Hamiltonian, Lagrangian, Dynamics and Singularity of the Compressible Fluid Flow

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ABSTRACT

The wave travels in the compressible fluid by a finite propagation speed. In the center-of-linear-momentum reference frame, the macroscopic velocity is zero, thereby, the macroscopic kinetic energy is also zero, the potential energy density (pressure) and the mass density are equivalent, and the proportional coefficient is the square of the wave speed. The macroscopic dynamic behavior of the compressible fluid flow cannot be depicted in CoM frame but is preferably described in the Lab frame – a pseudo-inertial frame. The system's velocity is, thus, reference frame dependent, however, wave propagation speed is frame independent. This relative velocity leads the fluid to be compressed and the mass density will increase, and the energy will increase, too, observed in the Lab frame. The increase factor is the Lorentz factor, which depends on the particle's relative flow velocity (relative to the reference frame) and the wave speed. This system is similar to a variable-mass system. The Hamiltonian, kinetic, and potential energy densities are not only the function of the relative velocity but also the function of wave speed. It is highlighted that for compressible fluids, when the flow velocity is equal to the wave speed, there exists a singularity, where the mass density, kinetic energy, and Hamiltonian increase infinitely great, while the potential energy goes to zero. This behavior is quite different from incompressible flow. According to the definition, the wave propagation speed is infinitely great for incompressible flow, and the potential energy is purely a function of position in flow field, not a function of wave speed. Any change will instantaneously propagate through the whole field without any time lag. The mathematical description cannot depict and thus disregards the wave behavior, just like the Newtonian mechanics. This is also the reason why

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the particles are instantaneously entangled in quantum mechanics, no matter how far they are in the field. At last, the dynamic equation is given out in Euler coordinates. It shows that the equation is not defined at the transonic point of $\beta = v/c = 1$, where there is a singularity point.

1. Introduction

As an approximation, the incompressible fluid model is widely applied under the low flow velocity condition, or exactly to say, when the flow velocity is far smaller than the wave propagation speed, by using the Newtonian mechanics, in which the change of the mass density is ignored, due to its simplification. However, the compressibility of a fluid is the intrinsic property, meaning it's a fundamental characteristic of the fluid itself. The compressibility of a fluid refers to its tendency to decrease in volume under an increase in pressure. Any fluid is compressible, to some degree. Even though some fluids, like liquids, are less compressible compared to gases, they still undergo some change in volume when subjected to pressure changes. Compressible fluid flow deals with the movement of fluids (liquids or gases) where changes in pressure and density significantly affect the flow behavior. This subject plays a crucial role in various fields of science and engineering, including aerospace, meteorology, and chemical engineering, such as the design of aircraft, rockets, gas turbines, and high-speed vehicles, and accurately describing the hurricane, etc. In compressible fluid field, changes in pressure and density propagate as waves, with a finite wave propagation speed determining how fast these changes occur. The speed of the wave is thus a critical parameter in compressible flow, this is essential difference from the incompressible flow model, where the wave propagation speed is infinitely great. Under this circumstance, the flow is always in the subsonic regime. We cannot use the incompressible fluid model to describe the transonic, supersonic, and shock wave dynamics. Thus, correctly and exactly describing compressible flow has great significance to help engineers and researchers understand complex flow phenomena, optimize designs, and predict performance under different operating conditions.

2. Observing in Center-of-Linear-Momentum Frame –“ Relative Rest Frame”

When discussing movements of macroscopic bodies, usually, the kinetic energy referred to is that of the macroscopic movement only. The velocities, and thus the kinetic energies of the system are frame-dependent (relative).

2.1 In Center-of-Linear-Momentum Coordinate (CoM frame)

At first, we choose the macroscopic body’s center-of-linear-momentum as a reference frame (CoM frame). In this coordinate frame, the total linear momentum of a system is zero. The net momentum of all the particles in the system cancels out, meaning that the system as a whole is not moving relative this coordinate frame. In this frame, no macroscopic motion can occur, as if the time were “frozen”. In the language of the relative theory, there is no relative motion for the observer – it is also called a co-moving frame in the special relativity.

