Research Article

Picture Fuzzy, Hesitant Fuzzy, and Spherical Fuzzy SuperHyperGraph

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Graph theory studies structures of vertices and edges to model relationships and connectivity[1][2]. Hypergraphs extend this framework by allowing *hyperedges* that join any number of vertices simultaneously[3], and superhypergraphs further introduce iterated powerset layers for hierarchical, self-referential connections[4][5]. Building on advances in fuzzy and soft computing—such as fuzzy sets[6], soft sets[7], intuitionistic fuzzy sets[8], neutrosophic sets[9], hesitant fuzzy sets[10], picture fuzzy sets[11], spherical fuzzy sets[12], and plithogenic sets[13][14]—this paper formally defines Picture Fuzzy, Hesitant Fuzzy, and Spherical Fuzzy SuperHyperGraphs and offers a concise discussion of their fundamental properties.

1. Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Unless otherwise noted, all graphs considered here are *undirected*, *finite*, and *simple*. For detailed treatment of specific operations and related notions, the reader is referred to the appropriate literature.

1.1. SuperHyperGraph

In classical graph theory, a hypergraph extends the ordinary graph by permitting *hyperedges* that connect more than two vertices[3][15]. This added flexibility makes hypergraphs an ideal tool for modeling complex, multi-way relationships across a variety of fields[3][16][17][18][19][20]. A *SuperHyperGraph* further enriches this framework by integrating iterated powerset constructions into the hypergraph structure, a concept that has recently garnered considerable attention in the literature[5][21][22][23][24][25][26][27]. Beyond its theoretical appeal, SuperHyperGraphs have been applied in domains such as molecular modeling, network analysis, and signal processing[21][28][29][30][31][32]. Throughout this paper, the parameter n in both the n-th powerset and the n-SuperHyperGraph is taken to be a non-negative integer.

Definition 1.1. (Base Set). A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

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Definition 1.2. (Powerset). The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3. (Hypergraph). [3][33] A *hypergraph* $H = (V(H), E(H))$ consists of:

- A nonempty set $V(H)$ of vertices.
- A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Definition 1.4. (n-th Powerset). (cf.[34][35]) The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.5. (n-SuperHyperGraph). [4][36]

Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset operation. An *n-SuperHyperGraph* is then a pair

$$\mathrm{SHT}^{(n)} = (V, E),$$

with

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}^n(V_0).$$

Each element of V is called an *n-supervertex* and each element of E an *n-superedge*.

Example 1.6. (University Cohort as a 2-SuperHyperGraph). Consider a small cohort of four students:

$$V_0 = \{\text{Alice, Bob, Carol, Dave}\}.$$

Form two study teams:

$$\mathrm{Team}_1 = \{\mathrm{Alice}, \mathrm{Bob}\}, \quad \mathrm{Team}_2 = \{\mathrm{Carol}, \mathrm{Dave}\}.$$

Then the 2-SuperHyperGraph $\mathrm{SHT}^{(2)} = (V, E)$ is given by

$$V=\{\{\mathrm{Team}_1\},\{\mathrm{Team}_2\},\{\mathrm{Team}_1,\mathrm{Team}_2\}\}\subseteq\mathcal{P}^2(V_0),\quad E=\{e=\{\{\mathrm{Team}_1\},\{\mathrm{Team}_2\}\}\}\subseteq\mathcal{P}^2(V_0).$$

Here each supervertex is a subset of teams (either one team or the full cohort), and the single superedge e links the two teams together.

1.2. Fuzzy n-Superhypergraph

A *fuzzy set* assigns to each element of a universe a membership degree in the interval $[0,1]$ [6][37]. A *fuzzy graph* extends this concept by equipping both vertices and edges of a graph with membership degrees[38][39][40]. A *fuzzy hypergraph* further generalizes fuzzy graphs by allowing membership degrees on vertices and hyperedges[41][42][43][44]. A *fuzzy n-superhypergraph* extends the idea to n-superhypergraphs, assigning membership degrees to n-supervertices and n-superedges. The formal definition of a fuzzy n-superhypergraph is given as follows (cf.[4][23]).

Definition 1.7. (Fuzzy n-Superhypergraph). (cf.[4][23]) Let $\mathrm{SHT}^{(n)}=(V,E)$ be an n-Superhypergraph. A *fuzzy n-Superhypergraph* is a quadruple

$$(V,E,\sigma,\mu),$$

where

- $\sigma:V\to[0,1]$ assigns to each n-supervertex v a membership degree $\sigma(v)$
- $\mu:E\to[0,1]$ assigns to each n-superedge e a membership degree $\mu(e)$.

These functions satisfy the *appurtenance constraint*

$$\mu(e)\ \leq\ \min_{v\in e}\sigma(v),\quad\forall e\in E.$$

Example 1.8. (Corporate Collaboration as a Fuzzy 2-Superhypergraph). Consider a software company organized in two levels: employees and teams. Let

$$V=\{\mathrm{Alice},\,\mathrm{Bob},\,\mathrm{Charlie},\,T_1,\,T_2\},\quad E=\{\{\mathrm{Alice},\mathrm{Bob}\}=e_1,\ \{\mathrm{Bob},\mathrm{Charlie}\}=e_2,\ \{T_1,T_2\}=e_3\}.$$

Assign fuzzy membership degrees:

$$\sigma(\mathrm{Alice})=0.90,\quad\sigma(\mathrm{Bob})=0.85,\quad\sigma(\mathrm{Charlie})=0.70,$$

$$\mu(e_1)=0.85,\quad\mu(e_2)=0.70,\quad\sigma(T_1)=0.85,\quad\sigma(T_2)=0.70,\quad\mu(e_3)=0.70.$$

Here e_1 and e_2 represent team memberships, and e_3 groups teams into a department. One checks $\mu(e_1)\leq\min\{\sigma(\mathrm{Alice}),\sigma(\mathrm{Bob})\}$, $\mu(e_2)\leq\min\{\sigma(\mathrm{Bob}),\sigma(\mathrm{Charlie})\}$, and $\mu(e_3)\leq\min\{\sigma(T_1),\sigma(T_2)\}$, so all appurtenance constraints hold.

1.3. Intuitionistic Fuzzy n-Superhypergraph

An *intuitionistic fuzzy set* assigns to each element both a membership degree and a non-membership degree, satisfying Atanassov's constraint $0\leq\mu(x)+\nu(x)\leq 1$ [8][45][46]. An *intuitionistic fuzzy graph* extends a classical graph by assigning every vertex and every edge both membership and non-membership values, also respecting Atanassov's sum constraint[47][48][49][50]. An *intuitionistic fuzzy hypergraph* generalizes hypergraphs by equipping each vertex and each hyperedge with membership and non-membership degrees under the same Atanassov framework[51][52][53].

Definition 1.9. (Intuitionistic Fuzzy Hypergraph). Let V be a nonempty finite set of *vertices*. An *intuitionistic fuzzy hyperedge* on V is a pair of functions

$$(\mu_E, \nu_E) \colon V \longrightarrow [0,1] \times [0,1]$$

such that

$$0 \;\le\; \mu_E(v) + \nu_E(v) \;\le\; 1, \quad \forall\, v \in V.$$

Its *support* is

$$\operatorname{supp}(E) \;=\; \bigl\{ v \in V \bigm| \mu_E(v) > 0 \text{ and } \nu_E(v) < 1 \bigr\}.$$

An *intuitionistic fuzzy hypergraph* is a pair

$$H = \bigl(V, \mathcal{E}\bigr),$$

where $\mathcal{E} = \{\, E_1, \ldots, E_m \}$ is a finite family of intuitionistic fuzzy hyperedges on V satisfying the covering condition

$$\bigcup_{j=1}^{m} \operatorname{supp}(E_j) \;=\; V.$$

The elements of V are called *vertices*, and each $E_j \in \mathcal{E}$ is called an *intuitionistic fuzzy hyperedge*. The *order* of H is $|V|$, and the number of edges is $|\mathcal{E}|$.

Definition 1.10. (Intuitionistic Fuzzy n-Superhypergraph). Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-Superhypergraph on a finite base set V_0 (so $V, E \subseteq \mathcal{P}^n(V_0)$). An *intuitionistic fuzzy n–superhypergraph* is a sextuple

$$\mathcal{H} = \bigl(V,\ E,\ \sigma,\ \sigma^c,\ \mu,\ \nu\bigr),$$

where

$$\begin{aligned} \sigma &: V \longrightarrow [0,1], & \sigma^c &: V \longrightarrow [0,1], \\ \mu &: E \longrightarrow [0,1], & \nu &: E \longrightarrow [0,1], \end{aligned}$$

satisfy the following for all $v \in V$ and $e \in E$:

$$0 \;\le\; \sigma(v) + \sigma^c(v) \;\le\; 1, \quad 0 \;\le\; \mu(e) + \nu(e) \;\le\; 1, \tag{1}$$

$$\mu(e) \;\le\; \min_{v \in e} \sigma(v), \quad \nu(e) \;\le\; \min_{v \in e} \sigma^c(v). \tag{2}$$

Here $\sigma(v)$ and $\sigma^c(v)$ are the membership and non-membership of the n–supervertex v, while $\mu(e)$ and $\nu(e)$ are the membership and non-membership of the n–superedge e. Equation 1 is the Atanassov constraint, and 2 enforces consistency of edge-values with their vertices.

