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Euler-Lagrangian Approach to Fluid Dynamics and the Incompleteness of the Navier-Stokes Equations

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Abstract

Navier-Stokes equations are based on Newton's second law and the Stokes hypothesis. In this paper, we have derived the fluid dynamic equations by applying the powerful tool of the Euler-Lagrangian approach, based on the principle of least action. The new equation highlights the incompleteness of the Navier-Stokes equations. The main reason is that the Stokes hypothesis uses engineering shear strain concept (through an average procedure for shear strain) to model the viscous stresses instead of using the tensorial shear strain. The general velocity gradient (tensorial shear strains) contains stretch, shear, and rotation deformations. The average procedure, based on the Stokes hypothesis, can only partially account for the shear strains. This deficiency should be remedied by adding an extra term – a pure spin tensor. Geometric interpretations and geometric algebra explanations are provided to show this deficiency and its counterbalance. One of the notable findings is that, both fluid flow and electromagnetic fields are, in essence, the same. All of them can be described by the same mathematical tools.

1. Introduction

The Navier-Stokes equations express momentum balance for Newtonian fluids. They are based on the applying Newton's second law to the fluid motion, together with the Stokes hypothesis that the viscous stresses in the fluid are proportional to the first spatial derivatives of the flow velocity. The applied force on a fluid parcel is the viscous stress tensor plus a pressure gradient term.

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The Newtonian method and the Euler-Lagrangian approach are both powerful tools used in classical mechanics to analyze the motion of particles and systems.

Newton's laws of motion are based on the concept of forces acting on objects and how these forces affect the motion of those objects, on the other hand, Euler-Lagrange mechanics focuses on the concept of minimizing or maximizing a quantity known as the action based on the variational principle, instead of dealing directly with forces.

Despite the differences in formulation, Euler-Lagrange mechanics and Newton's laws are fundamentally equivalent in the sense that they both describe the dynamic behavior of classical mechanical systems: they can be used interchangeably to describe classical mechanical systems.

However, the Euler-Lagrangian approach has some advantages compared to the Newtonian method:

The Euler-Lagrangian formulation is based on the principle of least action, where the path followed by a system between two points in configuration space minimizes the action integral. This provides a powerful and elegant framework for deriving the equations of motion.

Another advantage to the Euler-Lagrangian formulation is that it naturally leads to the identification of conserved quantities such as energy, momentum, etc. through Noether's theorem. These conservation laws can be derived systematically from the symmetries of the Lagrangian, which is fundamental and common in many branches of physics.

Furthermore, the Euler-Lagrangian approach readily extends to systems with curvilinear coordinates or so-called generalized coordinates. This makes it to be capable to naturally handle non-inertial forces.

This paper is organized as follows: at first, we introduce the principle of least action and derive the equation of motion in vector form by the Euler-Lagrangian approach. Inspired by this approach, the fluid dynamic equation was given for a non-relativistic, incompressible fluid. The gradient of kinetic energy can be decomposed into two parts: symmetric and antisymmetric parts. The main point is that the Stokes hypothesis only account for the symmetric part and ignores the antisymmetric part. This leads to the Navier-Stokes equation being incomplete. In the following section, we give out some scenarios to explain this incompleteness with geometric interpretation and geometric algebra explanation. It has been further shown that the fluid dynamic equation is similar to the electrical particle behavior in the electromagnetic field. At last, the summary and conclusion are given.

2. The Principle of Least Action and the Euler-Lagrangian Equation

2.1 The Principle of Least Action

The Principle of Least Action, also known as Hamilton's Principle or the Action Principle, is a fundamental concept in physics that plays a central role in physics.

The principle states that the path taken by a physical system between two points in configuration space is the one for which the action is minimized, or, more precisely, stationary. The action, denoted by the symbol S, is a quantity defined as the integral of the Lagrangian over time:

$$S = \int_{t_1}^{t_2} L dt, \qquad (1)$$

where L is the Lagrangian, a function that describes the difference between the kinetic and potential energies of the system.

The action is a quantity that depends on the trajectory of the particle as it moves through spacetime. The least action principle says that if a system starts out at point and ends up at another point in spacetime, it will “choose” a particular kind of path among all the possible paths. Specifically, it chooses the path that minimizes the quantity we call action, Mathematically, the Principle of Least Action is expressed as Hamilton's stationary action principle:

$$\delta S = 0. \qquad (2)$$

This implies that the true path taken by the system is such that any infinitesimally small variation (denoted by δ in above equation) of the path leads to zero change in the action [1].

Hamilton’s principle is more general and permits a natural extension to continuum systems (fields), such as fluid flow field.

2.2 The Euler-Lagrangian Equation of Motion

Given a mechanical system in a field, we can define the Lagrangian density function in this field as:

$$\mathcal{L}(\vec{r}, \vec{v}, t). \qquad (3)$$

The system is assumed to occupy the positions $\vec{r}_1$ and $\vec{r}_2$ at time t_1 and t_2 in the field, respectively. We fix positions of $\vec{r}(t_1) = \vec{r}_1$ and $\vec{r}(t_2) = \vec{r}_2$ at the initial and final times.

The Lagrangian density is a function of positions, velocities and time, where the velocity is:

$$\vec{v} = \frac{d\vec{r}}{dt}. \qquad (4)$$

Given two instants t_1 and t_2, then, we define the action

$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{r}, \vec{v}, t) dx^3 dt. \tag{5}$$

In field, the Lagrangian can be expressed as spatial integral of the Lagrangian density function:

$$L = \int \mathcal{L} dx^3. \tag{6}$$

Principle of least action (or Hamilton's principle) says that from time t_1 to t_2 the system moves in such a way that S is a minimum (extremum) over all paths:

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(\vec{r}, \vec{v}, t) dx^3 dt = 0. \tag{7}$$

This implies that the true path taken by the system is such that any infinitesimally small variation (denoted by δS) of the path leads to zero change in the action.

$$\delta \int_{t_1}^{t_2} \mathcal{L}(\vec{r}, \vec{v}, t) dx dt = \int_{t_1}^{t_2} [\mathcal{L}(\vec{r} + \delta\vec{r}, \vec{v} + \delta\vec{v}, t) - \mathcal{L}(\vec{r}, \vec{v}, t)] dx^3 dt = 0. \tag{8}$$