Initially, assuming a closed massive particle system, any two particles are at an infinite distance from each other. So, the interaction energies (forces) between any two particles are negligible. We define this initial system as a no-interacting particle system or as an infinitely dilute system, and thus no potential energy in this system.

In CoM frame, exerting an external force on this system, through a reversible adiabatic compression process, see Fig 1. (Isentropic: no entropy creation or destruction during the process, no entropy flows into or out of the system), the system is compressed from the infinitely dilute state to a box with a finite volume of V_0. The total mass of particles is m_0, it is invariant, thus mass density of the box is $\rho_0 = m_0/V_0$, and since the box is stationary in CoM frame, the net momentum flux across the boundary is zero.

According to the first law of thermodynamics and the definition of a reversible adiabatic compression process.

$$dU = \delta W, \tag{1}$$

where *dU* is the change in the internal energy of the system and *δW* is work done on the system by external forces. All the work (*δW*) done by the surroundings goes into the system, and thus, the system's internal energy of *U* increases, since no heat is being supplied from (or lost to) the surroundings. An increase in internal energy can be represented by the pressure (volumetric potential energy density – interaction energy between particles) increase, see Fig.2, accordingly,

$$dU = d(pV). \tag{2}$$

With the assumption that initially the interaction energy (pressure) between particles is negligible because particles are at an infinite distance from each other, thus the initial potential energy of the system can be assumed to be zero, without loss of generality. Integrating Eq. (2) from an infinitely dilute state (the initial volume of the system is assumed to be infinite great due to the infinite distance of the particles) to the present configuration with a finite volume of V_0, we have,

$$U_0 = p_0 V_0 \quad or \quad p_0 = U_0/V_0 \tag{3}$$

This is the stored potential energy (and volumetric energy density) inside the system, supplied by some external forces. Since no macroscopic relative motion to the observer in CoM frame, in fluid dynamics, this pressure is also called stagnation pressure: $p_0 = p_{stag}$.