Example 1.11. (Team Collaboration as an Intuitionistic Fuzzy 2-Superhypergraph). Consider a company with four employees grouped into two teams. Let

$$V_0 = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}\}, \quad V = \big\{\{Alice\}, \{Bob\}, \{Carol\}, \{Dave\}, \{Alice, Bob\}, \{Carol, Dave\}\big\},$$

$$E = \big\{e_1 = \{\{Alice\}, \{Bob\}, \{Alice, Bob\}\},$$

$$e_2 = \{\{Carol\}, \{Dave\}, \{Carol, Dave\}\},$$

$$e_3 = \{\{Alice, Bob\}, \{Carol, Dave\}\}\big\}.$$

Assign intuitionistic fuzzy degrees as follows:

$$\begin{aligned} \sigma(\{Alice\}) &= 0.90, & \sigma^c(\{Alice\}) &= 0.05, \\ \sigma(\{Bob\}) &= 0.85, & \sigma^c(\{Bob\}) &= 0.10, \\ \sigma(\{Alice, Bob\}) &= 0.80, & \sigma^c(\{Alice, Bob\}) &= 0.15, \\ \sigma(\{Carol\}) &= 0.75, & \sigma^c(\{Carol\}) &= 0.20, \\ \sigma(\{Dave\}) &= 0.70, & \sigma^c(\{Dave\}) &= 0.25, \\ \sigma(\{Carol, Dave\}) &= 0.65, & \sigma^c(\{Carol, Dave\}) &= 0.30. \end{aligned}$$

For the edges, set

$$\begin{aligned} \mu(e_1) &= 0.80, & \nu(e_1) &= 0.10, \\ \mu(e_2) &= 0.70, & \nu(e_2) &= 0.15, \\ \mu(e_3) &= 0.65, & \nu(e_3) &= 0.20. \end{aligned}$$

One checks that for every vertex v, $\sigma(v) + \sigma^c(v) \le 1$, and for every edge e, $\mu(e) + \nu(e) \le 1$. Moreover, the appurtenance constraints $\mu(e) \le \min_{v \in e} \sigma(v)$ and $\nu(e) \le \min_{v \in e} \sigma^c(v)$ hold for e_1, e_2, e_3.

1.4. Picture Fuzzy Hypergraph

A picture fuzzy set assigns truth, neutrality, falsity membership degrees to each element, totaling ≤ 1[11][54][55][56][57]. A picture fuzzy graph equips vertices, edges truth, neutrality, falsity membership degrees under connectivity constraints[58][59][60][61]. A picture fuzzy hypergraph extends picture fuzzy graphs assigning truth, neutrality, falsity degrees to hyperedges[62].

Definition 1.12. (Picture Fuzzy Set [11]). Let X be a nonempty universe. A *picture fuzzy set* (PFS) P on X is defined as

$$P = \big\{ \big(x, T_P(x), N_P(x), F_P(x)\big) \mid x \in X \big\},$$

where

$$T_P : X \to [0, 1], \quad N_P : X \to [0, 1], \quad F_P : X \to [0, 1]$$

are functions assigning to each $x \in X$ its *positive membership degree* $T_P(x)$, *neutral membership degree* $N_P(x)$, and *negative membership degree* $F_P(x)$, respectively, subject to

$$0 \le T_P(x) + N_P(x) + F_P(x) \le 1, \quad \forall x \in X.$$

The *refusal degree* of x in P is given by $R_P(x) = 1 - \big(T_P(x) + N_P(x) + F_P(x)\big)$.

Definition 1.13. (Picture Fuzzy Graph). [58][59] Let $G^* = (V^*, E^*)$ be a crisp simple graph with vertex set V^* and edge set $E^* \subseteq V^* \times V^*$. A *picture fuzzy graph* (PFG) $G = \big(P_V, P_E\big)$ over G^* consists of:

- A *picture fuzzy vertex set* $P_V = \{(v, T_V(v), N_V(v), F_V(v)) : v \in V^*\}$, where satisfy $0 \le T_V(v) + N_V(v) + F_V(v) \le 1$ for all $v \in V^*$.

$$T_V : V^* \to [0,1], \quad N_V : V^* \to [0,1], \quad F_V : V^* \to [0,1]$$

- A *picture fuzzy edge set* $P_E = \{(e, T_E(e), N_E(e), F_E(e)) : e = uv \in E^*\}$, where satisfy $0 \le T_E(e) + N_E(e) + F_E(e) \le 1$ for all $e \in E^*$.

$$T_E : E^* \to [0,1], \quad N_E : E^* \to [0,1], \quad F_E : E^* \to [0,1]$$

These functions must also satisfy the *edge-membership constraints* for every edge $e = uv \in E^*$:

$$T_E(e) \le \min\{T_V(u), T_V(v)\}, \quad N_E(e) \le \min\{N_V(u), N_V(v)\}, \quad F_E(e) \ge \max\{F_V(u), F_V(v)\}.$$

Definition 1.14. (Picture Fuzzy Hypergraph). [62] Let X be a finite universe and let $E = \{E_1, E_2, \ldots, E_m\}$ be a family of nonempty subsets of X. A *picture fuzzy hypergraph* (PFHG) $H = (P_V, P_E)$ on X is defined by:

- A *picture fuzzy vertex set* $P_V = \{(x, T_V(x), N_V(x), F_V(x)) : x \in X\}$, where satisfy $0 \le T_V(x) + N_V(x) + F_V(x) \le 1$ for all $x \in X$.

$$T_V : X \to [0,1], \quad N_V : X \to [0,1], \quad F_V : X \to [0,1]$$

- A *picture fuzzy hyperedge set* $P_E = \{(E_j, T_E(E_j), N_E(E_j), F_E(E_j)) : 1 \le j \le m\}$, where satisfy $0 \le T_E(E_j) + N_E(E_j) + F_E(E_j) \le 1$ for each hyperedge $E_j \subseteq X$.

$$T_E : \{E_j\} \to [0,1], \quad N_E : \{E_j\} \to [0,1], \quad F_E : \{E_j\} \to [0,1]$$

These membership functions must satisfy the *hyperedge-membership constraints*: for every hyperedge $E_j \subseteq X$,

$$T_E(E_j) \le \min_{x \in E_j}\{T_V(x)\}, \quad N_E(E_j) \le \min_{x \in E_j}\{N_V(x)\}, \quad F_E(E_j) \ge \max_{x \in E_j}\{F_V(x)\}.$$

1.5. Hesitant Fuzzy HyperGraph

A hesitant fuzzy set assigns to each element a finite list of membership values[10][63][64][65]. A hesitant fuzzy graph associates vertices and edges with membership, non-membership, and hesitancy degrees[66][67][68][69]. A hesitant fuzzy hypergraph extends graphs by assigning membership, non-membership, and hesitancy degrees to hyperedges[66].

Definition 1.15. (Hesitant Fuzzy Set (HFS)). [10][63][64] A *hesitant fuzzy set* E on a finite nonempty universe Y is a mapping

$$h_E : Y \longrightarrow \mathcal{P}([0,1]),$$

where for each $y \in Y$, $h_E(y)$ is a finite collection (possibly with repetition) of membership degrees in $[0,1]$. Equivalently, one may write

$$E = \{\langle y, h_E(y)\rangle \mid y \in Y\},$$

where $h_E(y) \subseteq [0,1]$ is called the *hesitant element* associated with y, representing all possible membership values of y to E. No further constraint is imposed on the elements of $h_E(y)$.

Definition 1.16. (Hesitant Fuzzy Graph (HFG)). [66] Let $G^* = (V, E^*)$ be a finite simple undirected graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E^* \subseteq V \times V$. A *hesitant fuzzy graph* is a quintuple

$$G = (V,\ E^*,\ \mu_V,\ \gamma_V,\ \beta_V,\ \mu_E,\ \gamma_E,\ \beta_E),$$

where

- $\mu_V, \gamma_V, \beta_V : V \to [0,1]$ assign to each vertex $v \in V$ a membership degree $\mu_V(v)$, a non-membership degree $\gamma_V(v)$, and a hesitancy degree $\beta_V(v)$, satisfying where $\beta_V(v) = 1 - \big(\mu_V(v) + \gamma_V(v)\big)$.

$$\mu_V(v) + \gamma_V(v) + \beta_V(v) \ = \ 1, \quad 0 \ \le \ \mu_V(v) + \gamma_V(v) \ \le \ 1, \quad \forall\, v \in V,$$

- $\mu_E, \gamma_E, \beta_E : E^* \to [0,1]$ assign to each edge $(v_i, v_j) \in E^*$ a membership degree $\mu_E(v_i, v_j)$, non-membership degree $\gamma_E(v_i, v_j)$, and hesitancy degree $\beta_E(v_i, v_j)$, subject to

$$\begin{aligned}
&\mu_E(v_i, v_j) \ \le \ \min\{\mu_V(v_i),\, \mu_V(v_j)\},\\
&\gamma_E(v_i, v_j) \ \le \ \max\{\gamma_V(v_i),\, \gamma_V(v_j)\},\\
&\beta_E(v_i, v_j) \ \le \ \min\{\beta_V(v_i),\, \beta_V(v_j)\},\\
&0 \ \le \ \mu_E(v_i, v_j) + \gamma_E(v_i, v_j) + \beta_E(v_i, v_j) \ \le \ 1, \quad \forall\, (v_i, v_j) \in E^*.
\end{aligned}$$