Using the first order Taylor expansion approximation, we have

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial \vec{r}} \cdot \delta\vec{r} + \frac{\partial \mathcal{L}}{\partial \vec{v}} \cdot \delta\vec{v}\right] dx^3 dt = 0. \tag{9}$$

Using the chain rule and the product rule, the second term in the integrand can be written as:

$$\frac{\partial \mathcal{L}}{\partial \vec{v}} \cdot \delta\vec{v} = \frac{\partial \mathcal{L}}{\partial \vec{v}} \cdot \frac{d(\delta\vec{r})}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{v}} \cdot \delta\vec{r}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{v}}\right) \cdot \delta\vec{r}. \tag{10}$$

Thus, the integrand of the eq. (9) is

$$\frac{\partial \mathcal{L}}{\partial \vec{r}} \cdot \delta\vec{r} + \frac{\partial \mathcal{L}}{\partial \vec{v}} \cdot \delta\vec{v} = \frac{\partial \mathcal{L}}{\partial \vec{r}} \cdot \delta\vec{r} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{v}} \cdot \delta\vec{r}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{v}}\right) \cdot \delta\vec{r}. \tag{11}$$

Hence, equation (9) can be rewritten as:

$$\delta S = \left[\frac{\partial \mathcal{L}}{\partial \vec{v}} \cdot \delta\vec{r}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{\left[\frac{\partial \mathcal{L}}{\partial \vec{r}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{v}}\right)\right] \cdot \delta\vec{r}\right\} dx^3 dt = 0, \tag{12}$$

Since the variation $\delta\vec{r}$ is arbitrary, equation (20) is only verified, when the integrand

$$\frac{\partial \mathcal{L}}{\partial \vec{r}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{v}}\right) = 0, \tag{13}$$

Because the boundaries are fixed, namely, $\delta\vec{r}(t_1) = \delta\vec{r}(t_2) = 0$, as all possible paths are such that $\vec{r}(t_1) = \vec{r}_1$ and $\vec{r}(t_2) = \vec{r}_2$.

The equation (13) is known as Euler-Lagrange's equation. When the partial derivative of Lagrangian density with respect to velocity is written as

$$\frac{\partial \mathcal{L}}{\partial \vec{v}} = \vec{p} = \rho\vec{v}, \tag{14}$$

Euler-Lagrange's equation can be written more concisely:

$$D_t\vec{p} = \nabla\mathcal{L}. \tag{15}$$

LHS of the equation (15) is the total derivative of momentum with respect to time (also called material derivative).

Lagrangian density for a closed system (not affected by external forces, such as gravitational force or more precisely, gravitational potential, etc.) reads

$$\mathcal{L} = T(\vec{r}, \vec{v}, t) - V(\vec{r}, t), \tag{16}$$

where $V(\vec{r}, t)$ represents the potential energy density of the interacting particles in the system.

T is the kinetic energy density. If T is written explicitly in terms of the $\vec{v}$ and $\vec{r}$, then these equations are just the equations of motions in terms of the $\vec{r}$. Since $\frac{\partial V}{\partial \vec{v}} = 0$, as $V(\vec{r}, t)$ depends only on the $\vec{r}$, not the $\vec{v}$.

The Lagrangian density function (16) and the equation (14) are substituted into equation (13), then, it reads:

$$\frac{d(\rho\vec{v})}{dt} = \frac{\partial T}{\partial \vec{r}} - \frac{\partial V}{\partial \vec{r}}. \tag{17}$$

The RHS of equation (17) represents the force. It is exactly the Newton's second law.

Actually, we can also derivate the Euler-Lagrangian equation from the Newton's equations of motion [2,3,4].

3. Fluid Dynamic Equation in the Flow Field

The Euler-Lagrangian approach can be directly applied to the motion of fluid in flow field. For simplicity, we apply Cartesian coordinate for the equation of motion. For the sake of illustration, the fluid is assumed to be incompressible as an approximation. This assumption implies the pressure wave propagation speed is infinitely great. For three-dimensional flow, the system has 3 freedoms of motion.

The (non-relativistic) kinetic energy density of a system may be written as

$$T(\vec{r}, \vec{v}, t) = \frac{1}{2}\rho\vec{v}\cdot\vec{v}. \tag{18}$$

The potential energy V in fluid is the pressure energy field:

$$V(\vec{r}, t) = p(\vec{r}, t). \tag{19}$$

It depends merely on the position of $\vec{r}$. This is a consequence of the assumption that the fluid is incompressible. The interactions (disturbances) between particles are instantaneously propagating through the whole field: a change of the fluid particle in position *a* is instantaneously experienced by the other fluid particle in position *b*, through an infinitely great pressure wave propagation speed, namely $c \approx \infty$, or $\vec{v} << c$. If the fluid is compressible, the wave will propagate in the field by a finite wave speed, in this case, we should turn to the relativistic fluid dynamics [5].

Equations (14), (16), (18) and (19) are substituted into equation (13), we can get the equation of motion:

$$\frac{d}{dt}(\rho\vec{v}) = -\nabla p + \nabla T. \tag{20}$$

where p is the thermodynamic pressure energy (static pressure, p_{static}). The kinetic energy T of the fluid parcels is the form of energy that it possesses due to its motion. It represents the amount of energy transformed from potential energy into kinetic energy (consumption of potential energy in the system). When the kinetic energy or the kinetic energy gradient equals zero, P is then equal to the total (stagnation) pressure ($p_{stagnation}$). The physical meaning of equation (20) is clear: T is the mechanical energy transformation from the potential energy into the kinetic energy, and p is the remaining potential energy (static pressure) after the transformation. The negative gradient of the potential energy ($-\nabla p_{static}$) is the force, according to the definition. Potential and kinetic energy can be changed from one form into another. The collective effects (gradients) of both energies are equal to the net force acting on the fluid parcel.