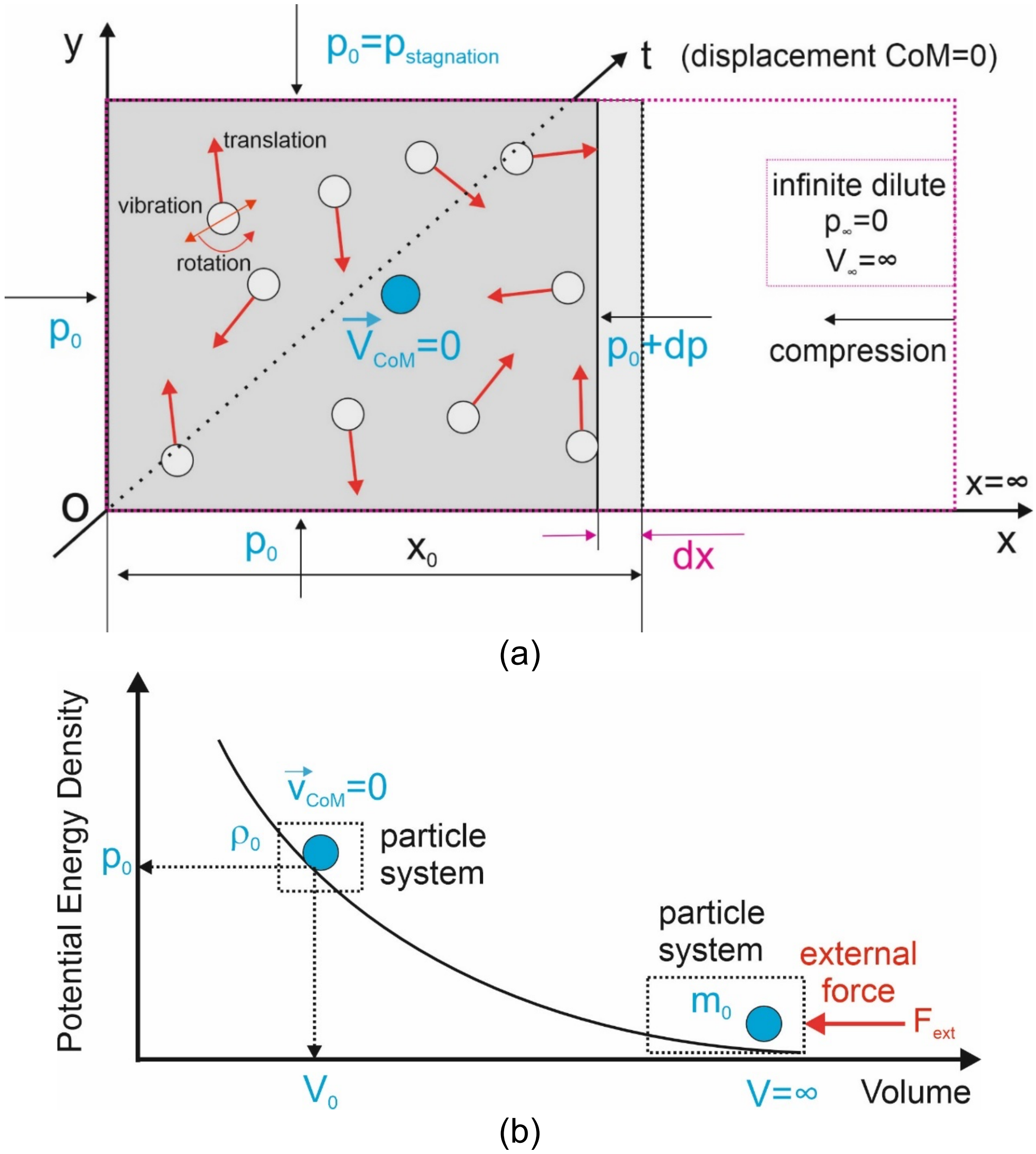


Fig. 1 (a) stationary compressible fluid particle box in CoM frame, the macroscopic linear momentum is zero relative to CoM frame. The control volume experiences an infinitesimal change of the volume (dV). The time is not considered as a variable. (b) illustration of the reversible adiabatic compression of the particle system from infinite distance to the present configuration of V_0 by external forces.

A macroscopic body that is stationary (i. e. corresponding to the body's CoM frame) may have various kinds of internal energy at the molecular or atomic level, which may be regarded as kinetic energy from a microscopic view, due to molecular translation, rotation, and vibration, etc.. These all contribute to the body's total internal energy. Since this compression is assumed to be reversible adiabatic, from a macroscopic view, the internal energy density is denoted as a pressure value (the entropy change of the system remains zero). Because of no macroscopic motion, and thereby no kinetic energy, relative to the observer in CoM frame, this is the minimal energy the system possesses.

The current box with the volume of V_0 is assumed to be compressible and the volumetric deformation (strain) is assumed still to be within the elastic limit. If we exert an infinitesimal pressure (dp) on the box, recalling the definition of elastic bulk modulus: the ratio of the infinitesimal pressure increases to the resulting relative decrease of the volume, accordingly, we have

$$B_0 = -\frac{dp}{\left(\frac{dV}{V_0}\right)}. \tag{4}$$

In an elastic fluid, the speed of sound (pressure wave propagation speed) depends on the bulk modulus and mass density:

$$c^2 = \frac{B_0}{\rho_0}. \tag{5}$$

Substituting Eq. (5) into Eq. (4), we can get:

$$-\frac{dp}{\left(\frac{dV}{V_0}\right)} = B_0 = \rho_0 c^2. \tag{6}$$

Rearranging it and recalling the mass density definition:

$$dp = -\rho_0 c^2 \left(\frac{dV}{V_0}\right) = -\frac{m_0 c^2}{V_0^2} dV. \quad (7)$$

Integrating from the infinitely dilute state to the present configuration state (p_0, V_0):

$$p_0 = \frac{m_0 c^2}{V_0} = \rho_0 c^2 \quad or \quad \rho_0 = \frac{p_0}{c^2}. \quad (8)$$

