Bending the Riemann critical strip to a lunula: no zeroes in $1/2 < Re(z) < 1$

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Abstract

The critical strip of the Riemann $\zeta(z)$ is transformed into a crescent-like lunula and the critical line into the unit circle by a conformal transformation. In the new extended complex plane, the argument principle is used to show that there are no zeroes outside of the unit circle, thus proving that there are no zeroes in the right half of the strip, $1/2 < Re(z) < 1$.

1 Introduction

The Riemann hypothesis states that all non-trivial zeroes of the $\zeta(z)$ function lie on the $Re(z) = 1/2$ critical line. This is an unsolved problem that has enormous consequences in many different fields, from number theory to cryptography[1]. It is already well-known that most zeroes lie on the critical line, but their presence or absence from the region $0 < Re(z) < 1$ is still under scrutiny. In this paper it is shown that there are no zeroes in the right half of the strip, i.e. for $1/2 < Re(z) < 1$. This fact, combined with the symmetry of the zeroes about the critical line, insures that, if any nontrivial zero exists, it must lie on the vertical critical line.

2 The *lunula*

The complex plane $z = x + iy$ is transformed into the w-plane, with $w = u + iv$, see Fig.1, by the following invertible conformal bilinear fractional transformation, that is also a Möbius transformation, $\mathcal{T} : z \to w/$

$$w = \frac{z+1}{2-z} , \qquad z = \frac{2w-1}{w+1} \tag{1}$$

that maps lines to circles. The Riemann ζ function, in the new plane, is mapped onto a new function $\mathcal{T}(\zeta(z)) \to \vartheta(w)$ such that:

$$\vartheta(w) = \zeta(z) = \zeta\left(\frac{2w-1}{1+w}\right) . \tag{2}$$

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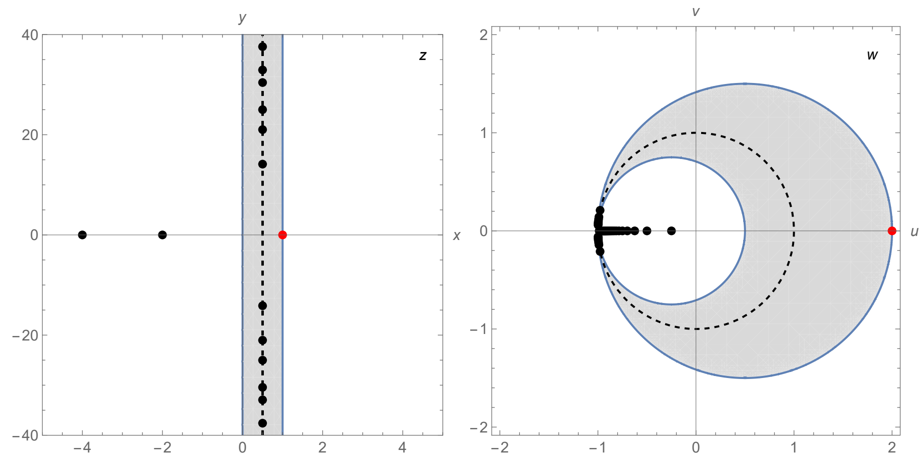

Figure 1: Left: z-plane, $z = x + iy$ with critical strip in gray, dashed black critical line $x = 1/2$, simple pole at $z = \{1, 0\}$ in red, trivial and critical zeroes as black dots. Right: transformed w-plane, $w = u + iv$ with the critical lunula in gray, dashed black critical line, simple pole at $w = \{2, 0\}$ in red, trivial and critical zeroes as black dots.

The simple pole at $z = \{1, 0\}$ is transformed into $w = \{2, 0\}$, the critical strip is bent into a *lunula*, a concave-convex crescent-like figure. The critical line $x = 1/2$ is mapped into the circumference of the unit circle. The trivial zeroes, that are found at $z = \{-2n, 0\}$ are mapped into $w = \{\frac{1-2n}{2+2n}, 0\}$, while the critical zeroes on the critical line are squeezed onto the corresponding points on the circumference, the first being located at an angle $\pm 167^{\circ}.885$ with respect to the positive u-axis. As Hardy proved [2], there are infinitely many of them. The black dots in the right panel of Fig. (1) are disconnected, but they are so close to form a hammer-shaped black figure.