3.1 Material Derivative and Convective Term

The LHS of equation (20) is the material derivative of momentum, it can be written as a partial derivative of momentum with respect to time plus a convective term:

$$\frac{d(\rho\vec{v})}{dt} = \frac{\partial(\rho\vec{v})}{\partial t} + (\vec{v}\cdot\nabla)(\rho\vec{v}). \tag{21}$$

The convective term can be written as the directional derivative of the momentum along the velocity vector $\vec{v}$ at a given point (x_0, y_0, z_0). It represents an instantaneous rate of change of the momentum in the direction $\vec{v}$. It is a second order tensor form, and can be defined as

$$\nabla_{\vec{v}}(\rho\vec{v}) = \nabla(\rho\vec{v})\frac{\vec{v}}{|\vec{v}|}. \tag{22}$$

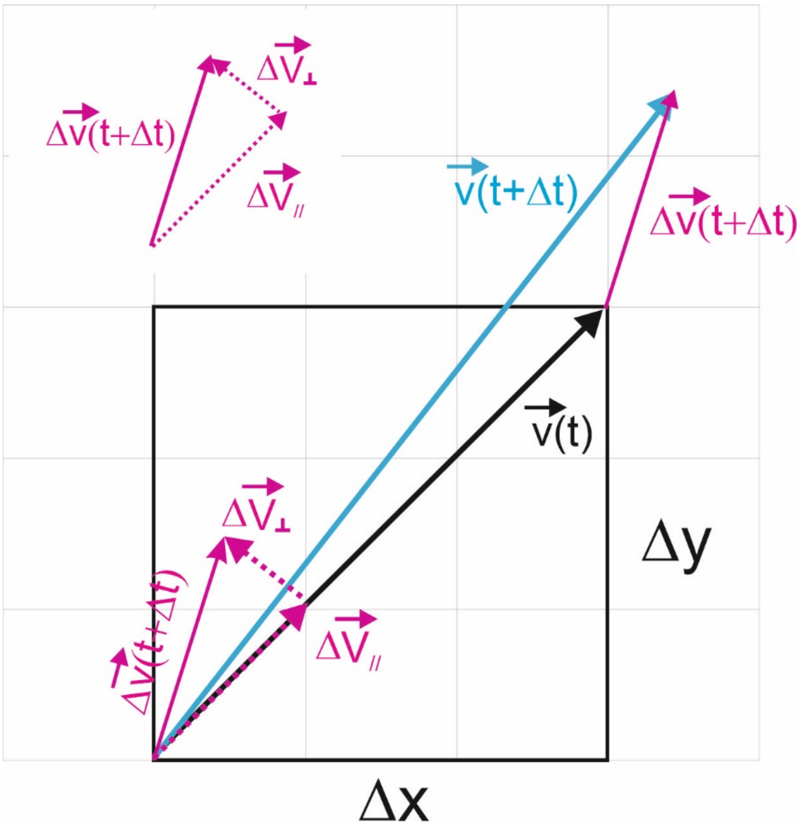

Fig. 1. The convective term $(\vec{v}\cdot\nabla)(\rho\vec{v}) = \nabla_{\vec{v}}(\rho\vec{v})\cdot\vec{v}$

The inner product of the equation (21) is the length of the momentum gradient tensor $\nabla(\rho\vec{v})$, projected onto $\vec{v}$, multiplied by the length of $\vec{v}$.

Geometrically, it will be equal to the product of the "projection" of the magnitude of $\nabla(\rho\vec{v})$ onto the $\vec{v}$ and multiplying the magnitude of the velocity vector $\vec{v}$, as shown by Fig. 1. Physically, it can be interpreted as the instantaneous rate of change of the momentum (stretching of the velocity) along the velocity direction while ignoring the rotational motion (perpendicular to the velocity vector). See the parallel component of the momentum gradient tensor in Fig.1.

3.2 Gradient of Kinetic Energy Density

The kinetic energy density is a scalar function, it is expressed as:

$$T = \frac{1}{2}\rho\vec{v}^T\vec{v} = \frac{1}{2}(\rho uu + \rho vv + \rho ww). \qquad (23)$$

where, the velocity vector in Cartesian coordinate is $\vec{v} = (u, v, w)$.

A scalar field's gradient is a vector field. The components of the vector show how quickly the kinetic energy is changing in each direction.

$$\nabla T = \begin{bmatrix} (\rho u)\partial_x u + (\rho v)\partial_x v + (\rho w)\partial_x w \\ (\rho u)\partial_y u + (\rho v)\partial_y v + (\rho w)\partial_y w \\ (\rho u)\partial_z u + (\rho v)\partial_z v + (\rho w)\partial_z w \end{bmatrix} = \begin{bmatrix} \partial_x u & \partial_x v & \partial_x w \\ \partial_y u & \partial_y v & \partial_y w \\ \partial_z u & \partial_z v & \partial_z w \end{bmatrix} \begin{bmatrix} \rho u \\ \rho v \\ \rho w \end{bmatrix}. \qquad (24)$$

The velocity gradient (derivative with respect to position, the Matrix form) is a measure of how the velocity of the fluid changes between infinitesimal distances within the flow field. The velocity gradient is a second-order tensor. It describes the rate of stretching, shearing and the rate of the rotation (spinning) of the fluid parcel in the flow field. Namely, the velocity gradient contains the total information about the stretching, shearing and spinning.

Recalling the Jacobian matrix definition of a vector-valued function, the velocity gradient can be written as:

$$\nabla T = J_{\vec{v}}^T(\rho\vec{v}). \qquad (25)$$

In three dimensions, the gradient of the velocity (the transpose of the Jacobian matrix) is called infinitesimal displacement in flow field. Thus, the kinetic energy gradient can be expressed as the infinitesimal displacement multiplying a momentum vector of $(\rho\vec{v})$.