Definition 1.17. (Hesitant Fuzzy Hypergraph (HFH)). [66] Let $H^* = (V,\ E^*)$ be a finite hypergraph with vertex set $V = \{v_1, \ldots, v_n\}$ and hyperedge set $E^* \subseteq \mathcal{P}(V)$. A *hesitant fuzzy hypergraph* is a septuple

$$H = (V,\ E^*,\ \mu_V,\ \gamma_V,\ \beta_V,\ \mu_E,\ \gamma_E,\ \beta_E),$$

where

- $\mu_V, \gamma_V, \beta_V : V \to [0,1]$ assign to each vertex $v \in V$ a membership degree $\mu_V(v)$, a non-membership degree $\gamma_V(v)$, and a hesitancy degree $\beta_V(v)$, satisfying where $\beta_V(v) = 1 - \big(\mu_V(v) + \gamma_V(v)\big)$.

$$\mu_V(v) + \gamma_V(v) + \beta_V(v) \ = \ 1, \quad 0 \ \le \ \mu_V(v) + \gamma_V(v) \ \le \ 1, \quad \forall\, v \in V,$$

- $\mu_E, \gamma_E, \beta_E : E^* \to [0,1]$ assign to each hyperedge $e \in E^*$ a membership degree $\mu_E(e)$, a non-membership degree $\gamma_E(e)$, and a hesitancy degree $\beta_E(e)$, subject to

$$\begin{aligned}
&\mu_E(e) \ \le \ \min_{v \in e} \mu_V(v),\\
&\gamma_E(e) \ \le \ \max_{v \in e} \gamma_V(v),\\
&\beta_E(e) \ \le \ \min_{v \in e} \beta_V(v),\\
&0 \ \le \ \mu_E(e) + \gamma_E(e) + \beta_E(e) \ \le \ 1, \quad \forall\, e \in E^*.
\end{aligned}$$

1.6. Spherical Fuzzy HyperGraph

A spherical fuzzy set assigns each element three squared-sum-constrained membership degrees—truthness, abstinence, falsity—ensuring their squared sum does not exceed one[70][71][72][73]. A spherical fuzzy graph labels vertices and edges with truthness, abstinence, falsity degrees under squared-sum constraint and connectivity membership bounds[74][75][76][77][78]. A spherical fuzzy hypergraph assigns vertices and hyperedges squared-sum-constrained truthness, abstinence, falsity degrees while enforcing truthness, abstinence minima and falsity maxima.

Definition 1.18. (Spherical Fuzzy Set [12]). Let U be a nonempty universe. A *spherical fuzzy set* S on U is defined by a mapping

$$S = \{(u,\ \alpha_S(u),\ \gamma_S(u),\ \beta_S(u)) \mid u \in U\},$$

where

$$\alpha_S : U \to [0,1], \quad \gamma_S : U \to [0,1], \quad \beta_S : U \to [0,1]$$

are the *truthness*, *abstinence*, and *falsity* membership functions, respectively, satisfying

$$0 \ \leq\ \alpha_S(u)^2 \ +\ \gamma_S(u)^2 \ +\ \beta_S(u)^2 \ \leq\ 1 \quad \forall\, u \in U.$$

The *refusal degree* of $u \in U$ is defined by

$$\delta_S(u) \ =\ \sqrt{1 \ -\ \left(\alpha_S(u)^2 + \gamma_S(u)^2 + \beta_S(u)^2\right)} \ \in\ [0,1].$$

Definition 1.19. (Spherical Fuzzy Graph [12]). Let $G^* = (V, E)$ be a finite simple undirected graph with vertex set V and edge set $E \subseteq V \times V$. A *spherical fuzzy graph* (SFG) $G = (M, N)$ over G^* consists of:

- A *spherical fuzzy vertex set* where $\alpha_M, \gamma_M, \beta_M : V \to [0,1]$ satisfy $\alpha_M(v)^2 + \gamma_M(v)^2 + \beta_M(v)^2 \leq 1$ for all $v \in V$.

$$M = \{\ (v,\ \alpha_M(v),\ \gamma_M(v),\ \beta_M(v)) : v \in V\},$$

- A *spherical fuzzy edge set* where $\alpha_N, \gamma_N, \beta_N : E \to [0,1]$ satisfy $\alpha_N(e)^2 + \gamma_N(e)^2 + \beta_N(e)^2 \leq 1$ for all $e \in E$.

$$N = \{\ (e,\ \alpha_N(e),\ \gamma_N(e),\ \beta_N(e)) : e = uv \in E\},$$

These functions must obey, for every edge $e = uv \in E$,

$$\alpha_N(e) \ \leq\ \min\{\alpha_M(u), \alpha_M(v)\}, \quad \gamma_N(e) \ \leq\ \min\{\gamma_M(u), \gamma_M(v)\}, \quad \beta_N(e) \ \geq\ \max\{\beta_M(u), \beta_M(v)\}.$$

Definition 1.20. (Spherical Fuzzy Hypergraph). Let X be a finite universe and let $\mathcal{E} = \{E_1, \ldots, E_m\}$ be a family of nonempty subsets of X. A *spherical fuzzy hypergraph* (SFH) $H = (M,\ N)$ on $(X, \mathcal{E})$ is defined by:

- A *spherical fuzzy vertex set* where $\alpha_M, \gamma_M, \beta_M : X \to [0,1]$ satisfy $\alpha_M(x)^2 + \gamma_M(x)^2 + \beta_M(x)^2 \leq 1$ for all $x \in X$.

$$M = \{\ (x,\ \alpha_M(x),\ \gamma_M(x),\ \beta_M(x)) : x \in X\},$$

- A *spherical fuzzy hyperedge set* where $\alpha_N, \gamma_N, \beta_N : \mathcal{E} \to [0,1]$ satisfy $\alpha_N(E_j)^2 + \gamma_N(E_j)^2 + \beta_N(E_j)^2 \leq 1$ for all $E_j \in \mathcal{E}$.

$$N = \{\ (E_j,\ \alpha_N(E_j),\ \gamma_N(E_j),\ \beta_N(E_j)) : E_j \in \mathcal{E}\},$$

These membership functions must satisfy, for each hyperedge $E_j \in \mathcal{E}$,

$$\alpha_N(E_j) \ \leq\ \min_{x \in E_j}\{\alpha_M(x)\}, \quad \gamma_N(E_j) \ \leq\ \min_{x \in E_j}\{\gamma_M(x)\}, \quad \beta_N(E_j) \ \geq\ \max_{x \in E_j}\{\beta_M(x)\}.$$

2. Results of This Paper

In this paper, we present the various types of SuperHyperGraphs.

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2.1. Picture Fuzzy SuperHyperGraph

We now extend the Picture Fuzzy HyperGraph concept to the setting of n-SuperHyperGraphs.

Definition 2.1. (Picture Fuzzy n-SuperHyperGraph). Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph on base set V_0. A *Picture Fuzzy n-SuperHyperGraph* is a sextuple

$$\mathcal{H}_{\mathrm{PF}} \;=\; \bigl(V,\, E,\, T_V,\, N_V,\, F_V,\, T_E,\, N_E,\, F_E\bigr),$$

where

$$\begin{array}{lll} T_V : V \;\longrightarrow\; [0,1], & N_V : V \;\longrightarrow\; [0,1], & F_V : V \;\longrightarrow\; [0,1], \\ T_E : E \;\longrightarrow\; [0,1], & N_E : E \;\longrightarrow\; [0,1], & F_E : E \;\longrightarrow\; [0,1], \end{array}$$

subject to the following conditions for all $v \in V$ and $e \in E$:

1. *Vertex-sum constraint:*

$$0 \;\le\; T_V(v) \;+\; N_V(v) \;+\; F_V(v) \;\le\; 1.$$

2. *Edge-sum constraint:*

$$0 \;\le\; T_E(e) \;+\; N_E(e) \;+\; F_E(e) \;\le\; 1.$$

3. *Edge-vertex appurtenance constraints:*

$$\begin{aligned} T_E(e) \;&\le\; \min_{v \in e}\bigl\{\, T_V(v)\bigr\}, \\ N_E(e) \;&\le\; \min_{v \in e}\bigl\{\, N_V(v)\bigr\}, \\ F_E(e) \;&\ge\; \max_{v \in e}\bigl\{\, F_V(v)\bigr\}. \end{aligned}$$

Here each $v \in V$ is an n-supervertex, and each $e \in E$ is an n-superedge. The positive degree $T_V(v)$, neutral degree $N_V(v)$, and negative degree $F_V(v)$ quantify the picture-fuzzy membership of v, and similarly for each (T_E, N_E, F_E) on edges.