The point $w = \{-1, 0\}$ is very remarkable as it is at the same time an accumulation point for trivial zeroes and an essential singularity. A theorem states [3] that, for an analytic function on a simply connected domain, when we have a sequence of zeroes converging to a limiting (or accumulation) point, then the function in that point is either vanishing identically or it is an essential singularity. It is obviously an accumulation point for trivial zeroes, because the formula above insures that these zeroes become denser and denser as n grows, approaching the point. If we now consider the Riemann sphere, i.e. the $\{x, y\}$ complex plane augmented with complex infinity, we have that the Riemann ζ has an essential singularity at $\infty_{\mathbb{C}}$ that is mapped onto the point $\{-1, 0\}$. On the w-plane the limits $\lim_{w \to -1} \theta(w)$ and $\lim_{w \to -1} 1/\theta(w)$ are both indeterminate, therefore the point $\{-1, 0\}$ is also an essential singularity. To see the behaviour of the function at these points, we can plot the modulus of the ζ function in the w-plane as in Fig. 2. It is cut at some finite height, and, on the left side,

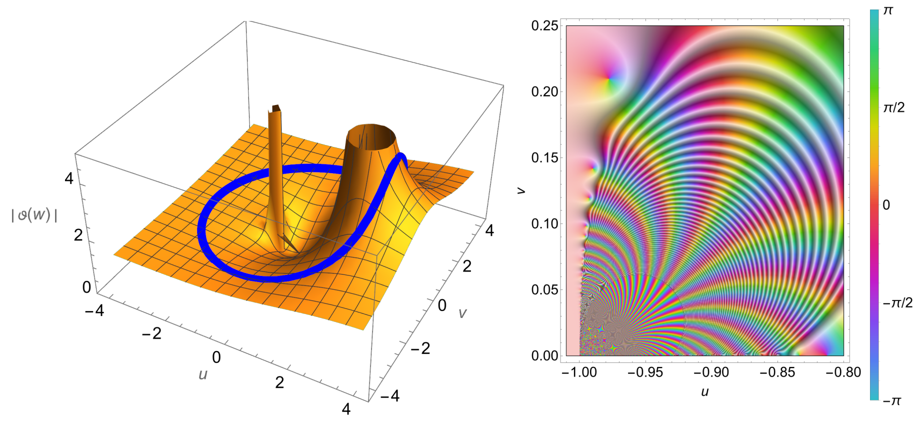

Figure 2: Left: Modulus of the theta function, $|\vartheta(w)|$, in the w-plane, showing the pole in $\{2, 0\}$ and the essential singularity in $\{-1, 0\}$. A circular path with radius $r = 3$ is shown in blue. Right: close up of the complex plot of the ϑ function near $\{-1, 0\}$. The trivial zeroes are visible as black dots on the arc, surrounded by a $2\pi i$ change in argument (rainbow). The internal region reaches all possible values and the argument winds more and more often as one approaches the essential singularity.

one sees a hint of the fact that $\lim_{w->\infty_{\mathbb{C}}} \theta(w) \to \frac{\pi^2}{6} \simeq 1.645$. One recovers the same constant on all sides, also on the right side, after the pole. The other part of the figure shows the complex plot, i.e. the plot of the argument of θ in the corner close to $\{-1, 0\}$. The rainbow colors appear whenever the argument winds by $2\pi i$, around zeroes. According to the Great Picard's theorem, any punctured neighborhood of an essential singularity attains all possible complex values infinitely often, with at most one exception. That's why the complex plot shows an intricate pattern close to the singularity.

3 Choudury's formula and the argument principle

B.K. Choudury gave a formula (in Ref. [4], unnumbered, just before Eq. 8) for the logarithmic derivative of the Riemann zeta-function :

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{1-z} + \gamma - \sum_{n=1}^{\infty} \bar{A}_n (z-1)^n \tag{3}$$

where γ is the Euler-Mascheroni constant and the coefficients A_n and $\bar{A}_n$ are connected to the Stieltjes' constants γ_n by:

$$\bar{A}_n = -(n+1)A_n - \sum_{k=0}^{n-1} \bar{A}_n; \qquad \bar{A}_0 = -\gamma; \qquad A_n = (-1)^n \gamma_n / n! \tag{4}$$

This is exact, but slowly converging. The corresponding formula in the w-plane reads

$$f(w) = \gamma + \frac{1+w}{2-w} - \sum_{n=1}^{\infty} \bar{A}_n \left(\frac{w-2}{1+w}\right)^n \tag{5}$$

In order to apply Cauchy's argument principle, we will need to evaluate the logarithmic derivative:

$$\frac{\vartheta'(w)}{\vartheta(w)} = \frac{\zeta'(\frac{2w-1}{w+1})}{\zeta(\frac{2w-1}{w+1})} \frac{3}{(w+1)^2} \tag{6}$$

where the last term is the derivative of the argument, or dz/dw. This approach works, and indeed this can be shown either numerically or by using the argument principle on small circular paths around the isolated zeroes or the pole.