Any square matrix can be decomposed into sum of a symmetric matrix and an antisymmetric matrix. This decomposition is often referred to as the "symmetric part" and "skew-symmetric part".

$$J_{\vec{v}}^{T} = \frac{1}{2}\left(J_{\vec{v}}^{T} + J_{\vec{v}}\right) + \frac{1}{2}\left(J_{\vec{v}}^{T} - J_{\vec{v}}\right) = \bar{\bar{S}} + \bar{\bar{A}}. \tag{26}$$

With this decomposition approach, thus, the kinetic energy gradient can be written as a symmetric part plus an antisymmetric part.

$$\nabla T = \bar{\bar{S}}(\rho\vec{v}) + \bar{\bar{A}}(\rho\vec{v}). \tag{27}$$

3.2.1 Symmetric Part of the Kinetic Energy Gradient

Accordingly, the symmetric part reads:

$$\bar{\bar{S}}(\rho\vec{v}) = \frac{1}{2}\left(\begin{bmatrix} \partial_x u & \partial_x v & \partial_x w \\ \partial_y u & \partial_y v & \partial_y w \\ \partial_z u & \partial_z v & \partial_z w \end{bmatrix} + \begin{bmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{bmatrix}\right)(\rho\vec{v}). \tag{28}$$

It can be explicitly rewritten as:

$$\bar{\bar{S}}(\rho\vec{v}) = \begin{bmatrix} \partial_x u & \frac{1}{2}\left(\partial_x v + \partial_y u\right) & \frac{1}{2}\left(\partial_x w + \partial_z u\right) \\ \frac{1}{2}\left(\partial_y u + \partial_x v\right) & \partial_y v & \frac{1}{2}\left(\partial_y w + \partial_z v\right) \\ \frac{1}{2}\left(\partial_z u + \partial_x w\right) & \frac{1}{2}\left(\partial_z v + \partial_y w\right) & \partial_z w \end{bmatrix}(\rho\vec{v}). \tag{29}$$

Or more compactly as:

$$\bar{\bar{S}}(\rho\vec{v}) = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{bmatrix} (\rho\vec{v}). \tag{30}$$

where, $\bar{\bar{S}}$ is the strain tensor:

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{31}$$

It can be seen that the shear strain components (off-diagonal elements) in the symmetric part are expressed as an average shear strain (historically, it was called engineering shear strain).

Recalling the Stokes hypothesis, the model of the viscous stress tensor in the Navier-Stokes equations:

$$\bar{\bar{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \mu(\nabla\vec{u} + \nabla\vec{u}^T) = 2\mu S_{ij}. \tag{32}$$

Comparing equation (30) with (32), it is recognized that

$$\bar{\bar{S}}(\rho\vec{v}) = \nabla \cdot \bar{\bar{\sigma}}. \tag{33}$$

From the perspective of mathematics, the symmetric part of the kinetic energy gradient represents the divergence of the viscose stress tensor in the Navier-Stokes equations.

3.2.2 Antisymmetric Part of the Kinetic Energy Gradient (spin)

It can see that the antisymmetric part of the kinetic energy gradient reads:

$$\bar{\bar{A}}(\rho\vec{v}) = \frac{1}{2}\left(\begin{bmatrix} \partial_x u & \partial_x v & \partial_x w \\ \partial_y u & \partial_y v & \partial_y w \\ \partial_z u & \partial_z v & \partial_z w \end{bmatrix} - \begin{bmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{bmatrix}\right)(\rho\vec{v}). \tag{34}$$

It can be written more compactly:

$$\bar{\bar{A}}(\rho\vec{v}) = -\frac{1}{2}\begin{bmatrix} 0 & -(\partial_x v - \partial_y u) & \partial_z u - \partial_x w \\ \partial_x v - \partial_y u & 0 & -(\partial_y w - \partial_z v) \\ -(\partial_z u - \partial_x w) & \partial_y w - \partial_z v & 0 \end{bmatrix}(\rho\vec{v}). \tag{35}$$

Recalling the definition of the vorticity (the curl of the flow velocity):

$$\vec{\omega} = \nabla \times \vec{v}. \tag{36}$$

Thus, the antisymmetric part can be expressed as:

$$\bar{\bar{A}}(\rho\vec{v}) = -\frac{1}{2}\vec{\omega} \times \rho\vec{v} = \frac{1}{2}(\rho\vec{v}) \times \vec{\omega}. \tag{37}$$

From equations (30) and (37), finally, the gradient of kinetic energy for an incompressible fluid is expressed as:

$$\nabla T = \rho\left[\bar{\bar{S}}(\vec{v}) + \frac{1}{2}\vec{v} \times \vec{\omega}\right]. \tag{38}$$

Substituting equations (21) and (38) into (20), the equation of motion of the fluid dynamic, derived from the Euler-Lagrangian approach, reads

$$\frac{\partial(\rho\vec{v})}{\partial t} + (\vec{v}\cdot\nabla)(\rho\vec{v}) = -\nabla p + \rho\left[\bar{\bar{S}}(\vec{v}) + \frac{1}{2}\vec{v}\times\vec{\omega}\right]. \tag{39}$$

The vorticity of a fluid element is a measure of the local rotation of the fluid. The vorticity field is just twice the local angular velocity at a point in a fluid flow field.

$$\vec{\omega} = 2\vec{\Omega}. \tag{40}$$

The factor of 2 before the local angular velocity arises from the mathematical formulation of vorticity and angular velocity in the context of fluid dynamics.

Thus, the equation of motion of the fluid dynamic can be also written as:

$$\frac{\partial(\rho\vec{v})}{\partial t} + (\vec{v}\cdot\nabla)(\rho\vec{v}) = -\nabla p + \rho\left[\bar{\bar{S}}(\vec{v}) + \vec{v}\times\vec{\Omega}\right]. \tag{41}$$

4. Geometric Interpretation of the Kinetic Energy Gradient

Compared to the equation (39) with the Navier-Stokes equations, the Stokes hypothesis only accounts for the symmetric part of the kinetic energy gradient and ignores the antisymmetric part. In the following, we will show some scenarios to illustrate the deficiency of the Navier-Stokes equations.

4.1 General Kinetic Energy Gradient

For illustration and simplicity, consider a 2-dimensional deformation of an infinitesimal square control volume in a flow field, as shown by Fig. 2. In the general case, the rate of change of the shear strain (the change in angle between two originally orthogonal control volume lines) is not necessarily equal to each other, namely, $\partial_y u \neq \partial_x v$.