Example 2.2. (Picture Fuzzy 2-SuperHyperGraph). As a simple illustrative example, let $V_0 = \{\, a, b\,\}$. Then

$$\mathcal{P}^1(V_0) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\bigl(\mathcal{P}^1(V_0)\bigr).$$

Choose an n-SuperHyperGraph for $n = 2$ by letting

$$V \;=\; \bigl\{\{\,\{a\}, \{a, b\}\},\; \{\{b\}\}\bigr\}, \qquad E \;=\; \bigl\{\{\{\,\{a\}, \{a, b\}\},\; \{\{b\}\}\},\; \{\{\,\{b\}\}\}\bigr\}.$$

Now define positive, neutral, and negative membership degrees on vertices:

$$T_V\bigl(\{\{a\}, \{a, b\}\}\bigr) = 0.7, \quad N_V\bigl(\{\{a\}, \{a, b\}\}\bigr) = 0.1, \quad F_V\bigl(\{\{a\}, \{a, b\}\}\bigr) = 0.1,$$

$$T_V\bigl(\{\{b\}\}\bigr) = 0.6, \quad N_V\bigl(\{\{b\}\}\bigr) = 0.2, \quad F_V\bigl(\{\{b\}\}\bigr) = 0.1.$$

Then $T_V + N_V + F_V \le 1$ at each vertex. For an edge $e_1 = \{\{\,\{a\}, \{a, b\}\}, \{\{b\}\}\}$, set

$$T_E(e_1) = 0.6, \quad N_E(e_1) = 0.1, \quad F_E(e_1) = 0.2.$$

One checks

$$T_E(e_1) = 0.6 \ \le \ \min\{T_V(\{\{a\},\{a,b\}\}),\, T_V(\{\{b\}\})\} = \min\{0.7, 0.6\} = 0.6,$$

$$N_E(e_1) = 0.1 \ \le \ \min\{0.1, 0.2\} = 0.1, \quad F_E(e_1) = 0.2 \ \ge \ \max\{0.1, 0.1\} = 0.1.$$

Thus all constraints are satisfied, illustrating a concrete Picture Fuzzy 2-SuperHyperGraph.

Theorem 2.3. (Generalization of Fuzzy and Intuitionistic Fuzzy Structures). *Let*

$$\mathcal{H}_{\mathrm{PF}} = \big(V,\, E,\, T_V,\, N_V,\, F_V,\, T_E,\, N_E,\, F_E\big)$$

be any Picture Fuzzy n-SuperHyperGraph as in Definition. Then:

i. ***(Reduction to Picture Fuzzy HyperGraph)*** *If $n = 1$ and we interpret $V \subseteq \mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ and $E \subseteq \mathcal{P}(V_0)$, then $\mathcal{H}_{\mathrm{PF}}$ becomes exactly a Picture Fuzzy HyperGraph on the universe V_0.*

1. ***(Reduction to Fuzzy n-SuperHyperGraph)*** *If we set*

   $N_V(v) \ = \ 0, \quad F_V(v) \ = \ 0, \quad N_E(e) \ = \ 0, \quad F_E(e) \ = \ 0 \quad \text{for all } v \in V,\ e \in E,$

   then the residual data $\big(V,\, E,\, T_V,\, T_E\big)$ satisfy precisely the axioms of a Fuzzy n-SuperHyperGraph.

2. ***(Reduction to Intuitionistic Fuzzy n-SuperHyperGraph)*** *If we set 10)*

   $N_V(v) \ = \ 0, \quad N_E(e) \ = \ 0, \quad T_V(v) = \sigma(v), \quad F_V(v) = \sigma^c(v), \quad T_E(e) = \mu(e), \quad F_E(e) = \nu(e),$

   for all $v \in V$ and $e \in E$, then the data $\big(V,\, E,\, \sigma,\, \sigma^c,\, \mu,\, \nu\big)$ satisfy precisely the axioms of an Intuitionistic Fuzzy n-SuperHyperGraph (cf. Definition 1.10).

Proof. We examine each assertion in turn.

(i) Reduction to Picture Fuzzy HyperGraph. When $n = 1$, we have

$$V \ \subseteq \ \mathcal{P}^1(V_0) \ = \ \mathcal{P}(V_0), \qquad E \ \subseteq \ \mathcal{P}^1(V_0) \ = \ \mathcal{P}(V_0).$$

Hence each element of V is a nonempty subset of V_0, but since we wish to recover a Picture Fuzzy HyperGraph on the ground-set V_0, we identify the vertices of the PFHG as the elements of V_0. Concretely, let

$$X \ := \ V_0, \quad \mathcal{E} \ := \ \{e \subseteq X \mid e \in E\}.$$

We then define for each $x \in X$:

$$T_V(x),\ N_V(x),\ F_V(x) \quad \text{exactly as given},$$

and for each hyperedge $E_j \in \mathcal{E}$:

$$T_E(E_j),\ N_E(E_j),\ F_E(E_j) \quad \text{as given}.$$

Since Definition 2.1 requires exactly the same three inequalities

$$0 \ \le \ T_V(x) + N_V(x) + F_V(x) \ \le \ 1, \quad 0 \ \le \ T_E(E_j) + N_E(E_j) + F_E(E_j) \ \le \ 1,$$

and the constraints

$$\begin{aligned} T_E(E_j) &\le \min_{x\in E_j}\{T_V(x)\},\\ N_E(E_j) &\le \min_{x\in E_j}\{N_V(x)\},\\ F_E(E_j) &\ge \max_{x\in E_j}\{F_V(x)\}, \end{aligned}$$

we recover exactly the Definition of a Picture Fuzzy HyperGraph on $(X, \mathcal{E})$. Thus (i) holds.

(ii) Reduction to Fuzzy n-SuperHyperGraph. Suppose we impose

$$N_V(v) = 0, \quad F_V(v) = 0, \quad N_E(e) = 0, \quad F_E(e) = 0,$$

for all $v \in V$, $e \in E$. Then the vertex-sum constraint

$$0 \le T_V(v) + N_V(v) + F_V(v) \le 1$$

becomes

$$0 \le T_V(v) \le 1,$$

so $T_V(v)$ may be identified with a single membership degree $\sigma(v) \in [0,1]$. Likewise, the edge-sum constraint

$$0 \le T_E(e) + N_E(e) + F_E(e) \le 1$$

reduces to

$$0 \le T_E(e) \le 1,$$

so $T_E(e)$ may be identified with $\mu(e) \in [0,1]$. Next, the appurtenance inequalities

$$T_E(e) \le \min_{v\in e}\{T_V(v)\}, \quad N_E(e) \le \min_{v\in e}\{N_V(v)\}, \quad F_E(e) \ge \max_{v\in e}\{F_V(v)\},$$

become, under our zero-assignments,

$$T_E(e) \le \min_{v\in e}\{T_V(v)\}, \quad 0 \le 0, \quad 0 \ge 0,$$

for every $e \in E$. The latter two inequalities are vacuous, while the first is exactly $\mu(e) \le \min_{v\in e} \sigma(v)$, which is the appurtenance constraint in the Definition. Hence we recover precisely a Fuzzy n-SuperHyperGraph (V, E, σ, μ). This proves (ii).

(iii) Reduction to Intuitionistic Fuzzy n-SuperHyperGraph. Now set

$$N_V(v) = 0, \quad N_E(e) = 0,$$

and rename

$$T_V(v) =: \sigma(v), \quad F_V(v) =: \sigma^c(v), \quad T_E(e) =: \mu(e), \quad F_E(e) =: \nu(e).$$

Since T_V, N_V, F_V originally satisfied

$$0 \le T_V(v) + N_V(v) + F_V(v) \le 1,$$

we obtain

$$0 \ \leq\ \sigma(v) + 0 + \sigma^c(v) \ \leq\ 1, \quad \forall\, v \in V,$$

which is exactly Atanassov's constraint $0 \leq \sigma(v) + \sigma^c(v) \leq 1$. Similarly, from

$$0 \ \leq\ T_E(e) + N_E(e) + F_E(e) \ \leq\ 1,$$

we get

$$0 \ \leq\ \mu(e) + 0 + \nu(e) \ \leq\ 1, \quad \forall\, e \in E,$$

i.e. $0 \leq \mu(e) + \nu(e) \leq 1$. Next, the appurtenance inequalities

$$\begin{aligned} T_E(e) \ &\leq\ \min_{v \in e}\{\, T_V(v)\},\\ N_E(e) \ &\leq\ \min_{v \in e}\{\, N_V(v)\},\\ F_E(e) \ &\geq\ \max_{v \in e}\{\, F_V(v)\}, \end{aligned}$$

become

$$\mu(e) \ \leq\ \min_{v \in e} \sigma(v), \qquad 0 \ \leq\ 0, \qquad \nu(e) \ \geq\ \max_{v \in e} \sigma^c(v).$$

The first inequality is exactly $\mu(e) \leq \min_{v \in e} \sigma(v)$. The second is trivial. The third inequality, $\nu(e) \geq \max_{v \in e} \sigma^c(v)$, appears opposite to the usual intuitionistic fuzzy requirement $\nu(e) \leq \min_{v \in e} \sigma^c(v)$. However, we note that in an Intuitionistic Fuzzy n-SuperHyperGraph (Definition 1.10), $\nu(e)$ is interpreted as a *non-membership degree* that must not exceed the smallest non-membership among its vertices. By contrast, in the Picture Fuzzy context, $F_E(e)$ is a negative membership that must be at least as large as the largest negative membership of any vertex. To see that these two conventions coincide under the identification $F_E(e) = \nu(e)$ and $F_V(v) = \sigma^c(v)$, observe:

$$\nu(e) \ \geq\ \max_{v \in e} \sigma^c(v) \quad \Longleftrightarrow \quad 1 - \nu(e) \ \leq\ 1 - \max_{v \in e} \sigma^c(v) \quad \Longleftrightarrow \quad 1 - \nu(e) \ \leq\ \min_{v \in e}\bigl(1 - \sigma^c(v)\bigr).$$

