, polyhedral, group-theoretically controlled, program-compatible substrate.

73. Degenerate comparison: complete graph K_{12}

The complete graph K_{12} is a useful comparison. K_{12} is vertex-transitive and maximally symmetric, but its distance structure is degenerate: any two distinct vertices are at distance 1. It therefore has neither a non-trivial finite shell ordering across $d = 1, 2, 3$ nor an antipodal layer.

This comparison shows that vertex-transitivity is not the only relevant property in this paper. The icosahedron seed is not merely symmetric but possesses

$$(N_0, N_1, N_2, N_3) = (1, 5, 5, 1),$$

together with diameter-three geometry and an A_5 -compatible polyhedral realization. The attractive Green ordering is therefore not a claim about every vertex-transitive graph; it is a claim about a specific seed with non-degenerate finite shell geometry.

74. Mass-overlap diagnostic: auxiliary, not the primary basis for the attractive sign

In the mass-overlap diagnostic, one uses

$$C(d) = \sum_x u_d(x)u_d(x).$$

This is not primarily a reading of physical gravitational energy but a diagnostic measuring where two one-center shell-uniform fields contract into a two-center axial structure.

For the regular icosahedron seed, the finite calculation gives

$$C(1) = \frac{11}{1080}, C(2) = -\frac{16}{1080}, C(3) = -\frac{28}{1080},$$

and hence

$$C(1) > C(2) > C(3).$$

However, this is not the principal basis for the gravitational-energy claim of this paper; the principal basis is the source-coupled Green ordering of §7.1. The mass-overlap ordering is only a compatibility check showing that the same two-center axial localization also appears in the finite diagnostic.

More detailed calculations of $C(d)$ are treated in the companion Lean infrastructure of this program; the attractive-sign claim in the main text is consolidated into the Green / on-shell reading.

75. The role of A_5 : not the sole generator but a program-compatible finite anchor

The role of A_5 in this paper becomes clear when separated into three steps (Table 14).

Step	Content	A_5 dependence
Source-position dependence localization	In a vertex-transitive finite quadratic medium, the self-energy cannot carry the placement dependence.	A_5 -independent
Attractive finite demonstration	The attractive sign is realized through the source-coupled Green ordering $G_1 > G_2 > G_3$.	Depends on the regular-icosahedron seed
Program compatibility	The same seed connects to the A_5 Hodge / Grad sector, the $A_5 \rightarrow 2I \rightarrow E_8$ ladder, and the $\beta_0 = 11$ graph-gauge marker.	Depends on the broader A_5 program

Table 14. Three steps and the dependence on A_5 .

A_5 is therefore not used to require the existence of cross terms; it is used as a *finite, polyhedral, group-theoretically controlled, program-compatible* compatibility anchor for instantiating the distance-kernel localization theorem and verifying the attractive sign within the declared model class.

A_5 is not the sole generator of the attractive ordering, but the smallest compatibility anchor connecting the cross-distance-kernel carrier mechanism to an existing finite-readout program.

This positioning means that comparisons with K_{12} and other 12-vertex-transitive graphs are not counterexamples that undermine the claim of this paper but a part of the Open-Bridge G_0 / finite-check program. This paper neither shows that “the attractive sign is produced by every graph” nor that “only A_5 produces the attractive sign”. It shows that the A_5 icosahedron seed provides a minimal finite demonstration that the carrier mechanism aligns with the attractive sign under the source-coupled Green reading.

The smallness of the icosahedron graph is a weakness, but it also clarifies the role of this paper. This is not a large-graph approximation of continuum gravity but a minimal demonstration that a two-body direction-sign carrier can be generated on a finite medium without primitive direction. Whether the same sign is stable on the truncated icosahedron, the snub dodecahedron, an A_5 Cayley graph, or larger A_5 -compatible graphs is left as a future finite-check program.

7.6. Finite verification example: clique-bounded cross-term frustration on the A_5 seed

The multi-source corollary of §4.6.1 can be finitely verified on the A_5 icosahedron seed. Let Γ_{12} denote the icosahedron graph and define the maximum number of sources whose pairwise distances are all bounded by k :

$$v_k(\Gamma_{12}) = \max \{ |S| : d(x,y) \leq k \text{ for all distinct } x,y \in S \}.$$

This invariant is a finite-graph-side index measuring the simultaneous saturation possible for pairwise cross-distance kernels. On the icosahedron graph,

$$v_1(\Gamma_{12}) = 3, v_2(\Gamma_{12}) = 6, v_3(\Gamma_{12}) = 12.$$

The meaning is as follows. $v_1 = 3$ means that the clique number is 3: at most three sources can simultaneously saturate the nearest-neighbor requirement $d = 1$ (on a triangular face), but with four sources at least one pair is forced to $d \geq 2$. $v_2 = 6$ means that, taking at most one element from each of the six antipodal pairs, all pair distances can be kept at ≤ 2 , but with seven sources the pigeonhole principle forces at least one antipodal pair $d = 3$. $v_3 = 12$ means that all vertices are saturated within diameter 3.

The two-body theorem of this paper therefore makes the following finite predictions (Table 15).

Sources	Finite-graph fact	Cross-term reading
$N = 2$	$d = 1$ realizable on an edge	Single cross kernel
$N = 3$	All pairs $d = 1$ realizable on a triangular face	Constrained triple kernel, but shallow
$N = 4$	Clique number 3 exceeded	Onset of nearest-neighbor frustration
$N = 7$	$v_2 = 6$ exceeded	Onset of antipodal frustration
$N = 12$	All vertices occupied	Full seed saturation

Table 15. Pairwise cross-kernel saturation thresholds on the A_5 seed.

