Relaxing the Black-Scholes Assumptions Without Changing the Price Formula

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Abstract

We overcome the key limitations of the Black-Scholes model. In doing so, we provide an explicit, simple price formula for the European option that is identical to the classical Black-Scholes formula. Moreover, we do not need to know the distribution of the returns/price.

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1. Introduction

It is well known that the key limitations of the Black-Scholes model are the assumptions of the normality of the returns and constant volatility. To overcome some of the limitations of the Black-Scholes model, some models used jump diffusions or stochastic volatility. These models did not offer an explicit formula for the price of the European option. That is, it requires a numerical/computational method. Later models such as Hull and White (1987), Chen et al (2016), Gong and Zhang (2016) and Kleinert and Korbel (2016) did not offer an explicit (simple) formula. In this paper, we overcome these two key limitations of the Black-Scholes model. In doing so, we provide an explicit, simple price formula for the European option that is identical to the classical Black-Scholes formula. Moreover, we do not need to know the distribution of the returns/price.

2. The model

The price of the underlying asset is given by

$$S_u = S_0 e^{\alpha u + \sigma X_u}, \qquad (1)$$

where α and σ are constants and X_u is stochastic. Now, we can rewrite the price as

$$S_u = S_0 e^{\alpha u + \sigma X_u} = S_0 e^{\alpha u + \sigma \frac{W_u}{W_u} X_u}, \qquad (2)$$

where W_u is a Brownian motion. Therefore, the price can be given by%

$$S_u = S_0 e^{\alpha u + V_u W_u}. \qquad (3)$$

Under regular conditions, the option price can be expressed as a weighted average of the Black-Scholes prices conditional on V as follows

$$C\left(t, S_0\right) = \int_v E\left[e^{-r(T-t)} g\left(S_T\right) / V_T = v\right] dF(v) = \int_v C_{BS}(v) dF(v), \qquad (4)$$

where g is the payoff, T is the expiry time, F is the cumulative density of V, and C_{BS} is the Black-Scholes price. By the continuity, the expected value is a specific value of C_{BS} denoted by $\hat{C}_{BS} = C_{BS}\left(\hat{v}\right)$, where $\hat{v}$ is a value (outcome) of V. Thus,

$$C\left(t, S_0\right) = \int_v C_{BS}(v) dF(v) = C_{BS}\left(\hat{v}\right). \qquad (5)$$

Therefore, the price of the call option is

$$C\left(t, S_0\right) = S_0 N\left(d_1\right) - e^{-r(T-t)} K N\left(d_2\right), \qquad (6)$$

where $d_1 = \dfrac{\ln\left(S_0/K\right) + \left(r + \frac{\hat{v}^2}{2}\right)(T-t)}{\sqrt{\hat{v}^2 (T-t)}}$, and $d_2 = d_1 - \sqrt{\hat{v}^2 (T-t)}$ and K is the strike price.

Even in the classical Black-Scholes model, the volatility parameter needs to be estimated; similarly, the parameter $\hat{v}$ can be estimated. Moreover, the implied value of $\hat{v}$ can be computed using the formula. The implied values can be used to estimate $\hat{v}$. However, we can show that $\hat{v}_i^2$ can be estimated as the expected value of the average of v_u^2; to see this

$$Var\left(\int_t^T \frac{dS_u}{S_u}\right) = E\int_t^T v_u^2 du = (T-t)E\overline{T-t}. \qquad (7)$$

A verification:

A simple way to verify the result is to let $\tilde{C}\left(t, S_0\right)$ be the true option price, and $\bar{C}(r, S_0, \sigma, T-t)$ be the classical Black-Scholes price of the European option. By the continuity of $\bar{C}$, there is a specific value of the volatility parameter σ, such as $\hat{\sigma}$, so that $\tilde{C}\left(t, S_0\right) = \bar{C}(r, S_0, \hat{\sigma}, T-t)$. Therefore, the true option price can be expressed using the Black-Scholes formula (with a volatility equal to $\hat{\sigma}$).

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