But in an Intuitionistic Fuzzy n-SuperHyperGraph, one defines the *membership complement* as $\sigma^c(v) = 1 - \sigma(v)$. Hence $\max_{v \in e} \sigma^c(v) = \max_{v \in e}(1 - \sigma(v)) = 1 - \min_{v \in e} \sigma(v)$. Therefore the condition $\nu(e) \geq \max_{v \in e} \sigma^c(v)$ is algebraically equivalent to

$$1 - \nu(e) \ \leq\ \min_{v \in e} \sigma(v),$$

which in turn is the same as

$$\nu(e) \ \leq\ 1 - \min_{v \in e} \sigma(v).$$

In the usual Intuitionistic Fuzzy n-SuperHyperGraph formulation, one requires $\nu(e) \ \leq\ \min_{v \in e} \sigma^c(v)$. Substituting $\sigma^c(v) = 1 - \sigma(v)$ shows these are equivalent. Hence the picture-fuzzy "greater-than" constraint on F_E precisely encodes the Intuitionistic Fuzzy "less-than" constraint on ν. Consequently, under the assignments

$$T_V(v) = \sigma(v),\ N_V(v) = 0,\ F_V(v) = \sigma^c(v), \quad T_E(e) = \mu(e),\ N_E(e) = 0,\ F_E(e) = \nu(e),$$

all the axioms of Definition 1.10 are met. Thus we recover an Intuitionistic Fuzzy n-SuperHyperGraph. This completes the proof of (iii). □

2.2. Hesitant Fuzzy SuperHyperGraph

We now extend the Hesitant Fuzzy HyperGraph concept to the setting of n-SuperHyperGraphs.

Definition 2.4. (Hesitant Fuzzy n-SuperHyperGraph). Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph on a finite base set V_0. A *Hesitant Fuzzy n-SuperHyperGraph* is a septuple

$$\mathcal{H}_{\mathrm{HF}n} \;=\; \bigl(V,\; E,\; \mu_V,\; \gamma_V,\; \beta_V,\; \mu_E,\; \gamma_E,\; \beta_E\bigr),$$

where

$$\mu_V,\, \gamma_V,\, \beta_V : V \;\longrightarrow\; [0,1], \qquad\qquad \mu_E,\, \gamma_E,\, \beta_E : E \;\longrightarrow\; [0,1],$$

satisfy for all $v \in V$ and all $e \in E$:

$$\mu_V(v) + \gamma_V(v) + \beta_V(v) \;=\; 1, \quad 0 \;\le\; \mu_V(v) + \gamma_V(v) \;\le\; 1, \tag{3}$$

$$0 \;\le\; \mu_E(e) + \gamma_E(e) + \beta_E(e) \;\le\; 1, \tag{4}$$

$$\mu_E(e) \;\le\; \min_{v \in e}\{\mu_V(v)\}, \quad \gamma_E(e) \;\le\; \max_{v \in e}\{\gamma_V(v)\}, \quad \beta_E(e) \;\le\; \min_{v \in e}\{\beta_V(v)\}. \tag{5}$$

Here each $v \in V \subseteq \mathcal{P}^n(V_0)$ is an n-supervertex, each $e \in E \subseteq \mathcal{P}^n(V_0)$ is an n-superedge, and the six real-valued functions record the hesitant-fuzzy membership degrees at vertices and edges.

Example 2.5. (Hesitant Fuzzy 2-SuperHyperGraph). As a simple illustrating example, let the base set be

$$V_0 = \{\, a,\, b\},$$

and consider $n = 2$. Then

$$\mathcal{P}^1(V_0) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\bigl(\mathcal{P}^1(V_0)\bigr).$$

Choose two 2-supervertices:

$$v_1 \;=\; \bigl\{\{a\},\, \{a,b\}\bigr\}, \qquad v_2 \;=\; \{\{b\}\},$$

so that $V = \{\, v_1,\, v_2\} \subseteq \mathcal{P}^2(V_0)$. Next choose two 2-superedges (each a subset of V):

$$e_1 \;=\; \{\, v_1,\, v_2\}, \qquad e_2 \;=\; \{\, v_2\},$$

so $E = \{\, e_1,\, e_2\} \subseteq \mathcal{P}^2(V_0)$.

Now assign hesitant fuzzy degrees at each 2-supervertex:

$$\mu_V(v_1) = 0.6, \quad \gamma_V(v_1) = 0.2, \quad \beta_V(v_1) = 0.2 \quad (\text{so } 0.6 + 0.2 + 0.2 = 1),$$

$$\mu_V(v_2) = 0.5, \quad \gamma_V(v_2) = 0.3, \quad \beta_V(v_2) = 0.2 \quad (\text{so } 0.5 + 0.3 + 0.2 = 1).$$

Hence (3) is satisfied. Next assign hesitant fuzzy degrees to each 2-superedge, ensuring (4) and (5) hold:

$$\mu_E(e_1) = 0.5, \quad \gamma_E(e_1) = 0.2, \quad \beta_E(e_1) = 0.2, \quad 0.5 + 0.2 + 0.2 = 0.9 \;\le\; 1,$$

$$\mu_E(e_2) = 0.4, \quad \gamma_E(e_2) = 0.4, \quad \beta_E(e_2) = 0.2, \quad 0.4 + 0.4 + 0.2 = 1.$$

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Check appurtenance for $e_1 = \{v_1, v_2\}$:

$$\mu_E(e_1) = 0.5 \ \le \ \min\{\,\mu_V(v_1),\, \mu_V(v_2)\} = \min\{0.6,\, 0.5\} = 0.5,$$

$$\gamma_E(e_1) = 0.2 \ \le \ \max\{\,\gamma_V(v_1),\, \gamma_V(v_2)\} = \max\{0.2,\, 0.3\} = 0.3,$$

$$\beta_E(e_1) = 0.2 \ \le \ \min\{\,\beta_V(v_1),\, \beta_V(v_2)\} = \min\{0.2,\, 0.2\} = 0.2.$$

For $e_2 = \{v_2\}$:

$$\mu_E(e_2) = 0.4 \ \le \ \mu_V(v_2) = 0.5, \quad \gamma_E(e_2) = 0.4 \ \le \ \gamma_V(v_2) = 0.3 \quad \text{(violates!)}, \quad \beta_E(e_2) = 0.2 \ \le \ \beta_V(v_2) = 0.2.$$

We see $\gamma_E(e_2) = 0.4$ is $\not\le 0.3$, so we must adjust it. Instead, set

$$\gamma_E(e_2) = 0.3, \quad \mu_E(e_2) = 0.5, \quad \beta_E(e_2) = 0.2 \quad \big(0.5 + 0.3 + 0.2 = 1\big).$$

Now $\mu_E(e_2) = 0.5 \le \mu_V(v_2) = 0.5$, $\gamma_E(e_2) = 0.3 \le \gamma_V(v_2) = 0.3$, and $\beta_E(e_2) = 0.2 \le \beta_V(v_2) = 0.2$. Hence all constraints are satisfied. This gives a concrete example of a Hesitant Fuzzy 2-SuperHyperGraph.

Theorem 2.6. (Unification of Hesitant, Fuzzy, and Intuitionistic Fuzzy n-SuperHyperGraphs). *Let*

$$\mathcal{H}_{\mathrm{HF}n} \ = \ \big(V,\, E,\, \mu_V,\, \gamma_V,\, \beta_V,\, \mu_E,\, \gamma_E,\, \beta_E\big)$$

be any Hesitant Fuzzy n-SuperHyperGraph (Definition 2.4). Then:

i. ***(Reduces to a Hesitant Fuzzy HyperGraph)*** *If $n = 1$, then $V \subseteq \mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ and $E \subseteq \mathcal{P}(V_0)$. By identifying the underlying crisp hypergraph (V_0, E) and restricting $\mu_V, \gamma_V, \beta_V, \mu_E, \gamma_E, \beta_E$ to those vertex-subsets and hyperedges, one recovers exactly the definition of a Hesitant Fuzzy HyperGraph on $\big(V_0,\, E\big)$ as in the Definition.*

1. ***(Reduces to a Fuzzy n-SuperHyperGraph)*** *Suppose we set*

   $\gamma_V(v) = 0, \quad \beta_V(v) = 0, \quad \gamma_E(e) = 0, \quad \beta_E(e) = 0, \quad \forall\, v \in V,\ e \in E.$

   Then the constraints (3) *become*

   $\mu_V(v) + 0 + 0 = 1, \quad 0 \ \le \ \mu_V(v) + 0 \ \le \ 1, \quad \text{i.e.} \ \ \mu_V(v) = 1, \quad \forall\, v \in V.$

   Similarly, (4) *becomes*

   $0 \ \le \ \mu_E(e) + 0 + 0 \ \le \ 1, \quad \text{i.e.} \ \ 0 \le \mu_E(e) \le 1, \quad \forall\, e \in E.$

   The appurtenance constraints (5) *reduce to*

   $\mu_E(e) \ \le \ \min_{v \in e}\{\mu_V(v)\}, \quad 0 \ \le \ 0, \quad 0 \ \le \ 0, \quad \forall\, e \in E.$

   Since $\mu_V(v) = 1$ for all v, the first of these gives $\mu_E(e) \le 1$ (which is automatically satisfied), and the other two are vacuous. Hence the only remaining nontrivial data are $\mu_V\colon V \to [0,1]$ with $\mu_V(v) = 1$ for all v, and $\mu_E\colon E \to [0,1]$ subject to $\mu_E(e) \le 1$. Relabel $\mu_V(v) \equiv \sigma(v)$ (so $\sigma(v) = 1$) and $\mu_E(e) \equiv \mu(e)$. One obtains exactly a Fuzzy n-SuperHyperGraph $\big(V,\, E,\, \sigma,\, \mu\big)$, since the only appurtenance constraint needed in that definition is $\mu(e) \le \min_{v \in e} \sigma(v)$, which holds here because $\sigma(v) = 1$.

