Review of: "Autonomous Second-Order ODEs: A Geometric Approach"

Glen Wheeler[1]

1 School of Mathematics and Applied Statistics, University of Wollongong, Australia

Potential competing interests: No potential competing interests to declare.

The article explores the relationship between autonomous second-order ordinary differential equations (ODEs) and Riemannian geometry by associating these ODEs with Riemannian metrics on a first-order jet bundle.

This is an interesting and novel approach to understanding the dynamics of autonomous ODEs. The authors relate solutions to geodesic curves of a specifically constructed Riemannian manifold. This geometrisation appears valuable, as it may provide deeper insights into the structure of these ODEs, and the associated manifold offers a new geometric perspective.

The introduction of first-order jet bundles to describe the ODE's structure and define a Riemannian metric is well-conceived. The choice of utilising the first-order jet bundle $J^1(\mathrm{R}, \mathrm{R})$ provides a natural geometric setting. The paper defines a Riemannian metric on an open subset of the jet bundle and successfully relates this metric to the integrability of the ODE. The resulting manifold, endowed with a metric involving terms from the original ODE, becomes a geometric playground for analysing the properties of the solutions as geodesics. The construction of the orthonormal frame and the subsequent calculations of sectional curvatures are clear and significant as they relate curvature properties to the dynamics of the system.

The relationship between the geodesics of the manifold and the solutions of the ODE is particularly insightful. By establishing that integral curves of a specific vector field (constructed from the ODE) are geodesics, the paper provides a clean geometric characterisation of the solution space.

The notion of energy foliation introduced in the paper is an intriguing addition. This foliation is connected to the concept of energy in the context of mechanical systems, and it highlights the foliation's geometry as consisting of minimal surfaces. This provides a bridge between the classical notion of energy conservation and the geometry of the associated manifold, a perspective that might be valuable for researchers studying Hamiltonian or Lagrangian dynamics through a geometric lens.

The study of the geometry of the leaves of the foliation, particularly showing that they are minimal surfaces, ties into the broader context of minimal foliations in differential geometry. This minimality suggests stability and a natural variational characterisation, which could lead to further investigations into the stability of the system's dynamics.

The applications to Lagrangian systems, including the particle in a gravitational field and the damped harmonic oscillator, serve as practical illustrations of the theoretical framework. Particularly notable is the treatment of the damped harmonic oscillator, a system typically viewed as non-conservative. By placing it in this geometric framework, the paper finds a conserved quantity, which is a significant result. This could open avenues for rethinking other non-conservative systems from a similar geometric perspective.

The calculations of the Levi-Civita connection, the Riemann curvature tensor, and the shape operator are presented in a detailed manner that will be appreciated by the reader. The discussion of specific examples like the second-order ODEs with explicit calculations of curvatures helps to ground the theoretical framework in concrete cases. The explicit nature of the calculations makes the paper accessible to those interested in applications, although a more detailed discussion of the implications of the curvature values on the dynamics of the system could add further depth.

The concluding remarks of the paper point out interesting directions for future research, such as extending the framework to higher-order ODEs or to Lagrangian systems in higher dimensions. These extensions are indeed promising, and they could significantly broaden the applicability of the geometric framework introduced in this paper.

In my view, the paper could benefit from:

1. A more thorough discussion on how the curvature of the manifold provides insight into the qualitative behavior of solutions, especially in relation to dynamical systems theory.
2. A discussion on how geometric invariants of the Riemannian manifold may give insights, or not, into the solution space of the system.
3. Adding more visual aids, such as diagrams of the jet bundle, geodesics, or foliations.
4. Discussion of possible physical interpretations or analogies for the "non-standard" Lagrangian approach.

Overall, the paper makes an interesting contribution, blending techniques from Riemannian geometry with the theory of ODEs.