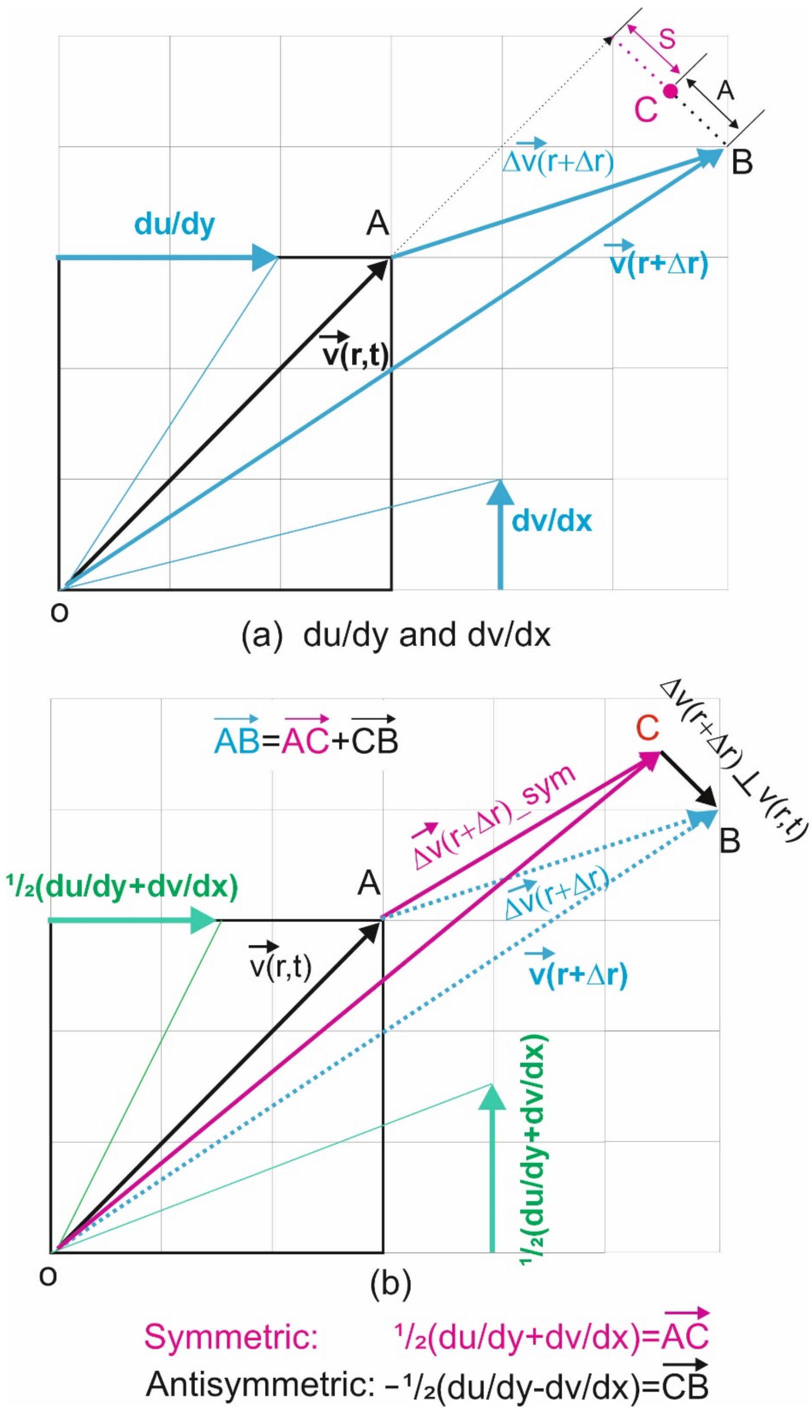

Fig. 2 General velocity gradient tensor, $\partial_y u \neq \partial_x v$.

As mentioned above, the infinitesimal velocity gradient of equation (29) in the symmetric part is called the strain rate tensor and describes the infinitesimal rate of stretching and shearing.

Its diagonal elements represent the rate of stretching (or extensional strain rate). By the definition, its off-diagonal elements represent an average rate of shear deformation (because of the averaged procedure). As shown by Fig. 2, the averaged shear strain rate accounts only for a part of the shear deformation.

As shown in Fig. 2, the real rate of change of the velocity vector is represented by the vector $\overrightarrow{AB}$. Due to the average procedure of the off-diagonal elements, $\frac{1}{2}(\partial_y u + \partial_x v)$, the averaged shear strain rate in the symmetric parts only represents a deformation vector of $\overrightarrow{AC}$. Thus, this averaged procedure accounts only partially for the spinning motion.

The velocity gradient in the antisymmetric part (pure spin) will supplement another part of the shear deformation and can correct this deficiency; this shear strain rate is represented by the vector of $\overrightarrow{CB}$. Thus, the collective effects of the symmetric and antisymmetric parts give the correct velocity gradient: $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$.

4.2 Simple Shear Flow

Simple shear flow will give a clearer image of this deficiency of the Navier-Stokes equations.

Fig. 3 shows a simple shear flow for 2-dimensional deformation; we assume the $\partial_x v = 0$ and $\partial_y u = 2\varepsilon = \overrightarrow{AB}$. Through the average procedure, the symmetric part represents an instantaneous rate of pure shear. It is represented by a vector of $\overrightarrow{AC}$.

The antisymmetric part represents a pure, rigid rotation of vector $\overrightarrow{AD}$. The direction of the rotational axes is also given in this figure; it is going into the page. By applying the right-hand rule of the vector cross product, it can be recognized that the resulted force by the antisymmetric part, $\bar{\bar{A}}(\rho\vec{v}) = \frac{1}{2}(\rho\vec{v}) \times \vec{\omega}$, of equation (37), is in an upward direction.