Comparison Eq. (3) with Eq. (8), we have

$$U_0 = p_0 V_0 = m_0 c^2. \quad (9)$$

This is the mass-energy equivalence principle in CoM frame. In other words, the mass density (ρ_0) and volumetric potential energy density (p_0) are equivalent, the proportional coefficient is the square of the wave speed, see Eq. (8). Since the net mass and energy across the boundary of the system is zero, it can be regarded as a closed reversible adiabatic system, the total internal energy is, thus, the Hamiltonian in the CoM frame:

$$H_0 = U_0 = p_0 V_0 = m_0 c^2. \quad (10)$$

Accordingly, the volumetric Hamiltonian density in the CoM frame thereby reads:

$$\mathcal{H}_0 = p_0 = \rho_0 c^2. \quad (11)$$

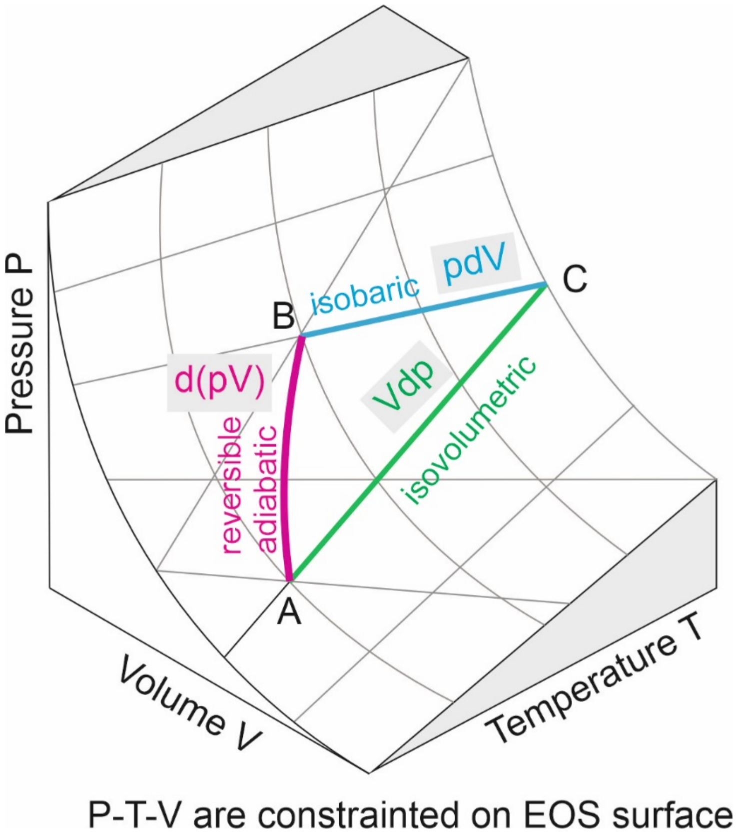


Fig. 2. An infinitesimal state change from A to B. Partial system is reversible adiabatically (isentropic) compressed from point A to B.

2.2 Wave Momentum and Energy Travelling in Field by a Finite Speed

Any disturbance (wave) will propagate in the field along the characteristic line with a speed of c, as shown in Fig.3. Waves, like any moving object, carry and transport momentum and energy as they propagate in field.

If another observer is also located in CoM coordinate system (having no relative motion to the disturbance source), though the observer cannot notice a macroscopic kinetic energy, but he will feel the momentum and energy, carried by the wave through the medium, with some time lag, due to the finite wave speed.