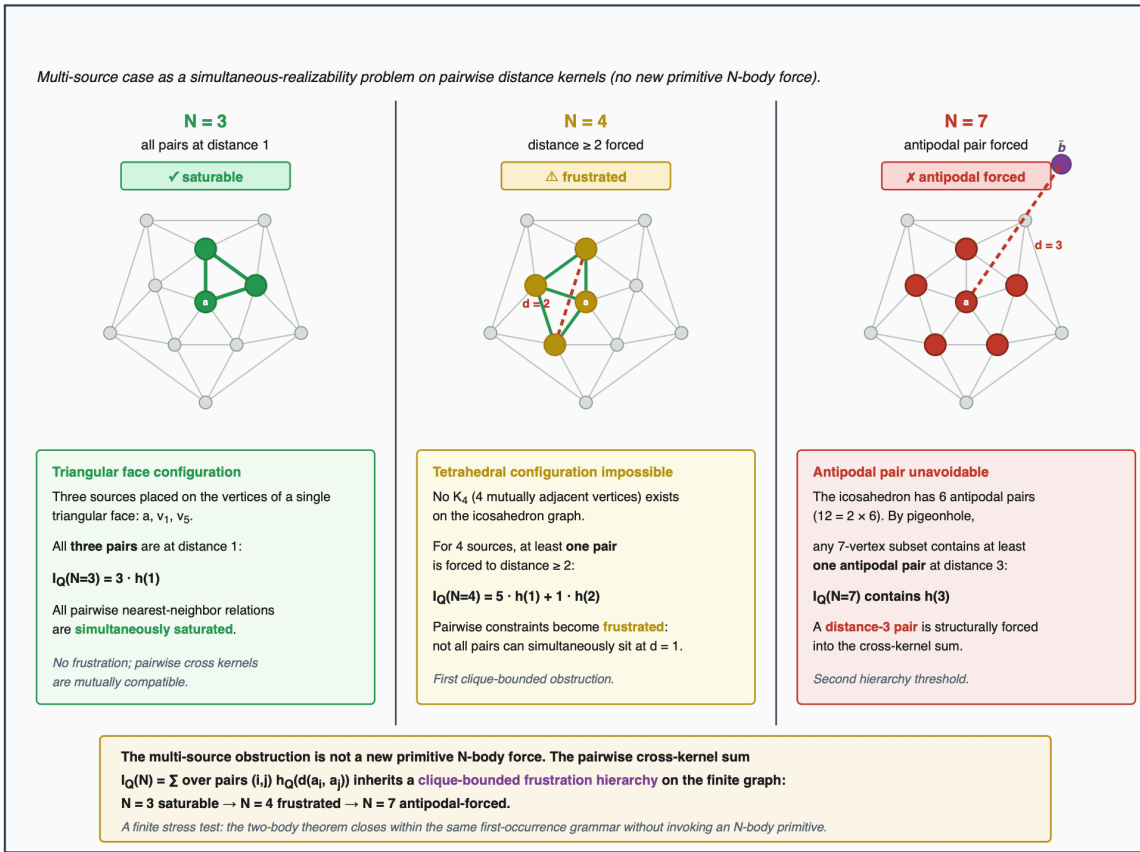


Figure 5. N -body frustration hierarchy on the icosahedron seed. The multi-source case is treated as a simultaneous-realizability problem on pairwise distance kernels; no new primitive N -body force is invoked. $N = 3$ (left, green): Placing three sources on the vertices of a single triangular face puts all three pairs at distance 1, so $I_Q^{(3)} = 3 \cdot h(1)$ and all pairwise nearest-neighbor relations are simultaneously saturated. $N = 4$ (center, yellow): The icosahedron contains no K_4 clique; hence, among any four sources, at least one pair is forced to distance ≥ 2 . The pairwise reading becomes $I_Q^{(4)} = 5 \cdot h(1) + 1 \cdot h(2)$, and the pairwise constraints become frustrated (first clique-bounded obstruction). $N = 7$ (right, red): The icosahedron has only six antipodal pairs ($12 = 2 \times 6$), so by the pigeonhole principle, any seven-vertex subset contains an antipodal pair, and $I_Q^{(7)}$ structurally inherits the term $h(3)$ (second hierarchy threshold). The multi-source obstruction therefore appears not as a new N -body primitive but as a clique-bounded frustration hierarchy on the finite graph, within which the two-body theorem closes inside the same first-occurrence grammar (§7.6).

This verification example does not amount to non-integrability or chaos in the classical N -body problem. What it demonstrates is more fundamental: given multiple sources on the same finite medium, even when the cross-distance kernels of individual pairs are simple, their simultaneous saturation is constrained by the clique/antipodal structure of the graph. The first finite-medium precursor of many-body hardness appears not as an additional primitive N -body force but as the impossibility of simultaneously realizing the pairwise carriers.

As a simulation prediction, the nearest-neighbor frustration index

$$F_1(S) = \binom{|S|}{2} - \#\{\{x,y\} \subset S: d(x,y) = 1\}$$

attains its first positive minimum at $N = 4$, while the antipodal frustration index

$$F_3(S) = \#\{\{x,y\} \subset S: d(x,y) = 3\}$$

attains its first positive minimum at $N = 7$. Whether a particular discrete relaxation simulation realizes these thresholds as dynamic frustration onsets remains a model- M / bridge problem depending on the update rule.

8. What is not derived in this paper: Newton scaling as a downstream-bridge problem

What is obtained in §7 is an attractive ordering with respect to source separation. This is not Newton's law of gravity itself.

To proceed to Newton's law

$$F \propto \frac{1}{r^2},$$

at least the following additional bridges are required:

- a bridge mapping the shell index n to a coarse-graining radius r ;
- the finite shell-weight reading $W_n = (n + 1)^2$;
- the shell-flux ansatz $W_{j_n} = C$;
- a force readout from energy ordering;
- a coupling-coefficient calibration to $G_{Newton}Mm$.

This paper does not solve these. Its claim is therefore not the derivation of the $1/r^2$ law but the identification of the attractive-direction carrier that precedes it.

8.1. Literal shell counts do not produce the inverse-square law

The literal shell counts of the regular icosahedron seed are

$$(N_0, N_1, N_2, N_3) = (1, 5, 5, 1).$$

This finite shell structure provides a non-trivial distance-kernel localization and Green ordering, but it does not directly provide the Newtonian continuum shell-area scaling. Reading the $1/r^2$ law directly off the literal seed shell counts is therefore not a claim of this paper.

8.2. Downstream bridge candidates

To connect the attractive-direction carrier to a Newtonian readout, the following downstream bridges are required at a minimum.

B_1 (shell-area parametrization):

$$W_n = (n + 1)^2.$$

B_4 (shell-uniform flux ansatz):

$$W_{j_n} = C.$$

B_2 (radius calibration):

$$r_n = a_0(n + 1).$$

B_3 (force readout):

$$F_n = \kappa j_n.$$

B_5 (coupling calibration):

$$\kappa C a_0^2 \mapsto G_{Newton}Mm.$$

These are not part of the main claim of this paper. They are registered here as the bridge candidates required by the downstream Newton-core extension.

8.3. Separating this paper from the main claim

This paper demonstrates the following self-contained chain:

$$\text{finite substrate} \rightarrow \text{quadratic energy} \rightarrow \text{distance-kernel localization} \rightarrow \text{source-coupled Green attractive sign}.$$

This paper does not demonstrate:

$$\text{cross-distance kernel} \rightarrow 1/r^2 \rightarrow G_{Newton} \rightarrow x = F \rightarrow \text{continuum / GR limit}.$$

The latter is delegated to the downstream Newton-core extension or to Open Bridges.

9. Open Bridge G_5 : connection to a continuum quadratic cross term

The cross-gradient mechanism established in §5–§8 is a structure on a finite A_5 medium. In this section, we do not immediately identify this finite structure with a specific continuum physical model; instead, we reformulate Open Bridge G_5 as a more general *continuum quadratic-form bridge*.

In an earlier version, the proposal that the finite cross term

$$\sum_{x \in V} u_x(x)u_x(x)$$

should be matched to the Eshelby-type inclusion-overlap energy of a continuous elastic medium [36][37],

$$\int_{\Omega} \varepsilon_a : C : \varepsilon_b dV,$$

was the central candidate of the main text. However, this proposal involves an additional lift from scalar-field overlap to strain-tensor overlap, which is too heavy for the main claim of this paper. To concentrate the present argument on the cross-gradient mechanism on finite graphs, the elastic/Eshelby correspondence is removed from the main bridge in the text and is left as a future bridge candidate.

G_5 in this paper is therefore redefined as

G_5 : the question of which kind of continuum quadratic form a finite quadratic cross term appears as in the continuum limit.

This is more than a mere graph-embedding problem. The question is to determine to which continuum quadratic form, and to which cross term within that form, a finite quadratic form maps.

9.1. G_5 is not merely an embedding problem

The correspondence between a finite graph and a continuum space is sometimes treated as the embedding problem

$$\Gamma_0 \hookrightarrow \mathbb{R}^3,$$

which is close to a candidate for the downstream kinematical bridge $B_{\gamma a}$. What is needed for the cross-term thesis of this paper, however, is not merely to place vertices in continuum space.