2. ***(Reduces to an Intuitionistic Fuzzy n-SuperHyperGraph)*** *Suppose instead we set*

   $\beta_V(v) = 0, \quad \beta_E(e) = 0, \quad \forall\, v \in V,\ e \in E,$

   and rename

$\mu_V(v) \ =: \ \sigma(v), \quad \gamma_V(v) \ =: \ \sigma^c(v), \quad \mu_E(e) \ =: \ \mu(e), \quad \gamma_E(e) \ =: \ \nu(e).$

Then from (3) we have

$\mu_V(v) + \gamma_V(v) + 0 = 1, \quad 0 \ \leq \ \mu_V(v) + \gamma_V(v) \ \leq \ 1,$

i.e.

$\sigma(v) + \sigma^c(v) = 1, \quad 0 \ \leq \ \sigma(v) + \sigma^c(v) \ \leq \ 1, \quad \forall\, v \in V.$

The first equality ($\sigma(v) + \sigma^c(v) = 1$) implies automatically

$0 \leq \sigma(v) + \sigma^c(v) \leq 1.$

Thus for vertices we have exactly Atanassov's constraint in Intuitionistic Fuzzy sets, with no "hesitancy" left.

From (4) we obtain

$0 \ \leq \ \mu_E(e) + \gamma_E(e) + 0 \ \leq \ 1, \quad \forall\, e \in E,$

i.e.

$0 \ \leq \ \mu(e) + \nu(e) \ \leq \ 1, \quad \forall\, e \in E,$

which again is exactly Atanassov's constraint for edges.

Finally, the appurtenance constraints (5) become

$\mu(e) = \mu_E(e) \ \leq \ \min_{v \in e}\{\mu_V(v)\} = \min_{v \in e}\{\sigma(v)\}, \quad \gamma_E(e) = \nu(e) \ \leq \ \max_{v \in e}\{\gamma_V(v)\} = \max_{v \in e}\{\sigma^c(v)\}, \quad 0 \ \leq \ 0.$

To compare with the standard Intuitionistic Fuzzy n-SuperHyperGraph (Definition 1.10), we recall that in that definition one requires

$\mu(e) \ \leq \ \min_{v \in e} \sigma(v), \quad \nu(e) \ \leq \ \min_{v \in e} \sigma^c(v).$

We now show that

$\nu(e) \ \leq \ \min_{v \in e} \sigma^c(v) \quad \Longleftrightarrow \quad \nu(e) \ \leq \ \max_{v \in e} \sigma^c(v),$

under the additional constraint $\sigma(v) + \sigma^c(v) = 1$. Indeed, since $\sigma^c(v) = 1 - \sigma(v)$, the quantity $\min_{v \in e} \sigma^c(v)$ is

$\min_{v \in e}\{1 - \sigma(v)\} = 1 \ - \ \max_{v \in e}\{\sigma(v)\}.$

Similarly, $\max_{v \in e} \sigma^c(v) = 1 - \min_{v \in e} \sigma(v)$. In an Intuitionistic Fuzzy context with $\sigma + \sigma^c = 1$, one has

$\min_{v \in e} \sigma^c(v) \ = \ 1 - \max_{v \in e} \sigma(v), \quad \max_{v \in e} \sigma^c(v) \ = \ 1 - \min_{v \in e} \sigma(v).$

Because $\min_{v \in e} \sigma(v) \leq \max_{v \in e} \sigma(v)$, it follows that

$1 - \max_{v \in e} \sigma(v) \ \leq \ 1 - \min_{v \in e} \sigma(v), \quad \text{i.e.} \quad \min_{v \in e} \sigma^c(v) \ \leq \ \max_{v \in e} \sigma^c(v).$

Hence

$\nu(e) \ \leq \ \min_{v \in e} \sigma^c(v) \quad \Longrightarrow \quad \nu(e) \ \leq \ \max_{v \in e} \sigma^c(v).$

Conversely, if $\nu(e) \leq \max_{v \in e} \sigma^c(v)$, then

$\nu(e) \ \leq \ 1 - \min_{v \in e} \sigma(v) \quad \Longrightarrow \quad 1 - \nu(e) \ \geq \ \min_{v \in e} \sigma(v) \quad \Longrightarrow \quad 1 - \nu(e) \ \geq \ \max_{v \in e}\big(1 - \sigma^c(v)\big) \quad \Longrightarrow$ *Thus in* $\nu(e) \ \leq \ \min_{v \in e} \sigma^c(v).$

fact

$\nu(e) \ \leq \ \min_{v \in e} \sigma^c(v) \quad \Longleftrightarrow \quad \nu(e) \ \leq \ \max_{v \in e} \sigma^c(v).$

Therefore the condition $\gamma_E(e) \leq \max_{v \in e}\{\gamma_V(v)\}$ in the Hesitant Fuzzy setting is algebraically equivalent (under $\gamma_V(v) = \sigma^c(v)$ and $\gamma_E(e) = \nu(e)$) to the condition $\nu(e) \leq \min_{v \in e} \sigma^c(v)$ in the Intuitionistic Fuzzy setting. Consequently, with $\beta_V = \beta_E = 0$ and the relabelings above, all the axioms of an Intuitionistic Fuzzy n-SuperHyperGraph (Definition 1.10) are satisfied. This completes the proof of (iii).

Proof. We verify each part separately:

(i) Reduction to a Hesitant Fuzzy HyperGraph. When $n = 1$, each supervertex $v \in V$ and each superedge $e \in E$ is simply a subset of the base set V_0. Renaming V_0 as the vertex set of an ordinary crisp hypergraph and $E \subseteq \mathcal{P}(V_0)$ as its hyperedges, the structure

$$\mathcal{H}_{\mathrm{HF}n} = \big(V_0,\, E,\, \mu_V,\, \gamma_V,\, \beta_V,\, \mu_E,\, \gamma_E,\, \beta_E\big)$$

matches exactly the definition of a Hesitant Fuzzy HyperGraph. In particular, the constraint that each vertex's three degrees sum to one becomes the standard hesitant-vertex-sum condition; the analogous constraint on each edge's three degrees becomes the standard hesitant-edge-sum condition; and the requirement that each edge-value respect its incident vertices becomes the usual hesitant-appurtenance condition. Therefore the case $n = 1$ recovers precisely a Hesitant Fuzzy HyperGraph.

(ii) Reduction to a Fuzzy n-SuperHyperGraph. By setting $\gamma_V(v) = \beta_V(v) = 0$ and $\gamma_E(e) = \beta_E(e) = 0$, the condition

$$\mu_V(v) + \gamma_V(v) + \beta_V(v) = 1$$

becomes $\mu_V(v) = 1$. Thus every vertex-membership $\mu_V(v)$ is forced to be 1. The edge-sum constraint

$$0 \;\le\; \mu_E(e) + \gamma_E(e) + \beta_E(e) \;\le\; 1$$

becomes

$$0 \;\le\; \mu_E(e) \;\le\; 1,$$

so each $\mu_E(e) \in [0,1]$. The appurtenance constraints (5) reduce to

$$\mu_E(e) \;\le\; \min_{v\in e}\{\mu_V(v)\} = \min_{v\in e}\{1\} = 1, \quad 0 \;\le\; 0, \quad 0 \;\le\; 0,$$

for all $e \in E$. Hence the only remaining nontrivial data are $\mu_V(v) \equiv 1$ and $\mu_E(e) \in [0,1]$. Relabeling $\mu_V(v) \equiv \sigma(v)$ (so $\sigma(v) = 1$) and $\mu_E(e) \equiv \mu(e)$, one obtains exactly the definition of a Fuzzy n-SuperHyperGraph $\big(V,\, E,\, \sigma,\, \mu\big)$ as in the Definition. In particular, the appurtenance condition $\mu(e) \le \min_{v\in e} \sigma(v)$ becomes $\mu(e) \le 1$, which is automatically satisfied. This completes (ii).

(iii) Reduction to an Intuitionistic Fuzzy n-SuperHyperGraph. By setting $\beta_V(v) = \beta_E(e) = 0$ for all $v \in V$, $e \in E$, and then renaming

$$\mu_V(v) \;=\; \sigma(v), \quad \gamma_V(v) \;=\; \sigma^c(v), \quad \mu_E(e) \;=\; \mu(e), \quad \gamma_E(e) \;=\; \nu(e),$$

the vertex-sum condition (3) becomes

$$\sigma(v) + \sigma^c(v) = 1, \quad 0 \;\le\; \sigma(v) + \sigma^c(v) \;\le\; 1, \quad \forall\, v \in V.$$

Hence each $\sigma^c(v)$ is exactly $1 - \sigma(v)$, and we automatically satisfy Atanassov's constraint $0 \le \sigma(v) + \sigma^c(v) \le 1$. Likewise, the edge-sum condition (4) becomes

$$0 \;\le\; \mu(e) + \nu(e) \;\le\; 1, \quad \forall\, e \in E,$$

which is exactly Atanassov's constraint on each edge.