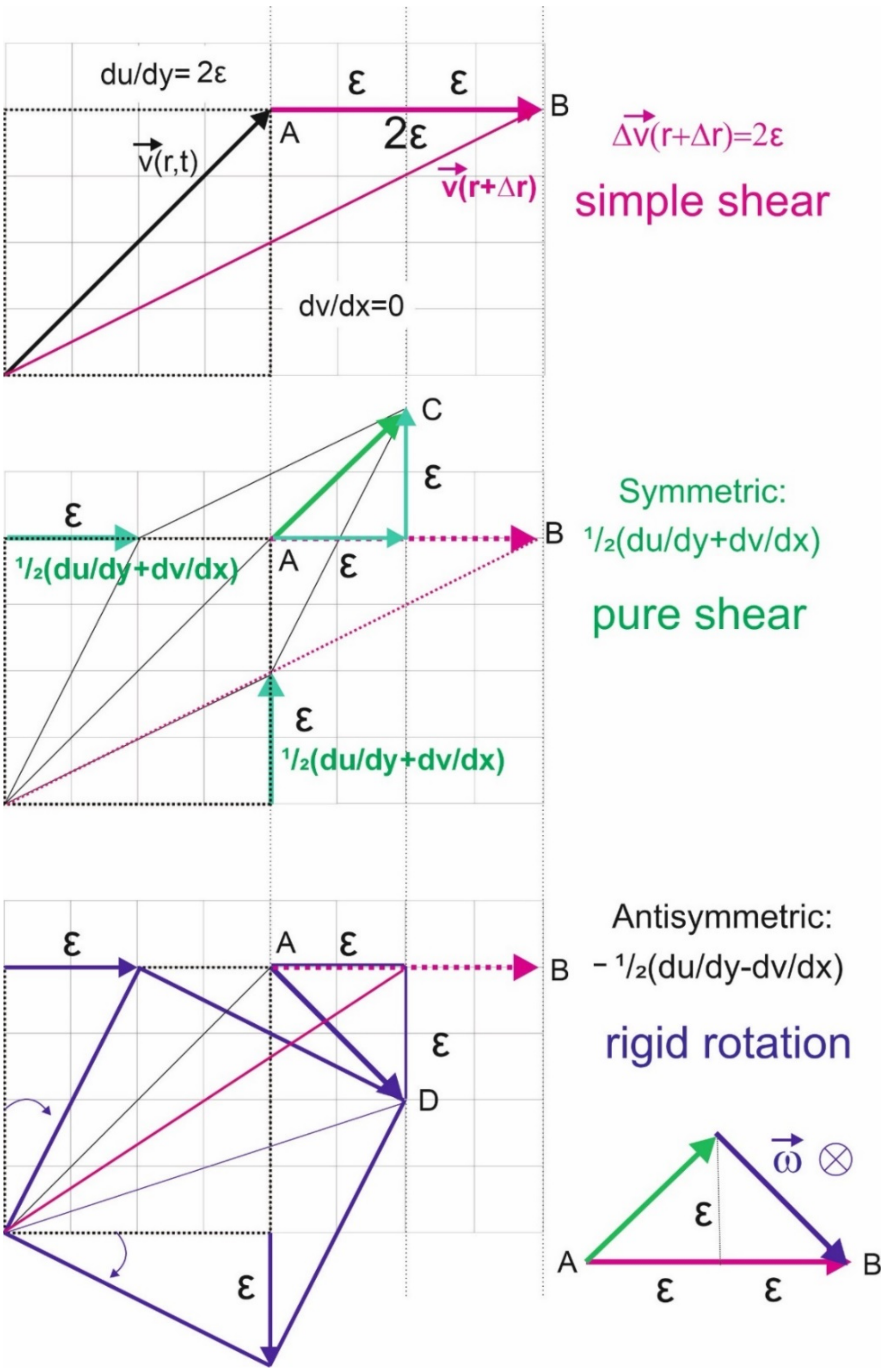

Fig. 3 simple shear flow is the combination of pure shear and rigid rotation.

4.3 Navier-Stokes Equations are a Special Case

As shown by Fig. 4, if the corresponding off-diagonal elements of the rate of velocity change are equal to each other, namely $\partial_y u = \partial_x v$.

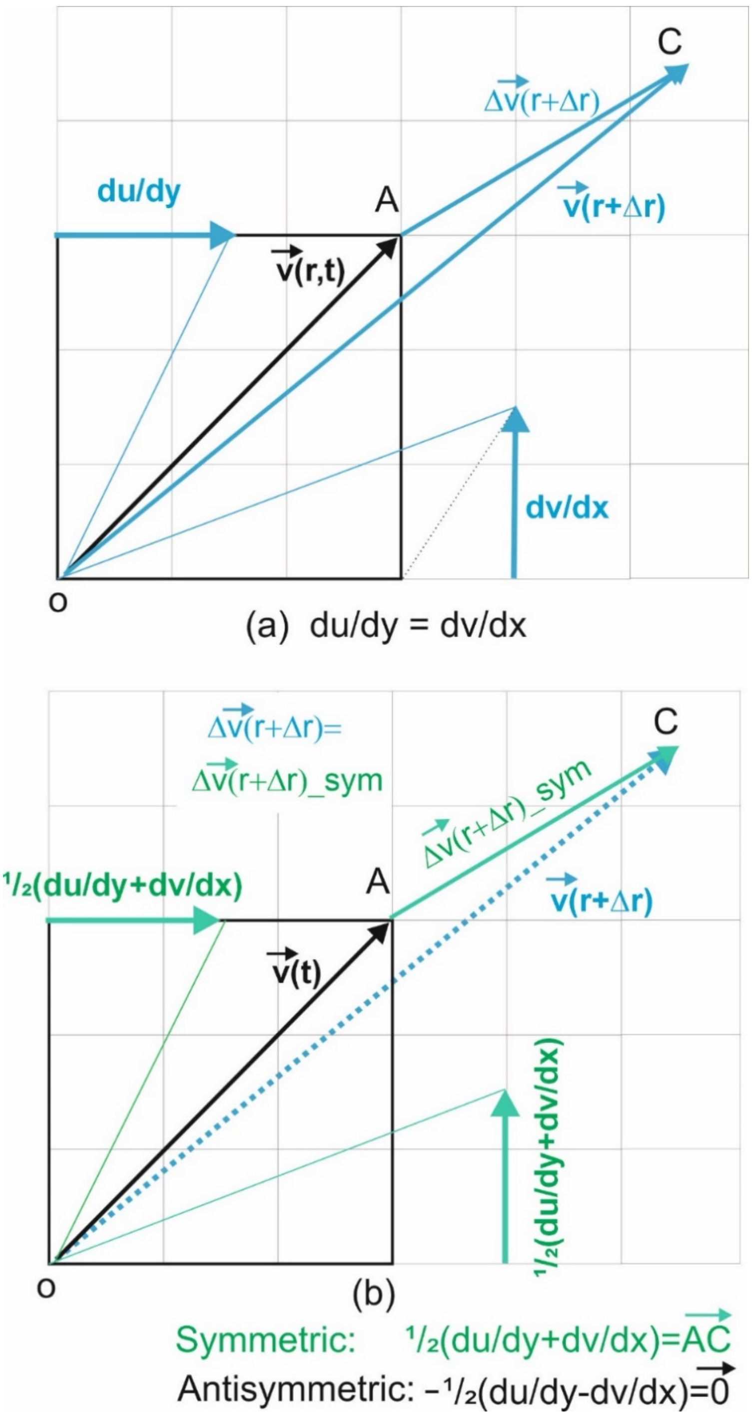

Fig. 4. The corresponding off-diagonal elements are equal to each other: $\partial_y u = \partial_x v$

The symmetric part really represents the actual shear strain rate. In this case, the antisymmetric part is equal to zero, because of $\partial_y u - \partial_x v = 0$. Under this circumstance, the Navier-Stokes equations can just describe the correct flow behavior.