What is needed is the correspondence by which the finite quadratic cross term maps to a continuum quadratic form. The minimal candidate is the scalar bridge corresponding to the mass-quadratic reading,

$$G_5^M : \sum_{x \in V} u_a(x) u_b(x) \longrightarrow \int_{\Omega} \phi_a(\mathbf{r}) \phi_b(\mathbf{r}) d\mu(\mathbf{r}),$$

where ϕ_a is the continuum scalar-field readout of u_a and $d\mu$ is the measure appearing in the continuum limit. This bridge is closest to the mass quadratic form

$$E_M(u) = \sum_x u(x)^2$$

used in the main text.

The second option is the Dirichlet bridge corresponding to the Laplacian-quadratic reading,

$$G_5^L : (u_a, Lu_b) \longrightarrow \int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV,$$

where A is the coefficient matrix on the continuum side, or the effective-medium tensor. This bridge is consistent with the Green-potential reading of §7, and in it $Cross_L(u_a, u_b) = G_{ab}^L$ corresponds to the Laplacian-quadratic reading.

A third option is the elastic/Eshelby bridge. This requires lifting the scalar field u_a to a displacement field \mathbf{u}_a and then forming a strain tensor

$$\varepsilon_a = \text{sym} \nabla \mathbf{u}_a,$$

which requires an additional lift. It is therefore not adopted in the main text and is treated as a future candidate (deferred).

9.2. Three continuum bridge candidates

The three continuum bridges to be examined in this paper are summarized in Table 16.

Bridge candidate	Finite side	Continuum side	Status
Scalar mass bridge G_5^M	$\sum_x u_a u_b$	$\int_{\Omega} \phi_a \phi_b d\mu$	Closest to the main text of this paper
Dirichlet/Laplacian bridge G_5^L	(u_a, Lu_b)	$\int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV$	Consistent with the Green ordering of §7
Elastic/Eshelby bridge G_5^{el}	Finite elastic analogue (TBD)	$\int_{\Omega} \varepsilon_a : C : \varepsilon_b dV$	Future candidate (deferred)

Table 16. Three continuum bridge candidates for G_5 .

The point of the table is not to treat the three as identical. They are all quadratic forms with cross terms, but they are not the same physical model. In particular, the natural continuum limit of the mass quadratic is

$$\int_{\Omega} \phi_a \phi_b d\mu,$$

not directly

$$\int_{\Omega} \varepsilon_a : C : \varepsilon_b dV.$$

To reach the latter, an additional lift from a scalar field to a displacement/strain field is required; this lift is not established in this paper.

9.3. Quartic decomposition of G_5 : general quadratic-form bridge

We decompose G_5 into four parts, treating it not as a specific Eshelby correspondence but as a general continuum quadratic-form bridge:

$$G_5 = G_{5a} + G_{5b} + G_{5c} + G_{5d}.$$

9.3.1. G_{5a} : shell geometry/measure convergence

The finite shell

$$S_n(a) = \{x \in V : d(x, a) = n\}$$

must be matched to a shell, or a measure layer, on the continuum side:

$$G_{5a} : S_n(a) \rightarrow S_r(\mathbf{a}), \text{ or } \sum_{x \in V} \rightarrow \int_{\Omega} d\mu.$$

This is an unresolved bridge. As confirmed in §8, the literal shell counts

$$(1, 5, 5, 1)$$

do not match the continuous spherical-area scaling

$$(1, 4, 9, 16);$$

a coarse-graining bridge is therefore required for measure convergence.

9.3.2. G_{5b} : single-source field convergence

The finite scalar field

$$u_a(x) = G(d(x, a))$$

must be matched to its continuum counterpart. The minimal candidate is the scalar-field readout

$$G_{5b}^M : u_a \rightarrow \phi_a.$$

In this case, the mass-quadratic bridge

$$\sum_x u_a u_b \rightarrow \int_{\Omega} \phi_a \phi_b d\mu$$

is closest to the main text.

In the Laplacian reading, u_a is treated as a scalar potential, and its gradient energy is read on the continuum side:

$$G_{5b}^L : \langle u_a, Lu_b \rangle \rightarrow \int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV.$$

The elastic/Eshelby reading requires the further lift

$$u_a \rightarrow \mathbf{u}_a \rightarrow \varepsilon_a = \text{sym} \nabla \mathbf{u}_a.$$

This issue is not resolved in the present paper and is deferred to future work.

9.3.3. G_{5c} : cross-energy convergence

The central question of this paper is whether a finite quadratic cross term can be matched to a continuum quadratic cross term.

In the mass-quadratic reading,

$$G_{5c}^M : \sum_{x \in V} u_a(x) u_b(x) \rightarrow \int_{\Omega} \phi_a(\mathbf{r}) \phi_b(\mathbf{r}) d\mu(\mathbf{r}).$$

In the Laplacian-quadratic reading,

$$G_{5c}^L: (u_a, Lu_b) \rightarrow \int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV.$$

We register these only as continuum bridge targets. We do not establish which is the physically correct readout, nor do we determine the limit conditions under which they relate to one another.

9.3.4. G_{5d} : effective-gradient convergence

On the finite side, the effective gradient is given as a discrete difference with respect to source separation,

$$F_{ab}^{eff} \sim -\Delta_{d,ab} U_{in}(d_{ab}).$$

On the continuum side, the corresponding quantity is the gradient in configuration space.

In the mass-quadratic bridge,

$$-\Delta_{d,ab} \sum_x u_a u_b \rightarrow -\nabla_a \int_{\Omega} \phi_a \phi_b d\mu.$$

In the Laplacian bridge,

$$-\Delta_{d,ab} (u_a, Lu_b) \rightarrow -\nabla_a \int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV.$$

The transition from this discrete difference to a continuum gradient requires specifying the lattice spacing, measure normalization, and field normalization. G_{5d} therefore remains an Open Bridge.

9.4. What can and cannot be claimed in this paper

The main claims that can be made are

A finite-graph cross term has bridge targets that can be connected to quadratic cross terms on the continuum side.

Or more cautiously,

The main claim of this paper is closed on the finite side; the continuum correspondence is Open Bridge G_5 .

What should not be claimed at this stage is

A derivation of continuum elasticity, Eshelby mechanics, or the GR limit from the finite A_5 graph.

Furthermore, this paper does not advocate the identification

$$\sum_x u_a u_b \equiv \int_{\Omega} \varepsilon_a : C : \varepsilon_b dV.$$

This would equate scalar overlap with strain-tensor overlap and cannot hold without the additional lift mentioned above. This correspondence is therefore removed from the main argument and left as a future candidate.

9.5. Final formulation of G_5

In this paper, G_5 is finally formulated as follows.

G_5 : the Open Bridge asking, for a quadratic cross term on a finite medium, $Cross(u_a, u_b)$, which continuum quadratic cross term it is read as in the continuum limit.

The minimal candidate is the scalar mass bridge,

$$\sum_x u_a u_b \rightarrow \int_{\Omega} \phi_a \phi_b d\mu.$$

The alternative candidate is the Dirichlet / Laplacian bridge,

$$(u_a, Lu_b) \rightarrow \int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV.$$

Support for the elastic / Eshelby type is handled separately by an additional lift,

$$u_a \mapsto \mathbf{u}_a \mapsto \varepsilon_a.$$

which is retained as a future candidate.

This reformulation clarifies the scope of the present argument: the main claim is the cross-gradient mechanism on a finite graph; the connection to the continuum is kept open as a quadratic-form bridge without presupposing any specific physical model.

10. Lean / claim-dependency map

In this section, the arguments of §1–§9 are organized as a Lean 4 [\[38\]\[39\]](#) / Mathlib [\[40\]](#) claim-dependency map. The methodological focus is to make explicit, in a *mechanically traceable* form, which epistemic layer ($M / B / P / E$) each claim relies on.