Now we want to apply the argument principle **in the w-plane** to the ϑ function, anticlockwise along circles C of radius R centered around the origin, i.e. along $Re^{i\omega}$:

$$\frac{1}{2\pi i} \oint_C \frac{\vartheta'(w)}{\vartheta(w)} dw = \frac{1}{2\pi i} \oint_C f(w) \frac{3}{(1+w)^2} dw = N - P \tag{7}$$

that is connected to the number of zeroes (N) and the number of poles (P) inside the path in a anticlockwise manner, or, that is the same because ϑ is analytic on circles far away from the origin, on the number of zeroes and poles outside of it, if run across clockwisely. On the surface of the Riemann sphere, inside and outside loose their meaning and the argument principle is valid on the simply connected portion of the sphere, where circles are contractible to a point. Our function is analytic on the "outside" of the path C, with the exception of a finite number of points, actually only the pole at $w = 2$ in this case.

Now, by plugging in the definition of f, it is easy to see that the first term (γ) evaluates to zero by the residue theorem,

$$\frac{1}{2\pi i} \oint_C \underbrace{\gamma \frac{3}{(1+w)^2}}_{f_\gamma} dw = Res(f_\gamma, -1) = 0 \tag{8}$$

The second term gives a residue of 1 if the circle does not encompass the simple pole at $w = 2$ and goes to zero when the circle is larger, because it takes in the residue at $w = 2$:

$$\frac{1}{2\pi i} \oint_C \underbrace{\frac{(1+w)}{(2-w)} \frac{3}{(1+w)^2}}_{f_f} dw = \begin{cases} Res(f_f, -1) = 1 & \text{if } 1 < R < 2 \\ Res(f_f, -1) + Res(f_f, 2) = 0 & \text{if } R > 2 \end{cases} \tag{9}$$

The third term gives a null residue, $\forall n > 0$, because the Laurent series expansion around $w = -1$ of each term of the type

$$3\frac{(2-w)^n}{(1+w)^{n+2}} \tag{10}$$

always starts from the term $(1+w)^{-2}$, therefore the residue is 0. Thus, since the sum

$$N - P = \begin{cases} 1 & \text{if } 1 < R < 2 \\ 0 & \text{if } R > 2 \end{cases} \tag{11}$$

the difference is a constant on all circles of radius R larger then 1. Here the value $N - P = 1$ is not counting the essential singularity and the infinity of trivial and nontrivial zeroes of the hammer shape, but rather is counting only the single pole outside of the circle, changed in sign because of the equivalence between the inside counted anticlockwise and the outside counted clockwise. There cannot be other zeroes in this region or the argument principle would count them when $R \to 1$. We have just proven that there are no zeroes in the annular region that goes from the black dashed circle in Fig. 1 to the circle touching $w = 2$. But that region comprises the whole outer part of the lunula that maps back to the right half of the critical strip, therefore we have just proven that there are no zeroes of the Riemann ζ function for $1/2 < Re(z) < 1$. This is a big step forward with respect to any estimate found so far (See [5, 6]).

Now, if a zero cannot exist in this region, because of the symmetry established by the functional relation between $\zeta(z)$ and $\zeta(1-z)$, this implies that there cannot be zeroes also on the left part of the strip, i.e. for $0 < Re(z) < 1/2$. The only place left is the critical line itself, as Riemann conjectured back in 1859.

References

[1] P. Borwein, S. Choi, B. Rooney and A. Weirathmueller, "The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike", Spinger (2007).

[2] G.H. Hardy, Comp. Rend. Acad. Sci. 158 (1914) 1012–1014.

[3] H.K. Pathak, Complex Analysis and Applications, Springer Nature Singapore Pte Ltd. (2019)

[4] B.K. Choudury, The Riemann zeta-function and its derivatives, Proc. R. Soc. Lond. A 450 (1995) 477–499

[5] K. Ford, Vinogradov's integral and bounds for the Riemann zeta function, Proc. London Math. Soc. 85 (3) (2002) 565–633

[6] M.J. Mossinghoff and T.S. Trudgian, Nonnegative trigonometric polynomials and a zero-free region for the Riemann zeta-function, J. Number Theory 157 (2015) 329–349