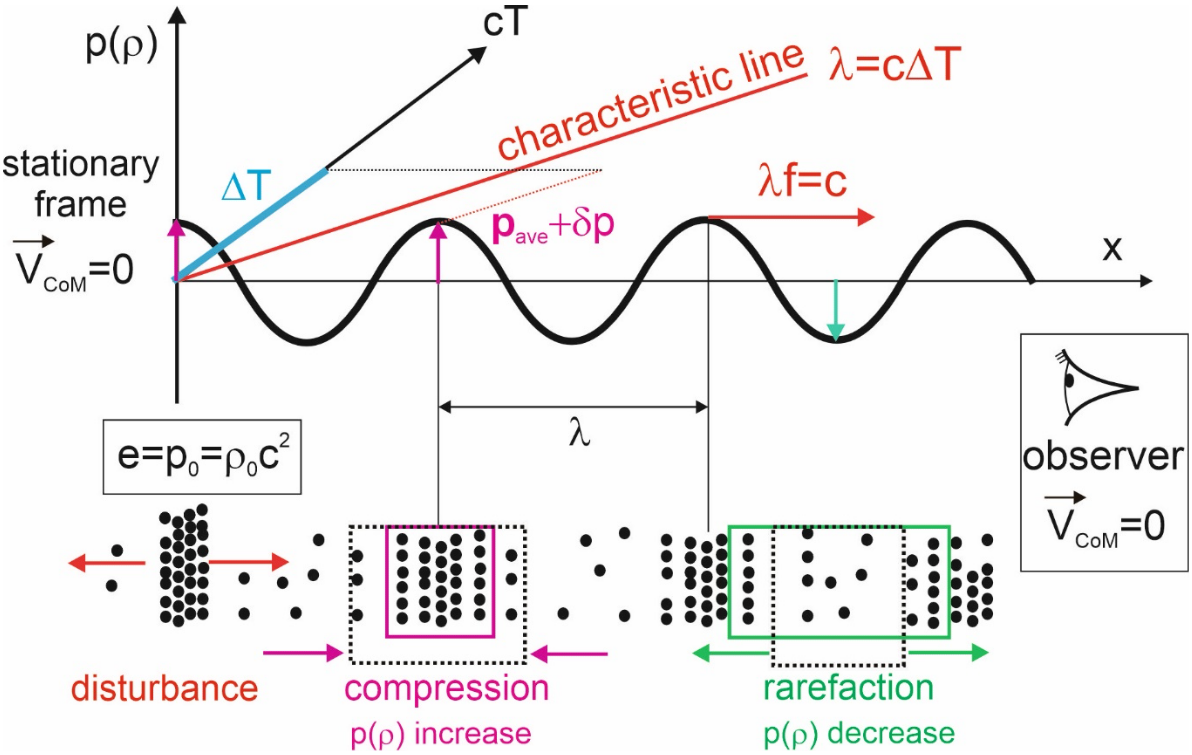


Fig.3. volumetric potential energy density, $p_0=\rho_0c^2$, is propagating in field in the form of wave by a finite wave speed of c along the wave prorogation characteristic line,

Wave momentum may be defined as the following form,

$$\vec{p}_{w0} = \frac{\mathcal{H}_0}{c} = \frac{\rho_0 c^2}{c} = \rho_0 c. \tag{12}$$

The wave momentum flux (wave propagation energy) is

$$\mathcal{H}_{w0} = \vec{p}_{c0} \cdot \vec{c} = \rho_0 c^2 = p_0. \tag{13}$$

Accordingly, mass or mass density is a scalar parameter, it is a carrier of wave momentum and energy in CoM frame, if there exists a wave in space.

Mass density can also be defined as the ratio of wave energy density to the square of wave speed:

$$\rho_0 = \frac{\mathcal{H}_{w0}}{c^2}. \tag{14}$$

As expressed by Eq. (8), the volumetric potential energy density (pressure) is a function of position in the compressible fluid field, and disturbance propagation speed (pressure wave):

$$p_0 = p(\vec{r}, c, t). \tag{15}$$

The interactions (disturbances) are propagating through the compressible fluid field by a finite speed of c: there is a time lag between when the state change of the fluid particle in position *a* and when other fluid particle in position *b* experience this disturbance.

2.3 Different from Incompressible Fluid in CoM Coordinate

There is a fundamental difference between the compressible and incompressible fluids. For incompressible fluid, according to definition, the compressibility (β) of a fluid is zero:

$$\beta = -\frac{1}{V_0}\frac{\partial V}{\partial p} = 0. \tag{16}$$

This implies:

$$\frac{\partial V}{\partial p} = 0, \tag{17}$$

and therefore:

$$dV = 0. \tag{18}$$

Under this condition, Eq. (4) and (5) are no longer valid (in fact the bulk modulus is infinitely great, $B_0 = \infty$, according to the definition). Any disturbance will instantaneously propagate the whole field by an infinitely great wave propagation speed, namely, $c \approx \infty$, without any time lag, in spite of how big the field is. Under this circumstance, the volumetric potential energy density in the fluid field depends merely on the position of $\vec{r}$.

$$p_0 = p(\vec{r}, t). \tag{19}$$

as is used in the classical Newtonian mechanics [1].

