A Note on Relaxing the Black-Scholes Assumptions Without Changing the Price Formula

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Abstract

We provide explicit, simple price formulas for the European options under stochastic volatility and stochastic interest rate. The formulas are as simple as the classical Black-Scholes formula. Moreover, the formulas do not require the normality of the returns. We do not need to know the distribution of the returns/price. Furthermore, this approach enables us to avoid the incomplete markets problem. That is, we relax the key assumptions of the classical Black-Scholes model without changing their price formula.

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1. Introduction

To overcome some of the limitations of the Black-Scholes model, some models used jump diffusions. These models did not offer an explicit, simple formula for the price of the European option. That is, it requires a numerical/computational method. Later models such as Hull and White (1987), Chen et al (2016), Gong and Zhang (2016) and Kleinert and Korbel (2016) and Fouque et al (2000) relied on approximation.

Empirical studies include Leippold and Schärer (2017), Zhang and Wang (2013), and Zhang et al (2012). Others used a numerical/computational approach. Examples include Zhou et al (2013) and Martino et al (2015). Alghalith (2020) used a different process and a different method.

Similarly, studies that dealt with option pricing under a stochastic interest rate relied on numerical/computational methods. They mainly relied on Monte Carlo simulations, finite difference and/or Fourier transforms. Examples include He and Zhu (2018) who adopted fast Fourier transforms and Sun and Xu (2018) who employed Monte Carlo methods. Alghalith (2021) used a different process and a different method.

In this paper, we overcome these limitations. In doing so, we provide explicit, simple price formulas for the European options under stochastic volatility and stochastic interest rate. The formulas are as simple as the classical Black-Scholes formula. Moreover, the formulas do not require the normality of the returns. We do not need to know the distribution of the returns/price. That is, we relax the key assumptions of the classical Black-Scholes model without changing their price formula.

2. The model

The dynamics of the price of the underlying asset are given by

$$dS_u = S_u\left[rdu + v_u dW_u\right], \qquad (1)$$

where r is the risk-free interest rate, v_u is the stochastic volatility (that meets regularity conditions), and W_u is a Brownian motion. We do not need to specify the form of stochastic volatility. Also, clearly, the conditional distribution of the price (given the volatility) is log-normal.

If the returns are not normal, under regular conditions, the option price can be expressed as a weighted average of the Black-Scholes prices conditional on the volatility as follows

$$C(t, S) = \int_{v_i} E\left[e^{-r(T-t)} g\left(S_T\right)/v = v_i\right] dF\left(v_i\right) = \int_{v_i} C_{BS}\left(v_i\right) dF\left(v_i\right), \qquad (2)$$

where g is the payoff, S is the price at time t, T is the expiry time, F is the cumulative density, and C_{BS} is the Black-Scholes price.

By the continuity, the expected value is a specific value of C_{BS} denoted by $\hat{C}_{BS} = C_{BS}\left(\hat{v}_i\right)$, where $\hat{v}_i$ is a value (outcome) of the volatility. Thus,

$$C(t, S) = \int_{v_i} C_{BS}\left(v_i\right) dF\left(v_i\right) = C_{BS}\left(\hat{v}_i\right). \qquad (3)$$

Therefore, the price of the call option is

$$C(t, S) = SN\left(d_1\right) - e^{-r(T-t)} KN\left(d_2\right), \qquad (4)$$

where $d_1 = \frac{\ln(S/K) + \left(r + \frac{\hat{v}_i^2}{2}\right)(T-t)}{\sqrt{\hat{v}_i^2 (T-t)}}$, and $d_2 = d_1 - \sqrt{\hat{v}_i^2 (T-t)}$ and K is the strike price.

Verification

A simple way to verify the result is to let $\tilde{C}(t, S)$ be the true (market) price of the option, and $\bar{C}(r, S, \sigma, T-t)$ be the classical Black-Scholes price of the European option. By the continuity of $\bar{C}$, there is a specific value of the volatility parameter σ, such as $\hat{\sigma}$, so that $\tilde{C}(t, S) = \bar{C}(r, S, \hat{\sigma}, T-t)$. Therefore, the true option price can be expressed using the Black-Scholes formula (with volatility equal to $\hat{\sigma}$).

Estimation Methods

Even in the classical Black-Scholes model, the volatility parameter needs to be estimated and the estimation method is arbitrary; similarly, the volatility parameter $\hat{v}_i$ can be estimated. Moreover, the implied value of $\hat{v}_i$ can be computed using the formula. The (historical) implied values can be used in the estimation of $\hat{v}_i$.