10.1. M -layer-formalizable propositions

The following are the M -layer propositions organized at the level of finite graphs, finite linear algebra, and discrete difference operators. The earlier “M19 / M20 / M12” are not dispersed here; the central M -kernel is integrated as the *distance-kernel localization theorem*. The previously listed item “M23” has been reclassified as a scope declaration of the first-occurrence grammar (§6.4, §2.2) and is no longer registered as an M -layer theorem; the corresponding Lean object `MinimalDirectionSign.lean` is retained as an infrastructure note (see “Lean infrastructure notes” below).

ID	Proposition	Section	Role
M1	Structural data of the regular icosahedron seed graph Γ_0 : $ V = 12, E = 30, F = 20$	§3.1	Seed data
M2	Shell counts from base vertex a : $(N_0, \dots, N_3) = (1, 5, 5, 1)$	§3.1	Seed data
M3	Zero-mode removal: existence and uniqueness of the Green profile G	§3.2	Green anchor
M4	Green values per shell: $G_0 = \frac{7}{36}, G_1 = \frac{1}{90}, G_2 = -\frac{7}{180}, G_3 = -\frac{1}{18}$	§3.2	Green anchor
M5	Strict Green ordering $G_1 > G_2 > G_3$	§3.2, §7.1	Attractive-sign anchor
M6	Single-source shell uniformity: $d(x, a) = d(y, a) \Rightarrow u_a(x) = u_a(y)$	§3.3	No primitive direction
M7	u_a is invariant under $Stab(\{a\})$	§3.3	No primitive direction
M8	Polarization bookkeeping: $E_Q(u+v) = E_Q(u) + E_Q(v) + (u, Qv)$	§4.1, §4.2	Preliminary bookkeeping
M9	Linear-response grammar $u_{a+b} = u_a + u_b$	§2.2, §4.2	Model grammar
M20*	Distance-kernel localization theorem. Vertex transitivity + distance transitivity + Q -equivariance imply $B_Q(a, b) = (u_a, Qu_b) = h_Q(d(a, b))$	§1.2.1, §4.4	Central M theorem
M20a	Self-energy independence: $E_Q(u_a) = \frac{1}{2}h_Q(0)$, hence source-position-independent	§4.5	Corollary of M20*
M12	Cross-distance kernel: in linear-response, quadratic grammar, $I_Q(a, b) = h_Q(d(a, b))$	§4.6	Corollary of M20*
M10	Definition of overlap density $\Phi_{ab}(x) = u_a(x)u_b(x)$ and its determination by the graph-distance pair	§5.1	Axial diagnostic
M11	Invariant-group reduction of Φ_{ab} : $Stab(a, b)$ for labeled pairs, $Stab(\{a, b\})$ for homogeneous sources	§5.2	Axial diagnostic
M13	Mass-overlap diagnostic ordering $h_f(1) > h_f(2) > h_f(3)$	§7.4	Auxiliary diagnostic
M14	Definition of the finite-difference gradient $\Delta_a^h Q$ and its localization to the a - b axis	§6.2	Effective gradient
M15	Mismatch between literal shell counts and $(n+1)^2$: $(1, 5, 5, 1) \neq (1, 4, 9, 16)$	§8.1	Inverse-square firewall
M17	Source-coupled on-shell sign: $L_{int}^{sm} = -(\rho_a, G\rho_b)$	§4.7, §7.1	Model- M sign reading
M18	Static channel trichotomy: attractive / repulsive / neutral classification under the static, symmetric, linear-quadratic framework	§6.3	Model- M classification
M21	Cross-profile non-vanishing on the A_5 icosahedron seed: $h_f(d) \neq 0$ for $d \in \{0, 1, 2, 3\}$	§7.4	Diagnostic support
M22	Resistive-channel exclusion: time-reversal-symmetric, static, symmetric, linear-quadratic frameworks structurally exclude time-reversal-antisymmetric resistive channels	§6.3	Grammar scope
M24	N -body finite frustration thresholds: $v_1 = 3, v_2 = 6, v_3 = 12$; three sources can saturate all-nearest pairs, four sources force a non-nearest pair, and seven sources force an antipodal pair	§7.6	Verified finite enumeration / N -body audit

Table 17. M -layer propositions registered in this paper.

The current implementation status is shown in Table 18.

Proposition group	Status
M1–M5	Registered as finite seed calculations / Lean audits in §3 and §7; Lean implementations are part of the companion infrastructure.
M8, M16	The minimal algebraic skeleton can be implemented directly within the companion Lean infrastructure of this program.
M6–M7, M9–M12, M14–M15, M19–M20	Lean implementation candidates registered in this paper.
M13, M21	Confirmed as finite calculations; Lean implementation is part of the broader program roadmap.
M17	Model- M theorem under the source-coupled variational framework.
M18, M22	Structural classifications (model- M) under the static, symmetric, linear-quadratic framework.
M24	<code>TcosaNBodyFrustrationAudit</code> . Lean build OK: verified finite enumeration of the v_i hierarchy and threshold statements for $N = 4, 7$.

Table 18. Implementation status of the M -layer propositions.

Lean infrastructure notes.

In addition to the propositions in Table 18, the companion Lean infrastructure registers the scope declaration of the first-occurrence grammar (§2.2, §6.4) as the inductive type `Carrier` and the equivalence `cross_unique_minimal_endogenous_direction_sign` in `MinimalDirectionSign.lean`. This is not registered here as an M -layer theorem; its size and content match those of the corresponding scope statement, and its role is to make any future expansion of the model class (i.e., admission of a non-cross carrier) explicit and mechanically traceable.

This paper therefore does not claim that “M1–M24 are all Lean-verified in the same sense”. It claims to separate the dependencies among verified M , direct M , candidate M , and model- M propositions, and to indicate clearly which parts have been audited as finite enumerations and which remain to be Leanified.

Strategic significance of M20*.

M20* is the clean core of this paper. M20* is not a method of choosing a carrier grammar; it is a distance-kernel localization theorem derived from vertex-transitivity, distance-transitivity, and Q -equivariance.

The source-pair bilinear quantity $B_Q(a, b)$ falls to $h_Q(d(a, b))$. Hence the self-energy is constant, and the source-placement dependence is localized to the cross-distance kernel.

This proposition shows that the mechanism of this paper is not merely “a grammar in which the author has chosen to leave only the cross operator”; it is anchored, first of all, to the symmetry fact that finite symmetry compresses the source-pair dependence into a distance kernel.

Strategic significance of M22.

M22 completes the elimination argument of §6.3. It is registered not as a narrow “frictional” category but as a single principle (the structural absence of time-reversal-symmetry breaking) that structurally excludes “resistive” channels (a general term for time-reversal-antisymmetric dissipative forces). As a result, the trichotomy of this paper (attractive / repulsive / neutral) becomes a closed classification with respect to microscopic conservative separation channels within the static, symmetric, linear-quadratic framework.

Companion Lean skeleton.

A minimal Lean skeleton, used in the companion Lean infrastructure of this program, reads:

```

def Energy (u : V -> Q) : Q :=
  Sum x, u x * u x

def Cross (u v : V -> Q) : Q :=
  Sum x, u x * v x

theorem energy_add_eq_energy_add_energy_add_two_cross
  (u v : V -> Q) :
  Energy (fun x => u x + v x)
  = Energy u + Energy v + 2 * Cross u v := by
  ring_nf

```

This implements the minimal core of M8. The Lean audit of this paper is organized as the process of sequentially formalizing M1–M18 starting from this skeleton.