For incompressible fluids, we cannot write out an equation to describe the wave dynamic behavior of the particles, since the infinitely great bulk modulus and wave propagation speed, thus, the disturbed field cannot be described by any wave propagation function. In other word, if we persist in using wave function to describe this field, any particle will feel the state change of other particles instantaneously, no matter how far apart they are, provided they are in the same field. Just like the quantum entanglement behavior, where the potential energy in the Schrödinger equation is merely a function of position, which implies that the Schrödinger equation assumes the wave propagation speed is infinitely great. That is the so-called “spooky action at a distance” by Einstein.

3.Observing in an Inertial Reference Frame – Lab Frame

In CoM coordinate, the particle system is stationary to an observer, who is moving with the same velocity as the particle system, and therefore, the system has zero momentum and kinetic energy in this frame, though he can feel the momentum and energy, carried by the wave, as shown in Fig. 3 (a). The CoM coordinate is simpler, but not applicable in practice: It is not convenient to measure the physical properties, co-coming with the fluid flow, furthermore, the CoM coordinate is even not an inertial reference frame, if we describe the particle motions in this coordinate, we may ignore some important dynamic behavior of the particles, even give out incorrect interpretations.

3.1 Relative Motion Causing Length Contraction

For describing the motion, we should choose a suitable inertial frame of reference, saying the Lab. frame as our reference frame, as we generally use every day, (strictly speaking, there is no universal inertial frame, but as an approximation, we can assume the Lab frame as an inertial frame).

By contrast to the CoM frame, the particle may have a velocity relative to the Lab frame and thus processes kinetic energy, when the particle is passing an observer, who is located in the Lab reference frame, as shown in Fig. 4 (b).

The velocity, and thereby the kinetic energy of a particle is frame-dependent (relative). On the other hand, the propagation speed of a pressure wave is a property of the medium through which the sound (pressure wave) travels. It is, however, frame independent.

As illustration and simplification, the Lab frame axis is so orientated that the flow is one-dimensional, saying along the x-axis, see Fig. 4 (b). We can consider two scenarios. The first scenario is the macroscopic flow velocity is zero, the particle has only oscillation motion around its equilibrium position, this is the CoM coordinate. As illustrated in Fig. 4 (a), after a time interval the wave propagates a distance in medium, saying from point O to point A: $x_0 = c\Delta t$ (observer and wave source have no relative motion and observer is located at point A), here Δt is the measured time interval in Lab frame. The second scenario is the macroscopic flow velocity is not zero, but rather has a velocity of $\vec{v}_{CoM}$, relative to the point O, after the same time interval, the particle moved towards the right, having a distance to O of $\Delta x_{CoM} = \vec{v}_{CoM}\Delta t$, however, the wave produced by the moving particle also travels to point A through the medium, $\Delta x_{mov} = c\Delta t_{mov}$, here Δt_{mov} is the measured time interval if the observer is co-moving with the particle, and $\Delta x_{mov} \leq x_0$. The wave traveling length appear foreshortened in the direction of motion, just because the particle has a relative velocity to the observer in the Lab frame, as shown in Fig 4 (b). This is the so-called length contraction effect in special relativity.

For an observer, who is located at point A in space, the frequencies and periods are indeed different for the two scenarios, but the wave velocity is the same, they share the same wave propagation characteristic line, in spite of whether particle is moving or not.

$$\lambda f = \lambda_{mov} f_{mov} = c. \tag{20}$$

where λ and f are the wave length and frequency observed by the observer in CoM frame, λ_{mov} and f_{mov} are wave length and frequency produced by the moving particle.