However, we can show that $\hat{v}_i^2$ can be replaced by the expected value of the average of v_u^2; to see this

$$Var\left(\int_t^T \frac{dS_u}{S_u}\right) = E\int_t^T v_u^2 du = (T-t)E\left[\frac{\int_t^T v_u^2 du}{T-t}\right]. \qquad (5)$$

Numerical Example 1:

If $S = K = 100$, $r = .05$, and $T - t = 90$ days, and the true (market) price of a call is 10, the implied value of $\hat{v}_i = 47.73\%$.

Numerical Example 2:

Using historical data for the S&P 500 Index call options[1], for a short maturity and at-the-money, the average of the implied values of $\hat{v}_i$ is 19%.

These historical averages can be used to price options.

3. Stochastic interest rate

In this section, we consider the case of stochastic interest rate but constant volatility. The dynamics of the price of the underlying asset are given by

$$dS_u = S_u\left(r_u du + \sigma dW_u\right), \qquad (6)$$

where σ is the constant volatility and r_u is the stochastic interest rate.

If the returns are not normal, the option price can be expressed as a weighted average of the Black-Scholes prices conditional on the interest rate as follows

$$C(t, S) = \int_{r_i} C_{BS}\left(r_i\right) dG\left(r_i\right) = C_{BS}\left(\hat{r}_i\right), \qquad (7)$$

where G is the cumulative density, C_{BS} is the Black-Scholes price and $\hat{r}_i$ is a value (outcome) of the interest rate.

Therefore, the price of the call option is

$$C(t, S) = SN\left(d_1\right) - e^{-\hat{r}_i(T-t)} KN\left(d_2\right), \qquad (8)$$

where $d_1 = \frac{\ln(S/K) + \left(\hat{r}_i + \sigma^2/2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$ and $d_2 = d_1 - \sqrt{\sigma^2(T-t)}$.

Similarly, we can show that $\hat{r}_i$ can be replaced by the expected value of the average of r_u; to see this

$$E\int_t^T \frac{dS_u}{S_u} = E\int_t^T r_u du = (T-t) E\left[\frac{\int_t^T r_u du}{T-t}\right]. \qquad (9)$$

4. Alternative method

The price of the underlying asset is given by

$$S_u = Se^{\alpha u + \sigma X_u}, \qquad (10)$$

where α and σ are constants and X_u is stochastic. Now, we can rewrite the price as

$$S_u = Se^{\alpha u + \sigma X_u} = Se^{\alpha u + \sigma \frac{W_u}{W_u} X_u}, \qquad (11)$$

Therefore, the price can be given by

$$S_u = Se^{\alpha u + V_u W_u}, \qquad (12)$$

where W_u is a Brownian motion and V_u is a random variable.

Under regular conditions, the option price can be expressed as a weighted average of the Black-Scholes prices conditional on V as follows

$$C(t,S) = \int_V E\left[e^{-r(T-t)} g\left(S_T\right)/V = v\right] dF(v) = \int_V C_{BS}(v) dF(v), \qquad (13)$$

where g is the payoff, T is the expiry time, F is the cumulative density, and C_{BS} is the Black-Scholes price. By the continuity, the expected value is a specific value of C_{BS} denoted by $\hat{C}_{BS} = C_{BS}\left(\hat{v}\right)$, where $\hat{v}$ is a value (outcome) of V. Thus,

$$C(t,S) = \int_V C_{BS}(v) dF(v) = C_{BS}\left(\hat{v}\right). \qquad (14)$$

Therefore, the price of the call option is

$$C(t,S) = SN\left(d_1\right) - e^{-r(T-t)} KN\left(d_2\right), \qquad (15)$$

where $d_1 = \frac{\ln(S/K) + \left(r + \frac{\hat{v}2}{2}\right)(T-t)}{\sqrt{\hat{v}2(T-t)}}$, and $d_2 = d_1 - \sqrt{\hat{v}2(T-t)}$.

Conclusion

We showed that the key assumptions of the Black-Scholes model can be relaxed without complicating the analysis. Not only we relaxed the assumptions of normality, and constant volatility/interest rate, we provided price formulas that are as simple as the classical Black-Scholes formula. This makes option pricing much easier.

Footnotes

[1] Obtained from https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=97da7138bfd3fdda62fc9c9ddc6b74454ee2759c

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