Review of: "Grönwall's Theorem implies the Riemann Hypothesis"

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In the article [M23] it is proposed a short proof of the Riemann hypothesis (RH) based on the works of Grönwall [G13] and Robin [R84].

In 1913, Grönwall proved that the function

$$G(n) = \sigma(n)/(n \log\log n)$$

satisfies the equation

$$\lim_{n\to\infty} \sup G(n) = e^{\gamma_E}$$

where $\sigma(n)$ is the sum of divisors function and γ_E is the Euler constant.

In 1984, Robin proved theorem 1 [R84]

$$RH \iff G(n) \leq e^{\gamma_E}, \quad \forall n > 5040$$

In his proof Robin used Proposition 1: If N and N' are successive colossally abundant numbers then

$$G(n) \leq \mathrm{Max}(G(N), G(N')), \qquad 3 \leq N \leq n \leq N'$$

This proposition appears in section 3 entitled "Behaviour of $\sigma(n)$ if the RH is true" (page 192 of [R84]). At the beginning of this section one reads that the RH will be assumed to hold throughout the paragraph that includes the previous equation.

The proof proposed in [M23] does not hold for two reasons.

The first one is that Grönwall's theorem is understood in [M23] as saying that for a very large colosally abundant number B, one should have $G(B) \leq e^{\gamma_E}$. This is not correct because that theorem implies the limit of a supremum so this B does not have to satisfy that $G(B) \leq e^{\gamma_E}$.

The second, and more important reason is that the proposition 1 in [R84], assumes the truth of the RH. Therefore, it cannot be used to prove what one just wants to prove.

Grönwall's theorem is based on Mertens' third theorem, which is closely related to the prime number theorem although it appeared earlier in time. On general grounds one should not expect Grönwall theorem combined with the Robin's theorem to provide a proof of the RH.

[G13] T. H. Grönwall, Some Asymptotic Expressions in the Theory of Numbers.Transactions of the Am. Math. Soc. 14(1), 113-122 (1913).

[R84] Guy Robin, Grandes valeurs de la fonction somme des diviseurs et hypoth\'ese de Riemann. J. Math. pures appl, 63(2): 187-213 (1984).