The Observer – who is located at point A, will feel the same wave crest and trough "passing through" him, regardless of whether the wave source moving or not. Along the wave characteristic line:

$$\begin{cases} \dfrac{x_0}{t_0} = \dfrac{\lambda}{T} = c \\ \dfrac{x_{mov}}{t_{mov}} = \dfrac{\lambda_{mov}}{T_{mov}} = c \end{cases}. \tag{21}$$

where T and T_{mov} are wave periods produced by CoM and moving particles, respectively, see Fig. 4(a), and Fig. 4(b).

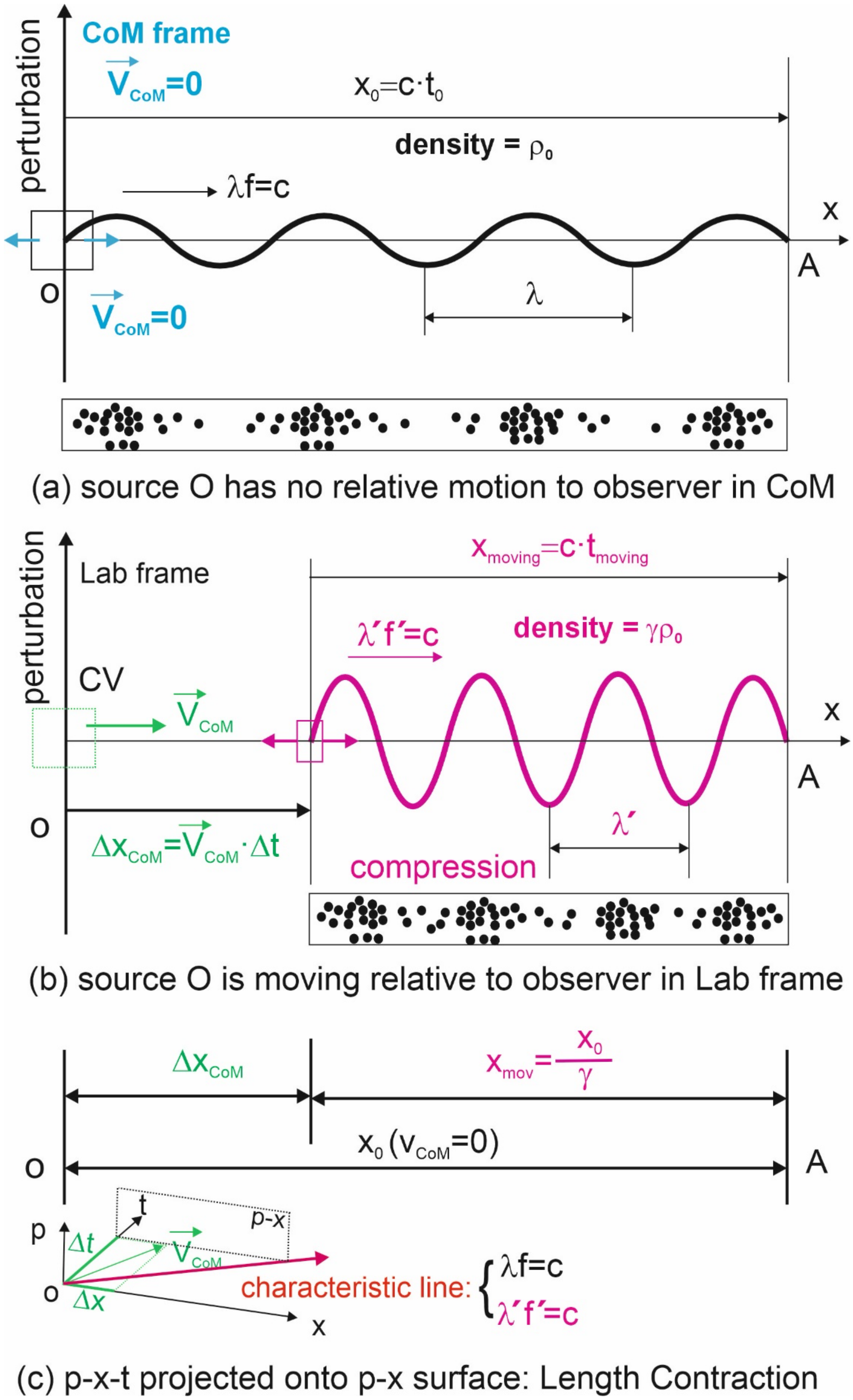


Fig. 4. The wave traveling length appear foreshortened in the direction of motion, if the particle has a relative velocity to the Lab frame.