4.4 Geometric Algebra Explanation

Assuming the velocity has an infinitesimally small variation in space from $\vec{r}$ to $\vec{r} + \delta\vec{r}$ in flow field (time keeps constant), it is denoted by $\delta\vec{v}(\vec{r} + \delta\vec{r})$. For two vectors $\rho\vec{v}(\vec{r})$ and $\delta\vec{v}(\vec{r} + \delta\vec{r})$, we may write the geometric product of two vectors, $\rho\vec{v}(\vec{r})$ and $\delta\vec{v}(\vec{r} + \delta\vec{r})$, as the sum of an inner product (a scalar field) and an exterior product of vectors (also called wedge product), it is a bivector field (in three dimensions, it represents a vector rotation, physically, it represents a vorticity field).

$$\rho\vec{v}\delta\vec{v} = \rho\vec{v} \cdot \delta\vec{v} + \rho\vec{v} \times \delta\vec{v}. \tag{42}$$

Both sides divided by the small variation of $\delta\vec{r}$:

$$\rho\vec{v}\frac{\delta\vec{v}}{\delta\vec{r}} = \rho\vec{v} \cdot \frac{\delta\vec{v}}{\delta\vec{r}} + \rho\vec{v} \times \frac{\delta\vec{v}}{\delta\vec{r}}. \tag{43}$$

Take a limit, $\delta\vec{v}/\delta\vec{r}$ approaches the velocity gradient tensor, $\nabla\vec{v}$, see equation (24).

$$\rho\vec{v}(\nabla\vec{v}) = \rho\vec{v} \cdot (\nabla\vec{v}) + \rho\vec{v} \times (\nabla\vec{v}). \tag{44}$$

Re-arrange it:

$$(\rho\vec{v}) \cdot (\nabla\vec{v}) = \rho\vec{v}(\nabla\vec{v}) - \rho\vec{v} \times (\nabla\vec{v}). \tag{45}$$

The LHS of equation (45) represents the projection of the velocity gradient onto the vector of $\rho\vec{v}$. It is parallel to $\rho\vec{v}$, while the cross-product term of

$\rho\vec{v} \times (\nabla\vec{v})$ is perpendicular to $\rho\vec{v}$. See Fig. 5b. Mathematically, it is the orthogonal decomposition.

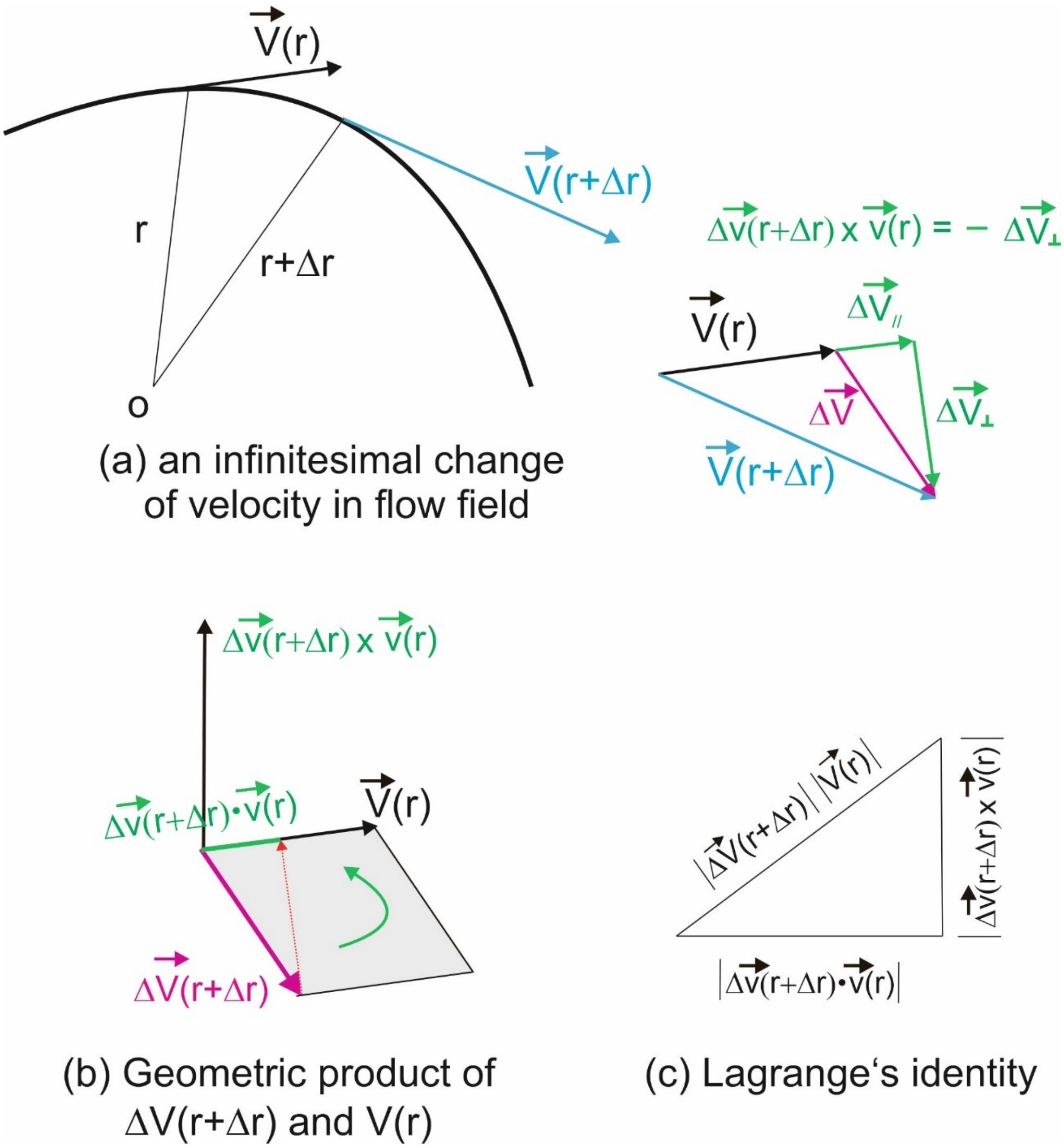

Fig. 5 Geometric product of two vectors contains all the information of the stretching (dot product) and spinning (cross product or vorticity) motion of the flow field.