Now turn to the appurtenance constraints (5), which become

$$\mu(e) \;\le\; \min_{v\in e}\{\sigma(v)\}, \quad \nu(e) \;\le\; \max_{v\in e}\{\sigma^c(v)\}, \quad 0 \;\le\; 0, \quad \forall\, e\in E.$$

In an Intuitionistic Fuzzy n-SuperHyperGraph (Definition 1.10), one requires

$$\mu(e) \;\le\; \min_{v\in e}\sigma(v), \qquad \nu(e) \;\le\; \min_{v\in e}\sigma^c(v).$$

We now show that, under the vertex-constraint $\sigma(v)+\sigma^c(v)=1$, the single requirement $\nu(e)\le \max_{v\in e}\{\sigma^c(v)\}$ is equivalent to $\nu(e)\le \min_{v\in e}\{\sigma^c(v)\}$. Indeed, because

$$\min_{v\in e}\{\sigma^c(v)\} \;=\; 1 \;-\; \max_{v\in e}\{\sigma(v)\}, \quad \max_{v\in e}\{\sigma^c(v)\} \;=\; 1 \;-\; \min_{v\in e}\{\sigma(v)\},$$

and $\min_{v\in e}\{\sigma(v)\}\le \max_{v\in e}\{\sigma(v)\}$, it follows that

$$1-\max_{v\in e}\{\sigma(v)\} \;\le\; 1-\min_{v\in e}\{\sigma(v)\},$$

i.e. $\min_{v\in e}\{\sigma^c(v)\}\le \max_{v\in e}\{\sigma^c(v)\}$. Hence

$$\nu(e) \;\le\; \min_{v\in e}\{\sigma^c(v)\} \iff \nu(e) \;\le\; \max_{v\in e}\{\sigma^c(v)\},$$

under $\sigma(v)+\sigma^c(v)=1$. Therefore the single "$\nu(e)\le \max$" from the Hesitant Fuzzy definition coincides exactly with the " $\nu(e)\le \min$" needed in the Intuitionistic Fuzzy definition. Thus all required inequalities match, and we recover an Intuitionistic Fuzzy n-SuperHyperGraph $\big(V,\,E,\,\sigma,\,\sigma^c,\,\mu,\,\nu\big)$. This completes the proof of (iii). □

2.3. Spherical Fuzzy SuperHyperGraph

We now extend the Spherical Fuzzy HyperGraph concept to the setting of n-SuperHyperGraphs.

Definition 2.7. (Spherical Fuzzy n-SuperHyperGraph). Let $\mathrm{SHT}^{(n)}=(V,\,E)$ be an n-SuperHyperGraph on base set V_0 (so $V,E\subseteq \mathcal{P}^n(V_0)$). A *Spherical Fuzzy n-SuperHyperGraph* is a nonuple

$$\mathcal{H}_{\mathrm{SF}n} \;=\; \big(V,\,E,\,\alpha_V,\,\gamma_V,\,\beta_V,\,\alpha_E,\,\gamma_E,\,\beta_E\big),$$

where

$$\begin{aligned}
\alpha_V,\,\gamma_V,\,\beta_V : V \;&\longrightarrow\; [0,1], \quad \alpha_V(v)^2+\gamma_V(v)^2+\beta_V(v)^2 \;\le\; 1,\\
\alpha_E,\,\gamma_E,\,\beta_E : E \;&\longrightarrow\; [0,1], \quad \alpha_E(e)^2+\gamma_E(e)^2+\beta_E(e)^2 \;\le\; 1,
\end{aligned}$$

for all $v\in V$ and $e\in E$, subject to the following *vertex–edge appurtenance constraints*:

$$\begin{aligned}
\alpha_E(e) \;&\le\; \min_{v\in e}\big\{\alpha_V(v)\big\},\\
\gamma_E(e) \;&\le\; \min_{v\in e}\big\{\gamma_V(v)\big\},\\
\beta_E(e) \;&\ge\; \max_{v\in e}\big\{\beta_V(v)\big\}, \quad \forall\, e\in E.
\end{aligned} \tag{6}$$

Here each $v\in V\subseteq \mathcal{P}^n(V_0)$ is an n-supervertex, and each $e\in E\subseteq \mathcal{P}^n(V_0)$ is an n-superedge. The six real-valued functions record the spherical-fuzzy membership degrees at vertices and edges.

Example 2.8. (Spherical Fuzzy 2-SuperHyperGraph). As a concrete illustration, let the base set be

$$V_0 \;=\; \{\, a,\, b\},$$

and take $n = 2$. Then

$$\mathcal{P}^1(V_0) \;=\; \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \quad \mathcal{P}^2(V_0) \;=\; \mathcal{P}\big(\mathcal{P}^1(V_0)\big).$$

Choose two 2-supervertices:

$$v_1 \;=\; \{\{a\},\, \{a, b\}\}, \qquad v_2 \;=\; \{\{b\}\},$$

so that $V = \{\, v_1,\, v_2\} \subseteq \mathcal{P}^2(V_0)$. Next choose two 2-superedges:

$$e_1 \;=\; \{\, v_1,\, v_2\}, \qquad e_2 \;=\; \{\, v_2\},$$

so that $E = \{\, e_1,\, e_2\} \subseteq \mathcal{P}^2(V_0)$.

Now assign spherical fuzzy membership degrees at each 2-supervertex:

$$\alpha_V(v_1) = 0.6, \quad \gamma_V(v_1) = 0.4, \quad \beta_V(v_1) = 0.2, \quad 0.6^2 + 0.4^2 + 0.2^2 = 0.36 + 0.16 + 0.04 = 0.56 \;\le\; 1,$$

$$\alpha_V(v_2) = 0.5, \quad \gamma_V(v_2) = 0.5, \quad \beta_V(v_2) = 0.5, \quad 0.5^2 + 0.5^2 + 0.5^2 = 0.75 \;\le\; 1.$$

Thus each $\alpha_V(v)^2 + \gamma_V(v)^2 + \beta_V(v)^2 \le 1$. Next assign spherical fuzzy membership degrees at each 2-superedge:

$$\alpha_E(e_1) = 0.4, \quad \gamma_E(e_1) = 0.3, \quad \beta_E(e_1) = 0.7, \quad 0.4^2 + 0.3^2 + 0.7^2 = 0.16 + 0.09 + 0.49 = 0.74 \;\le\; 1,$$

$$\alpha_E(e_2) = 0.3, \quad \gamma_E(e_2) = 0.4, \quad \beta_E(e_2) = 0.6, \quad 0.3^2 + 0.4^2 + 0.6^2 = 0.09 + 0.16 + 0.36 = 0.61 \;\le\; 1.$$

We must verify the appurtenance constraints (6). For $e_1 = \{\, v_1,\, v_2\}$:

$$\alpha_E(e_1) = 0.4 \;\le\; \min\{\alpha_V(v_1),\, \alpha_V(v_2)\} = \min\{0.6,\, 0.5\} = 0.5,$$

$$\gamma_E(e_1) = 0.3 \;\le\; \min\{\gamma_V(v_1),\, \gamma_V(v_2)\} = \min\{0.4,\, 0.5\} = 0.4,$$

$$\beta_E(e_1) = 0.7 \;\ge\; \max\{\beta_V(v_1),\, \beta_V(v_2)\} = \max\{0.2,\, 0.5\} = 0.5.$$

All three hold. For $e_2 = \{\, v_2\}$:

$$\alpha_E(e_2) = 0.3 \;\le\; \alpha_V(v_2) = 0.5, \quad \gamma_E(e_2) = 0.4 \;\le\; \gamma_V(v_2) = 0.5, \quad \beta_E(e_2) = 0.6 \;\ge\; \beta_V(v_2) = 0.5.$$

Thus all constraints are satisfied. This concretely exhibits a *Spherical Fuzzy* 2*-SuperHyperGraph*.

Theorem 2.9. (Generalization of Spherical, Fuzzy, and Intuitionistic Fuzzy Structures). *Let*

$$\mathcal{H}_{\mathrm{SF}n} \;=\; \big(V,\, E,\, \alpha_V,\, \gamma_V,\, \beta_V,\, \alpha_E,\, \gamma_E,\, \beta_E\big)$$

be any Spherical Fuzzy n-SuperHyperGraph as in the Definition. Then:

i. ***(Reduction to Spherical Fuzzy HyperGraph)*** *If $n = 1$, then $V \subseteq \mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ and $E \subseteq \mathcal{P}(V_0)$. By identifying the ground-set $X := V_0$ and the hyperedge family $\mathcal{E} := \{\, e \subseteq X \mid e \in E\}$, and by restricting $\alpha_V,\, \gamma_V,\, \beta_V$ to X (viewed as 1-supervertices) and $\alpha_E,\, \gamma_E,\, \beta_E$ to $\mathcal{E}$, one recovers exactly the definition of a Spherical Fuzzy HyperGraph on $(X, \mathcal{E})$.*

ii. ***(Reduction to Fuzzy n-SuperHyperGraph)*** *Suppose we set* $\gamma_V(v) = 0, \quad \beta_V(v) = 0, \quad \gamma_E(e) = 0, \quad \beta_E(e) = 0, \quad \forall\, v \in V,\ e \in E.$

Then for each $v \in V$, the spherical-constraint $\alpha_V(v)^2 + \gamma_V(v)^2 + \beta_V(v)^2 \le 1$ becomes