10.2. Bridges and interpretation contracts to be registered as P / Open layers

This paper decomposes what was previously B_{cross} into a polarization bookkeeping lemma, a distance–kernel localization theorem, and a scope declaration of the first-occurrence grammar. The algebraic appearance of the cross term is standard M -layer bookkeeping; the part that is read as a physical gravity channel remains in P / model- M layers. The bridges and interpretation contracts registered here are therefore the downstream bridges and the physical-interpretation principles.

B / Open-layer bridges.

ID	Content	Section	Source
B_1	Shell-area parametrization $W_n = (n + 1)^2$	§8.2	Downstream bridge candidate
B_4	Shell-uniform flux ansatz $W_{d/n} = C$	§8.2	Downstream bridge candidate
B_2	Radius calibration $r_n = a_0(n + 1)$	§8.2	Downstream bridge candidate
G_5	Continuum quadratic-form bridge	§9	Open Bridge registered here

Table 19. B / Open-layer bridges registered in this paper.

P -layer principles.

ID	Content	Section	Source
P_{med}	Medium-as-active principle	§2.3, §3, §6	Adopted in this paper
P_{grav} / $P_{energy \rightarrow force}$	Read the energy-lowering direction as the physical gravity channel	§2.3, §6.5, §7.3	Adopted and restricted in this paper
$P_{force-channel}$	Force channel as the macroscopic readout of an energy-lowering update tendency, not a primitive substrate vector	§2.3, §6.5	Adopted and restricted in this paper
P_{read}	Finite readout / admissible extraction commitment	§2.3	Adopted in this paper
P_{axial}	Read ϕ_{ab} as a bicenter axial localization diagnostic	§2.3, §5, §6	Newly registered in this paper

Table 20. P -layer principles registered in this paper.

As summarized in §2.6, these bridges and principles are treated separately from the M -layer theorems on the Lean side, either as explicit parameter / structure fields or as unverified interpretive contracts.

10.3. The Lean significance of this paper

The Lean audit of this paper does not prove any physical uniqueness of the cross term, nor does it dress the standard polarization identity as a non-trivial theorem. The role of Lean is narrower: to mechanically separate the grammar, claim scope, and bridge / P -layer dependencies declared here, and to prevent the introduction of hidden assumptions.

The Lean audit has the following five characteristics.

(i) Layer separation as a claim firewall.

The polarization identity is treated as a bookkeeping lemma in the M layer. The parts that read the energy-lowering direction as a real gravity channel, that adopt the source-coupled Green reading, and that read the finite graph as an active medium remain in the P / model layers. This avoids the error of “disguising a physical claim as a mathematical theorem”.

(ii) Scope-statement transparency.

`MinimalDirectionSign.lean` does not prove physical uniqueness. It encodes the scope declaration of the first-occurrence grammar (§2.2, §6.4) as the inductive type `Carrier`, making explicit which carriers (self / external / higher-order / dynamic / nonlinear-response) are placed outside the model class of this paper, leaving the cross-distance kernel as the admitted carrier. The role is a scope-bookkeeping / hidden-assumption detector: any future expansion of the model class must update `Carrier` explicitly rather than silently.

(iii) Transparency of the dependency map.

All derivations from the central proposition of §1.2 to the skeleton of §6.5 can be traced on the Lean side as a dependency graph among M -layer propositions, model- M propositions, and P / Open-layer elements (P_{grav} , P_{axial} , G_5 , B_1/B_4 , etc.). The earlier B_{cross} is not simply promoted to “ M_{cross} ”; it is decomposed into a polarization bookkeeping lemma and a scope declaration.

(iv) Confirmation of non-claim of substrate uniqueness.

The A_5 compatibility-anchor claim of §7.4 does not assert substrate uniqueness of the A_5 icosahedron seed. The Lean audit explicitly indicates, in the hypothesis structure, which parts of the main claims are specific to the A_5 seed and which are general structures holding for any vertex-transitive graph or symmetric quadratic medium.

(v) Honest disclosure of Open Bridge G_5 .

The continuum quadratic-form bridge developed in §9 is an Open Bridge not completed here. In the Lean audit, G_5 is not registered as a theorem; it is shown explicitly as a hypothesis / structure field.

Summary of the Lean significance.

This paper does not use Lean to prove physical gravity. What this paper does with Lean is to indicate clearly, in a mechanically traceable manner, where mathematical bookkeeping ends, where the central distance-kernel localization theorem sits, where the scope declaration of the first-occurrence grammar is registered, where the P -layer interpretation begins, and where the Open Bridges begin.

11. Non-claims: what this paper does not assert

This section explicitly lists the points that the arguments of this paper do *not* establish. In a paper structure centered on the reformulation thesis, this is a *methodological center*: an honest disclosure of the limited scope of this paper.

11.1. Ten items not established by this paper

#	Claim not made by this paper	Treatment
1	Derivation of Newton's constant G	Open Bridge G_1
2	Derivation of mass normalization	Open Bridge G_2
3	Derivation of the equivalence principle	Open Bridge G_6
4	Derivation of the inverse-square law from the literal A_5 shell counts	$B_1 + B_4$ coarse-graining bridge candidates (§8)
5	Complete derivation of continuum space	Open Bridge G_5 (internal decomposition incomplete; see §9)
6	Derivation of the Eshelby solution for a continuous elastic body from the finite A_5 graph	Elastic / Eshelby support is not advocated; future candidate (deferred)
7	General-relativistic limit	Open Bridge G_7
8	Full integration with quantum-side address readout	Open Bridge G_8 (a separate-paper item; not decomposed here)
9	A completed quantum-gravity theory	The reformulation thesis as a whole; awaits cumulative resolution of future Open Bridges
10	Realization of an antigravity channel	Explicitly non-implemented; see §12.3

Table 21. Ten items not established by this paper.

11.1.1. Note: G_8 is excluded from this paper

This paper does not establish a unification between quantum-side address readout and gravity-side relaxation / cross-gradient readout. A complete unification remains as Open Bridge G_8 .

While the previous draft elaborated on a partial bridging of G_8 , the technical content of this paper is the demonstration of an attractive sign for the cross-distance-kernel carrier on a finite quadratic medium and the A_5 seed. A detailed decomposition of G_8 is therefore not undertaken here; it is postponed to a separate paper.

The only permitted statement here is the following scope note:

The integration of address readout and relaxation readout is outside the scope of this paper and remains as Open Bridge G_8 .

The connection to specific probability laws, the Born rule, quantum measurement, gravitational back-reaction, and empirical quantum-gravity phenomena is entirely outside the scope of this paper.

11.2. What this paper does and does not assert about the cross term

This paper introduces no new algebraic identity. It defines the interaction excess in a finite linear-response quadratic medium and uses, as a bookkeeping lemma, the fact that this excess equals the cross term by the polarization identity.

Substantively, this paper demonstrates that, within a finite, static, symmetric, linear-response, quadratic medium grammar, the encounter of two omnidirectionally compressed clouds generates a source-pair-dependent cross carrier on which the minimal endogenous carrier of the two-body direction sign is localized; and that, on the A_5 icosahedron seed under the source-coupled Green reading, this carrier selects an attractive sign as a minimal finite demonstration.

The points open to criticism are therefore not the algebraic existence of the cross term itself. They are localized to (i) the physical adoption of the linear-response quadratic grammar, (ii) the adoption of the source-coupled Green reading as gravity-channel energy, (iii) the P -layer reading of the finite quadratic medium as a real gravity substrate, and (iv) the downstream bridge to Newton / continuum / GR.