With the help of Lagrange's identity (Pythagorean theorem), in three dimensions, equation (42) has the following relation:

$$|\rho\vec{v}|^2|\delta\vec{v}|^2 = |\rho\vec{v} \cdot \delta\vec{v}|^2 + |\rho\vec{v} \times \delta\vec{v}|^2. \quad (46)$$

where $|\rho\vec{v}\cdot\delta\vec{v}|$ is the length of the inner product of $\rho\vec{v}$ and $\delta\vec{v}$, and $|\rho\vec{v}\times\delta\vec{v}|$ represents the length of the exterior (wedge) product. Equation (46) forms a right-angled triangle (see Fig. 5c). Thus, the geometric product of two vectors contains the total information between two vectors: stretching (dot product term) and local rotation (cross-product term).

5. Similarity between Flow Field and Electromagnetic Field

For vectors in three-dimensional Cartesian coordinate, we have the following vector calculus identity:

$$\nabla T = \frac{1}{2}(\rho\vec{v}\cdot\vec{v}) = (\vec{v}\cdot\nabla)(\rho\vec{v}) + (\rho\vec{v})\times(\nabla\times\vec{v}). \quad (47)$$

The first term of the RHS of the equation (47) is the convective term and parallel to the velocity vector ("projected" onto the velocity vector), the second term is the cross product of velocity and vorticity vectors (perpendicular to both the vectors of velocity and vorticity). Geometrically, it just is the orthogonal decomposition of the kinetic energy gradient. See Fig. 5. Physically, it includes local stretching (dot product) and rotational motions (cross product of the velocity and vorticity) of the flow field.

Equation (47), together with the material derivative of equation (21), are substituted into the Euler-Lagarangian equation (20), yielding:

$$\frac{\partial(\rho\vec{v})}{\partial t} + (\vec{v}\cdot\nabla)(\rho\vec{v}) = (\vec{v}\cdot\nabla)(\rho\vec{v}) + (\rho\vec{v})\times(\nabla\times\vec{v}) - \nabla p. \quad (48)$$

The convective terms in both sides cancel out. The resulting equation reads:

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla p - (\rho\vec{v}) \times (\nabla \times \vec{v}) = 0. \tag{49}$$

For incompressible flow, it can be written as:

$$\rho\left[\left(-\frac{\partial\vec{v}}{\partial t} - \frac{1}{\rho}\nabla p\right) + \vec{v} \times (\nabla \times \vec{v})\right] = 0. \tag{50}$$

If we define a force field per unit mass density as:

$$\vec{E} = -\frac{1}{\rho}\nabla p - \frac{\partial\vec{v}}{\partial t}. \tag{51}$$

Similar to the definition of magnetic field, the vorticity field is written as

$$\vec{B} = \vec{\omega} = \nabla \times \vec{v}. \tag{52}$$

Then the force in the flow field can be written as following:

$$\vec{F} = \rho\vec{E} + \rho\vec{v} \times \vec{B}. \tag{53}$$

Mathematically, it is similar to the Lorentz force (electromagnetic force) expression for charged particles in an electromagnetic field.

It says that the force on a fluid particle, with mass density of ρ, in flow field is a combination of

(1) a force in the direction of the translational force density of $\vec{\boldsymbol{E}}$ (proportional to the magnitude of the field strength and the fluid mass density, parallel to the negative gradient of pressure, similar to the electric field), and
(2) a force at right angles to both the vorticity field $\vec{\boldsymbol{\omega}}$ and the velocity $\vec{\boldsymbol{v}}$ of the fluid particle (proportional to the magnitude of the vorticity

field strength, the mass density, and the velocity, similar to the charged particle moving in magnetic field).

It can be concluded that the trajectory of the fluid particle in the flow field will show the combination of a translation and a rotation motion (a vortex tube or helical shape).

It may be induced that an electromagnetic field is produced by a mixture of free-flowing particles with positive and negative charges, such as plasma flow in three-dimensions.

In this case, the fluid flow field, electromagnetic field, or, plasma flow field are, in essence, the same. All of them can be integrated within the same mathematical frame. Both can be written as the same field tensor:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}, \quad (54)$$

where, c represents the mechanical wave propagation speed for the fluid flow field (compressible fluid, incompressible flow is just an approximation that v<<c), while for the electromagnetic field it is the speed of light [5].

The equation (53), namely the force components in the spatial directions (x, y, z), can be rewritten as:

$$\vec{F} = -F_{\mu\nu}\left(\rho\vec{V}\right). \quad (55)$$

Here $\left(\rho\vec{V}\right)$ represents the four-velocity vectors:

$$\rho\vec{V} = (\rho c, \rho u, \rho v, \rho w). \quad (56)$$

6 Summary and Conclusion

In this work, we explore the Euler-Lagrangian approach to flow fields and highlight the incompleteness of the Navier-Stokes equations. The Euler-Lagrangian approach and the Newtonian method are both equivalent in classical mechanics, but the Euler-Lagrangian approach has advantages such as handling non-inertial forces and deriving equations of motion for systems with a curvilinear trajectory. The incompleteness of the Navier-Stokes equations is explained by geometric interpretations and geometric algebra explanations. The Stokes hypothesis can only account for a part of the spin motion of the fluid particles in the flow field because of the average procedure of the shear strain. In order to describe the flow field correctly, a spinning term (represented by an antisymmetric tensor multiplying the linear momentum) cannot be ignored. At last, we discuss the similarities between flow fields and electromagnetic fields. It is an amazing thing that fluid flow fields and electromagnetic fields, in essence, are the same. All of them can be described by the same mathematical frame.

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