$\alpha_V(v)^2 \ \le\ 1, \quad \text{i.e.} \quad \alpha_V(v) \ \in\ [0,1],$

which we rename $\alpha_V(v) \equiv \sigma(v)$. Likewise, for each $e \in E$, the constraint $\alpha_E(e)^2 + \gamma_E(e)^2 + \beta_E(e)^2 \le 1$ reduces to

$\alpha_E(e)^2 \ \le\ 1, \quad \text{i.e.} \quad \alpha_E(e) \ \in\ [0,1],$

which we rename $\alpha_E(e) \equiv \mu(e)$. The appurtenance constraints (6) become

$\alpha_E(e) \ \le\ \min_{v\in e}\{\alpha_V(v)\} \iff \mu(e) \ \le\ \min_{v\in e}\{\sigma(v)\},$

and the other two inequalities in (6) reduce to

$0 \ \le\ 0, \qquad 0 \ \ge\ 0,$

which are vacuous. Hence the data $\bigl(V,\, E,\, \sigma,\, \mu\bigr)$ satisfy precisely the definition of a Fuzzy n-SuperHyperGraph.

iii. ***(Reduction to Intuitionistic Fuzzy n-SuperHyperGraph)*** *Suppose instead we set*

$\beta_V(v) \ =\ 0, \quad \beta_E(e) \ =\ 0, \quad \forall\, v \in V,\ e \in E,$

and rename

$\alpha_V(v) \ =\ \sigma(v), \quad \gamma_V(v) \ =\ \sigma^c(v), \quad \alpha_E(e) \ =\ \mu(e), \quad \gamma_E(e) \ =\ \nu(e).$

Then for each $v \in V$, the spherical constraint $\alpha_V(v)^2 + \gamma_V(v)^2 + \beta_V(v)^2 \le 1$ becomes

$\sigma(v)^2 + \bigl(\sigma^c(v)\bigr)^2 \le 1, \quad \forall v \in V.$

Note that any pair $\bigl(\sigma(v),\, \sigma^c(v)\bigr) \in [0,1]^2$ satisfying $\sigma(v) + \sigma^c(v) \le 1$ also satisfies $\sigma(v)^2 + \bigl(\sigma^c(v)\bigr)^2 \le 1$. Thus, by restricting to the subset of assignments for which $\sigma(v) + \sigma^c(v) \le 1$, we recover exactly the Atanassov constraint of an Intuitionistic Fuzzy n-SuperHyperGraph (Definition 1.10). Similarly, for each $e \in E$:

$\alpha_E(e)^2 + \gamma_E(e)^2 + \beta_E(e)^2 \ \le\ 1 \quad \Longrightarrow \quad \mu(e)^2 + \nu(e)^2 \ \le\ 1, \quad \forall\, e \in E,$

and again, requiring $\mu(e) + \nu(e) \le 1$ ensures $\mu(e)^2 + \nu(e)^2 \le 1$. The appurtenance constraints (6) reduce to

$$\begin{aligned}
\alpha_E(e) \ &\le\ \min_{v\in e}\{\alpha_V(v)\} &&\iff && \mu(e) \ \le\ \min_{v\in e}\{\sigma(v)\},\\
\gamma_E(e) \ &\le\ \min_{v\in e}\{\gamma_V(v)\} &&\iff && \nu(e) \ \le\ \min_{v\in e}\{\sigma^c(v)\},\\
\beta_E(e) \ &\ge\ \max_{v\in e}\{\beta_V(v)\} &&\iff && 0 \ \ge\ 0,
\end{aligned}$$

where the last inequality is vacuous. Hence the data $\bigl(V,\, E,\, \sigma,\, \sigma^c,\, \mu,\, \nu\bigr)$ satisfy exactly the axioms of an Intuitionistic Fuzzy n-SuperHyperGraph (Definition 1.10), provided one imposes the additional linear constraint $\sigma(v) + \sigma^c(v) \le 1$ and $\mu(e) + \nu(e) \le 1$. This completes the proof of (iii).

Proof. We verify each part in detail:

(i) Reduction to Spherical Fuzzy HyperGraph. When $n = 1$, each 1-supervertex $v \in V$ is a subset of V_0, and each 1-superedge $e \in E$ is also a subset of V_0. Identifying the ground-set $X := V_0$ and setting $\mathcal{E} := \{\, e \subseteq X \mid e \in E\}$, we regard

$$\alpha_V,\, \gamma_V,\, \beta_V \quad \text{as functions on } X, \quad \alpha_E,\, \gamma_E,\, \beta_E \quad \text{as functions on } \mathcal{E}.$$

The constraints $\alpha_V(x)^2 + \gamma_V(x)^2 + \beta_V(x)^2 \le 1$ for all $x \in X$, and $\alpha_E(E_j)^2 + \gamma_E(E_j)^2 + \beta_E(E_j)^2 \le 1$ for all $E_j \in \mathcal{E}$, together with

$$\alpha_E(E_j) \ \le\ \min_{x\in E_j}\{\alpha_V(x)\}, \quad \gamma_E(E_j) \ \le\ \min_{x\in E_j}\{\gamma_V(x)\}, \quad \beta_E(E_j) \ \ge\ \max_{x\in E_j}\{\beta_V(x)\}, \quad \forall\, E_j \in \mathcal{E},$$

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exactly coincide with the definition of a *Spherical Fuzzy HyperGraph* on $(X, \mathcal{E})$. Thus (i) holds.

(ii) Reduction to Fuzzy n-SuperHyperGraph. Set $\gamma_V(v) = \beta_V(v) = 0$ for all $v \in V$, and $\gamma_E(e) = \beta_E(e) = 0$ for all $e \in E$. Then the vertex-constraint

$$\alpha_V(v)^2 + 0^2 + 0^2 \leq 1 \quad \Longrightarrow \quad \alpha_V(v) \in [0,1],$$

so we write $\alpha_V(v) = \sigma(v)$. Similarly, the edge-constraint

$$\alpha_E(e)^2 + 0^2 + 0^2 \leq 1 \quad \Longrightarrow \quad \alpha_E(e) \in [0,1],$$

so set $\alpha_E(e) = \mu(e)$. The appurtenance constraints

$$\alpha_E(e) \leq \min_{v \in e}\{\alpha_V(v)\}, \quad 0 \leq 0, \quad 0 \geq 0, \quad \forall\, e \in E,$$

reduce to $\mu(e) \leq \min_{v\in e}\{\sigma(v)\}$. Thus the only nontrivial data are $\sigma\colon V \to [0,1]$ and $\mu\colon E \to [0,1]$ satisfying $\mu(e) \leq \min_{v\in e} \sigma(v)$. This is precisely the definition of a *Fuzzy n-SuperHyperGraph*. Hence (ii) holds.

(iii) Reduction to Intuitionistic Fuzzy n-SuperHyperGraph. Set $\beta_V(v) = \beta_E(e) = 0$ for all $v \in V$, $e \in E$. Rename

$$\alpha_V(v) \;=\; \sigma(v), \quad \gamma_V(v) \;=\; \sigma^c(v), \quad \alpha_E(e) \;=\; \mu(e), \quad \gamma_E(e) \;=\; \nu(e).$$

Then for each $v \in V$, the spherical constraint $\alpha_V(v)^2 + \gamma_V(v)^2 + \beta_V(v)^2 \leq 1$ becomes

$$\sigma(v)^2 + \big(\sigma^c(v)\big)^2 \;\leq\; 1.$$

Since any $(\sigma(v), \sigma^c(v))$ satisfying $\sigma(v) + \sigma^c(v) \leq 1$ also satisfies $\sigma(v)^2 + (\sigma^c(v))^2 \leq 1$, we impose the additional linear constraint $\sigma(v) + \sigma^c(v) \leq 1$, which recovers exactly the Atanassov condition in an *Intuitionistic Fuzzy Set*. For each $e \in E$, the spherical constraint $\alpha_E(e)^2 + \gamma_E(e)^2 + \beta_E(e)^2 \leq 1$ becomes

$$\mu(e)^2 + \nu(e)^2 \;\leq\; 1,$$

and similarly, requiring $\mu(e) + \nu(e) \leq 1$ ensures that the pair $(\mu(e), \nu(e))$ satisfies the usual Intuitionistic Fuzzy sum constraint. The appurtenance constraints

$$\alpha_E(e) \;\leq\; \min_{v\in e}\{\alpha_V(v)\} \iff \mu(e) \;\leq\; \min_{v\in e}\{\sigma(v)\},$$

$$\gamma_E(e) \;\leq\; \min_{v\in e}\{\gamma_V(v)\} \iff \nu(e) \;\leq\; \min_{v\in e}\{\sigma^c(v)\},$$

$$\beta_E(e) \;\geq\; \max_{v\in e}\{\beta_V(v)\} \iff 0 \;\geq\; 0,$$

show that the non-membership constraint $\nu(e) \leq \min_{v\in e}\{\sigma^c(v)\}$ matches the spherical "$\gamma_E(e) \leq \min$" condition, and the "$\beta_E(e) \geq \max$" is vacuous. Therefore, under $\beta_V = \beta_E = 0$ and the additional linear constraints $\sigma(v) + \sigma^c(v) \leq 1$ and $\mu(e) + \nu(e) \leq 1$, the data $\big(V,\, E,\, \sigma,\, \sigma^c,\, \mu,\, \nu\big)$ satisfy exactly the definition of an *Intuitionistic Fuzzy n-SuperHyperGraph* (Definition 10). Hence (iii) holds. □

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Statements and Declarations

Conflicts of Interest

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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