This paper does *not* assert any of the following:

- that this finite quadratic medium is empirically established as a real gravitational medium;
- that the attractive channel automatically inherits a Newtonian $1/r^2$ scaling;

- that frictional / dissipative channels emerge from a static cross-term mechanism;
- that the same cross-term identity is preserved under nonlinear elasticity, mutual screening, or wake / memory responses;
- that the continuum / GR / quantum-readout unification program is complete.

The core of this paper is therefore not the “discovery of a cross-term identity” but the *de-axiomatization of the two-body direction-sign carrier within the linear-response quadratic finite-medium grammar*.

11.3. What this paper does claim

More specifically, this paper claims the following.

When two omnidirectionally compressed clouds meet on a static, linear-response, secondary finite medium, the source-pair-dependent cross carrier is separated by polarization bookkeeping; its pointwise diagnostic has a two-center axial structure; within the static quadratic endogenous grammar, this carrier is the minimal endogenous carrier of the two-body direction sign; and on the $\sim A_5$ -seed, the source-coupled Green reading selects an attractive sign.

This is the *only* technical claim of this paper. The structure developed in §1–§10 is intended to support this claim; other claims (including the ten items of §11.1) lie outside its scope.

11.4. Relation to the reformulation thesis

In §1.2, the reformulation thesis was shared as the background commitment of this paper: “the quantum-gravity problem is misposed insofar as both spacetime geometry and quantum wave ontology are taken as primitive continua, and both should be reformulated as finite readouts from a common finite substrate”. This paper contributes to that thesis as follows.

- This paper does *not* establish the thesis as a whole.
- It provides a *first-stage technical instance on the gravity-side branch* of the thesis: the cross-gradient generation mechanism.
- The second stage—Newtonian $1/r^2$ scaling, second-order motion, quasi-static identification—is the target of the downstream Newton-core extension.
- The quantum-side branch (address readout) is treated in Paper III, and the full integration of both branches remains as Open Bridge G_8 . A detailed decomposition of G_8 is not provided here.

The contribution of this paper to the thesis is therefore *a portion of the supporting evidence*; the soundness of the thesis as a whole is to be judged only by the cumulative resolution of future Open Bridges.

Structural note: branch coupling moved to a separate paper.

The coupling of address readout and relaxation readout is an important issue for the broader A_5 program. However, the focus of this paper is the attractive sign of the cross-distance-kernel carrier and the source-coupled Green reading on a finite quadratic medium. The structural details of two-branch coupling are therefore omitted from the claims of this paper and moved to a separate paper as Open Bridge G_8 .

11.5. §11 as honest disclosure

§11 is the methodological center of this paper. To make the claim firewall function while remaining anchored to the reformulation thesis, what is required is, equally—if not more—than “what one claims”, a precise specification of “*what one does not claim*”.

This paper is not a paper that completes the theory of gravity; it is a paper that isolates the mechanism by which the attractive channel is generated.

Each of the ten items of §11.1 has independent research value and may be addressed in the future through downstream extensions of this program or through resolution of Open Bridges. The role of this paper is to clarify its own scope so as not to be confused with such unresolved issues.

12. Discussion: why this is not “preservation of symmetry”

The terminology used in this paper differs in several respects from conventional physical language. This section clarifies the intent behind these choices. The central point is that the conventional terms “preservation of symmetry” and “symmetry breaking” do not accurately describe the phenomenon discussed here, and that an alternative terminology is essential for the structural argument of this paper.

12.1. The problem with “preservation of symmetry”

The statement “the medium preserves its symmetry” leaves the following ambiguities:

- Which symmetry?
- At which scale of symmetry?
- What breaks the symmetry?
- Where does the “preferred direction” arise?
- Why does a force in a single direction emerge from a field that is uniform in all directions?

In the terminology of this paper, these questions are sharpened by way of §5.2 (invariant-group reduction) as in Table 22.

Question	Answer in this paper	Source
Which symmetry?	One-center graph-distance shell-uniformity (each individual field u_a, u_b enjoys its own)	§3.3
Where does it break?	In the overlap product $\Phi_{ab}(x) = u_a(x)u_b(x)$	§5.1
To what extent is it reduced?	Two-center axial structure (invariant-group reduction to $Stab(\{a, b\})$)	§5.2
Where does the direction come from?	From the overlap density along the $a-b$ axis, via the difference $\Delta_{ab} C$ in source-configuration space	§6.2
What is force?	The effective-gradient readout of the separation-dependent cross term	§6.3, §7.2

Table 22. Sharpening conventional “symmetry” questions in the language of this paper.

This refinement reveals that what is referred to by the term “symmetry breaking” is, in fact, an *invariant-group-theoretically well-defined structural reduction*.

Where the “rupture” is localized.

It is particularly important where the “rupture” is localized. By the localization-of-the-breaking proposition (Proposition 8, §5.3):

- u_a alone: fully shell-uniform, $Stab(\{a\})$ -invariant;
- u_b alone: fully shell-uniform, $Stab(\{b\})$ -invariant;
- $\Phi_{ab} = u_a u_b$: invariant only up to $Stab(\{a, b\})$.

That is, the “rupture” does not occur in the context of a single source. It is a derived phenomenon arising only in the context of a *secondary algebraic operation* on two fields.

12.2. Where gravity is located

In the structure presented in this paper, the attractive force is not placed primitively between objects. The thesis of this section is the following:

Gravity is localized to the cross-distance kernel that arises when two source fields overlap on the same medium.

This differs in descriptive level from Newton’s “force between two objects”. Newton’s law is an effective law after readout; this paper describes the finite-medium cross-gradient mechanism that precedes it.

Hierarchical organization.

Layer	Description	Source
Substrate layer	Deformation field u_a on a finite graph Γ_0	§3
Quadratic energy layer	$E(u_a + u_b) = E(u_a) + E(u_b) + 2Cross$	§4
Overlap density layer	Axial structure of $\Phi_{ab}(x)$	§5
Effective gradient layer	$-\Delta_{d_{ab}} U_{int}$	§6
<i>(End of the scope of this paper.)</i>		
Newton-scaling layer	$F \approx 1/r^2$	Downstream bridge problem
Continuum layer	Continuum quadratic-form readouts (scalar / Dirichlet / elastic)	Open Bridge G_5
GR layer	Curved-spacetime gravity	Open Bridge G_7

Table 23. Hierarchy of layers and the scope of this paper.

This paper deals only with the first four layers (substrate, quadratic energy, overlap density, effective gradient) and defers Newton scaling and beyond to downstream extensions or Open Bridges. The location of the gravitational force is therefore the *effective-gradient layer*: the hierarchical position of this paper is that gravity is already defined at a stage prior to Newton's law.

12.3. On antigravity

In the present configuration, there is little room to place antigravity primitively, because:

- the field u_a created by a single source is shell-uniform on its own (§3.3);
- when two such fields overlap, a cross term arises algebraically (§4);
- on the A_5 icosahedron seed, the cross-profile ordering $C(1) > C(2) > C(3)$ holds, and an attractive readout is obtained under the attractive-sign convention (§7).

To create a repulsive channel, at least one of the following is required:

- a sign reversal of the cross term (a substrate selection in which the cross-profile ordering is reversed, e.g., $C(1) < C(2) < C(3)$);
- a change in the energy-to-force readout convention (e.g., the anti-convention $F^{eff} \propto +\nabla U_{int}$);
- a different sign-structural branch of the source field (e.g., source inversion $u_a \rightarrow -u_a$);
- a different stability criterion for the medium response (e.g., driving in the energy-increasing direction).

None of these is introduced here. The default channel of this paper is therefore *attractive readout only*.

Note: this is not a claim of antigravity impossibility.

It should be emphasized that this paper does not assert the *impossibility* of antigravity; it asserts that, under the default convention, an attractive sign is generated. A repulsive channel can be constructed by any of the four alternative routes above; whether these are realized in the physical world is outside the scope of this paper.

This is consistent with non-claim #10 (realization of an antigravity channel) of §11.1: this paper neither excludes nor implements antigravity. It merely confirms that the default channel is attractive.

12.4. What can and cannot be said about the three-body problem

The N -body verification example of §7.6 does not prove the non-integrability of the classical three-body problem. It identifies, instead, the obstruction on the finite-medium side that precedes it. In the two-body case, the source-pair dependence reduces to a single cross-distance kernel $h_{\mathcal{O}(d_{ab})}$. In the three-body case, the number of pairwise kernels increases to three, but the obstruction remains shallow because all pair nearest-neighbor requirements can be simultaneously

13.2. Central conclusion

This paper does not derive the $1/r^2$ law, Newton's constant, the equivalence principle, the continuum limit, or the GR limit. The contribution of this paper is the *de-axiomatization of the attractive direction sign*: it shifts the source-separation dependence from a primitive force postulate to a cross-distance kernel within a symmetric finite quadratic medium.

In a vertex-transitive and distance-transitive finite quadratic medium, the source-pair dependence is localized as a cross-distance kernel; on the A_5 icosahedral seed, the source-coupled Green reaction is localized as a cross-distance kernel. This is the central conclusion of this paper.

13.3. Continuum limit: Open Bridge G_5

For the continuum limit, this paper formulates Open Bridge G_5 as a *continuum quadratic-form bridge*.

The minimal candidate is the scalar mass bridge,

$$\sum_{x \in V} u_a(x)u_b(x) \rightarrow \int_{\Omega} \phi_a(\mathbf{r})\phi_b(\mathbf{r})d\mu(\mathbf{r}).$$

The alternative candidate is the Dirichlet/Laplacian bridge,

$$(u_a, Lu_b) \rightarrow \int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV.$$

Support for the elastic/Eshelby type requires an additional lift $u_a \mapsto \mathbf{u}_a \mapsto \epsilon_{ab}$ and is therefore removed from the main text and left as a future candidate.

This paper does not, then, derive continuum elasticity or Einstein gravity. It addresses the task of passing an attractive cross-gradient channel on a finite quadratic medium to a future continuum/Newton/GR bridge.

13.4. What this paper hands downstream

The order of the paper series announced in §1.3 is finalized upon completion of this paper:

Cross-Term paper (this paper) \rightarrow Downstream Newton-core extension.

This paper addresses the "location of the gravitational direction sign"; the downstream extension connects this channel to a Newtonian readout stack.

What is not established here.

The following are explicitly outside the scope of this paper and are deferred to the downstream extension or to Open Bridges:

- $1/r^2$ scaling and Newton's constant G ;
- mass normalization and the equivalence principle;
- second-order motion and the dynamical readout;
- the continuum limit and the GR limit;
- retardation/wake kernels;
- the quantum-side address readout and quantum-gravity unification.

Chain to a Newtonian readout.

At a minimum, the downstream extension requires the following chain:

cross-distance kernel \rightarrow coarse-grained shell area
 \rightarrow flux readout \rightarrow force magnitude
 \rightarrow second-order motion \rightarrow continuum / GR bridge.

None of these is part of the main claim of this paper; this paper establishes only the attractive-direction carrier that precedes them, in self-contained form. The remaining future work is summarized in Table 24.

Future work	Content
G_5	Connection of the finite quadratic cross term to a continuum quadratic form
G_3	Wake/retardation/moving-source extension
B_1, B_4	Coarse-grained shell bridge for the inverse-square readout
B_{7a}, B_{7b}	Substrate embedding and quasi-static dynamical identification
G_7	GR limit
G_8	Integration of the quantum-side address readout with the gravity-side relaxation readout

Table 24. Remaining future work.

Minimal interface statement.

The minimal statement for handing the results of this paper to the downstream extension is

$$I_L(a, b) = h_L(d(a, b)) = G(d(a, b)), G_1 > G_2 > G_3,$$

together with the on-shell convention

$$U_{in}(d) = -\lambda h_L(d), \lambda > 0,$$

under which the proximate-source configuration is the lower-energy configuration. This is the self-contained deliverable of this paper; the downstream extension addresses the bridge work connecting this static attractive ordering to the $1/r^2$ scaling, second-order motion, the continuum limit, and the GR limit.

This paper is therefore not a final theory. It is a preliminary paper confirming, within a finite-readout program leading toward Newtonian gravity, GR, and quantum gravity, the conclusion of the *first carrier of the attractive direction sign*.

13.5. Final thesis

The final thesis of this paper is as follows.

Gravity is not inserted between bodies. The attractive direction sign is read off from the source-configuration-dependent cross carrier generated when two shell-uniform compressed clouds coexist.

Appendix A. List of Symbols

The main symbols used in this paper are listed below. The “From” column indicates the section in which the symbol first appears.

A.1. Substrate and graph structure

Symbol	Meaning	From
Γ_0	A_5 -seeded icosahedral graph (seed substrate)	§3.1
V	Vertex set of Γ_0 ($ V = 12$)	§3.1
E	Edge set of Γ_0 ($ E = 30$)	§3.1
F	Face set of Γ_0 ($ F = 20$)	§3.1
$a, b \in V$	Source vertices	§3.2
$d(x, a)$	Graph distance from x to a	§3.2
$d_{ab} = d(a, b)$	Graph distance between sources ($\in \{0, 1, 2, 3\}$)	§6.1
$S_n(a) = \{x: d(x, a) = n\}$	Distance shell of radius n around a	§9.3.1
$N_n = S_n(a) $	Shell counts: $(N_0, \dots, N_3) = (1, 5, 5, 1)$	§3.1
$Aut(\Gamma_0)$	Graph automorphism group of Γ_0	§3.3
$Stab\{a\}$	Subgroup of $Aut(\Gamma_0)$ that fixes the vertex a	§3.3
$Stab\{a, b\}$	Subgroup that fixes the source pair $\{a, b\}$ as a set	§5.2

Table A1. Symbols for the substrate and graph structure.

A.2. Field and Green's function

Symbol	Meaning	From
$L = D - A$	Graph Laplacian of Γ_0 (D : degree, A : adjacency)	§3.2
G_a	Zero-mode-removed Green profile sourced at a	§3.2
$G(d)$	Green value as a function of graph distance d	§3.2
G_0, G_1, G_2, G_3	Green values per distance layer: $\frac{7}{36}, \frac{1}{90}, -\frac{7}{180}, -\frac{1}{18}$	§3.2, §7.1
$u_a(x) = G(d(x, a))$	Single deformation field generated by source a	§3.2
$u_{a+b}(x) = u_a(x) + u_b(x)$	Composite field under linear-response grammar	§4.2

Table A2. Symbols for the field and Green's function.

A.3. Quadratic forms and cross term

Symbol	Meaning	From
$E(u) = \sum_x u(x)^2$	Mass quadratic energy (minimal model in this paper)	§4.1
$E_L(u) = \frac{1}{2}(u, Lu)$	Laplacian quadratic energy (alternative model; see §7.2 note)	§4.1, §7.2
$\langle u, v \rangle = \sum_x u(x)v(x)$	Standard inner product on \mathbb{R}^V	§4.1
$Cross(u, v) = \langle u, v \rangle$	Cross term of the mass quadratic	§4.1
$Cross_L(u_a, u_b) = G_{d_{ab}}$	Cross term of the Laplacian quadratic	§7.2 note
$\Phi_{ab}(x) = u_a(x)u_b(x)$	Pointwise overlap density	§5.1
$C(d_{ab}) = Cross(u_a, u_b)$	Total cross term (depends only on graph distance)	§6.1
$N(n_1, n_2 \mid d_{ab})$	Combinatorial count $\#\{x: d(x, a) = n_1, d(x, b) = n_2\}$	§7.1

Table A3. Symbols for quadratic forms and the cross term.

A.4. Energy/force readout

Symbol	Meaning	From
$U_{in}(a, b) = -\lambda C(d_{ab})$	Interaction energy readout ($\lambda > 0$)	§6.3, §7.2
λ	Dimensional coupling: only the sign is fixed in this paper; magnitude is deferred to Open Bridge G_1	§6.3
$\Delta_{d_{ab}} f = f(d_{ab} + 1) - f(d_{ab})$	Discrete difference in source separation	§6.2
$F^{eff} \sim \lambda \Delta_{d_{ab}} C(d_{ab})$	Effective force readout (attractive sign: §7.2)	§6.3, §7.2

Table A4. Symbols for the energy/force readout.

A.5. Downstream bridge candidates

Symbol	Meaning	From
W_n	Shell weight, bridge candidate (B_1 ; $W_n = (n + 1)^2$)	§8.2
j_n	Shell flux density, bridge candidate (B_1 ; $W_n j_n = C$)	§8.2
$r_n = a_0(n + 1)$	Radius calibration, bridge candidate (B_2)	§8.2
a_0	Lattice spacing for downstream radius calibration	§8.2

Table A5. Symbols for downstream bridge candidates.

A.6. Continuum quadratic-form bridge

Symbol	Meaning	From
$\Omega \subset \mathbb{R}^3$	Region of continuum elasticity	§9.1
$A, B \subset \Omega$	Elastic inclusions	§9.2
$\mathbf{r} = (x, y, z)$	Point in continuous space	§9.2
\mathbf{a}, \mathbf{b}	Inclusion-center position vectors	§9.2
$R = \mathbf{a} - \mathbf{b} $	Inter-inclusion Euclidean distance	§9.2
$\varepsilon_a(\mathbf{r})$	Strain field generated by inclusion A	§9.2
\mathbf{C}	Elasticity tensor (fourth-order tensor)	§9.1
$\varepsilon_a : \mathbf{C} : \varepsilon_b$	Strain double contraction by the elasticity tensor	§9.1
$\int_{\Omega} \phi_a \phi_b d\mu$	Continuum-side candidate of a scalar mass bridge	§9.1, §9.3.3
$\int_{\Omega} \nabla \phi_a \cdot A \nabla \phi_b dV$	Continuum-side candidate of a Dirichlet/Laplacian bridge	§9.1, §9.3.3
$\int_{\Omega} \varepsilon_a : \mathbf{C} : \varepsilon_b dV$	Elastic / Eshelby future candidate (not used in the main argument)	deferred

Table A6. Symbols for the continuum quadratic-form bridge.

A.7. Bridges and principles mobilized in this paper

Symbol	Kind	Meaning	From
M_{half}	M -layer bookkeeping lemma	In linear-response, quadratic grammar, the interaction excess decomposes via the cross term.	§2.2, §4.2
Scope decl. (first-occurrence grammar)	Model-class declaration (not a theorem)	Within the static, symmetric, endogenous, linear-response, quadratic grammar with no primitive direction or external witness, the admitted carrier of the source-pair dependence is the cross-distance kernel.	§2.2, §6.4
M_{static}	model- M theorem	Channels in a static, symmetric, quadratic medium are classified as attractive, repulsive, or neutral; no resistive channel is produced.	§6.3
P_{with}	P -layer principle	Medium-as-active reading.	§2.3
$P_{grav} / P_{energy \rightarrow force}$	P -layer principle	Read the energy-lowering direction as the physical gravity channel.	§2.3, §6.5
P_{read}	P -layer principle	Finite readout / admissible extraction commitment.	§2.3
P_{axial}	P -layer principle	Read Φ_{ab} as a bicenter axial-localization diagnostic.	§2.3 (newly introduced here)
P_{ζ}	P -layer selector	No-bias selector (operational).	Downstream dynamical extension candidate
$P_{\zeta}^{A_5}$	P -layer substrate Role B	A_5 -equivariant no-bias capacity.	Downstream no-bias extension candidate
G_5	Open Bridge	Continuum quadratic-form bridge.	§9
$G_{5a/b/c/d}$	G_5 sub-bridges	Shell geometry / single-source field / cross-energy / effective-gradient convergence.	§9.3
G_{NB}	Open Bridge	Dynamical-realization bridge for finite cross-term frustration thresholds (discrete relaxation, soft embedding, or continuum N -body approximation).	§7.6, §12.4

Table A7. Bridges and principles mobilized in this paper.

A.8. Open Bridges registered for downstream work

Symbol	Meaning	Applicable
G_1	Newton constant calibration	§11.1 #1
G_2	Mass normalization	§11.1 #2
G_6	Equivalence principle	§11.1 #3
G_7	GR limit	§11.1 #7
G_{NB}	N -body dynamical-realization bridge	§7.6, §12.4

Table A8. Open Bridges registered for downstream work.

A.9. Evacuation of G_8

This paper does not address the integration of the quantum-side address readout with the gravity-side relaxation/cross-gradient readout. Accordingly, G_8 is not decomposed in detail here but is simply registered as an Open Bridge.

Symbol	Status	Content
G_8	Open Bridge/special topic	Integration of address readout and relaxation readout.

Table A9. Evacuated Open Bridge G_8 .

Note. The detailed decomposition of G_8 from earlier work has been omitted in order to narrow the focus of this paper, which deals with cross-distance-kernel carriers in a finite, static, quadratic medium and with the attractive-sign demonstration on the A_3 seed.

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Potential Competing Interests

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Ethics

Not applicable.

Consent for Publication

Not applicable.

Data Availability

This paper contains no experimental data. All numerical values referenced in this work are exact rational results of finite computations on the A_3 icosahedron seed graph and are reproducible directly from the definitions given in the main text.

Materials Availability

Not applicable.

Code Availability

The complete Lean 4 source code for the formal verification (`sorry = 0`, `axiom = 0`) of the M -layer claims of §10, together with the theorem–identifier correspondence table provided in the supplementary material, is publicly available at <https://github.com/nuu/GravityCrossTerm> and permanently archived at Zenodo <https://doi.org/10.5281/zenodo.20049580> under the Apache License 2.0. The repository includes all files necessary to reproduce the verification: `lakefile.lean`, `Mathlib` dependency specifications, the root module `GravityCrossTerm.lean`, and the per-claim audit files (e.g., `IcosaLaplacianMatrix.lean`, `IcosaGreenFunctionTable.lean`, `DistanceKernelLocalization.lean`, `MinimalDirectionSign.lean`, `IcosaNBodyFrustrationAudit.lean`). Successful compilation of the root module via `lake build` constitutes end-to-end verification.

Author Contribution

M. Numagaki is the sole author and is responsible for the conception of the theoretical framework, all mathematical proofs and their Lean 4 formalization, the numerical analysis, and the writing of the manuscript.

Use of AI-Assisted Tools

During the preparation of this manuscript, AI-assisted tools were used for copyediting (LaTeX typesetting, English grammar and style refinement, and improvements to the clarity of mathematical exposition of human-generated text). All mathematical content, theorems, proofs, physical interpretations, and scientific conclusions are solely the author's own work. The author reviewed and edited all AI-assisted outputs and takes full responsibility for the content of the published article.

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