This relation has a fundamental meaning: the position and time are not two independent variables, both variables are just constrained by Eq. (21), only one variable can be chosen as an independent variable, and another one is expressed by a function of the first one, namely through the wave speed of Eq. (21), either in CoM frame or in the Lab frame. That means the wave speed is a key parameter for compressible flow. Furthermore, the position and time pairs in different frames are also related to each other.

3.2 Lorentz Length Contraction and Density Increase Effect

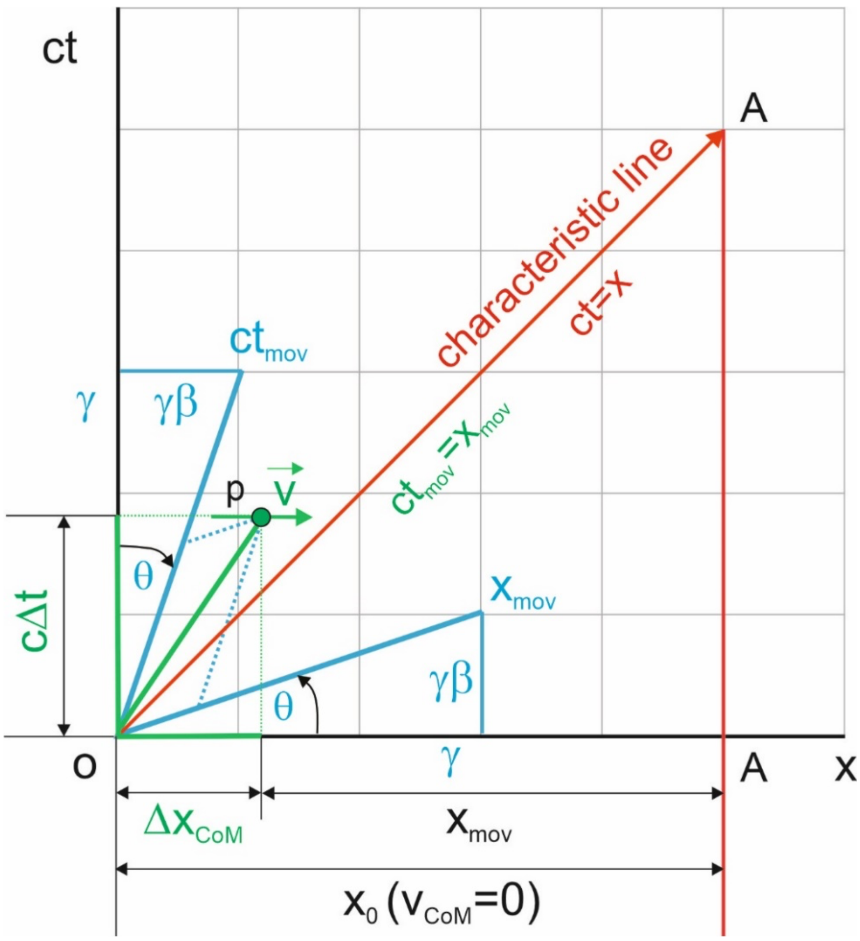


Fig. 5. Lorentz Transformation for two scenarios

Either for an observer in Lab frame or from the view of a moving particle, the wave speed is invariant. The Lorentz transformation guarantees the wave speed to be invariant in different frames.

Fig. 5 shows the Lorentz transformation. (O-x-ct) is the Lab reference frame, and (O-x_{mov}-ct_{mov}) is the moving particle reference frame. The wave travels from point O to point A along the characteristic line in some time interval for two scenarios. The Lab observer and moving particle share the same characteristic line.

From the view of the Lab frame, the particle flows into the positive direction of x, introducing a dimensionless factor of

$$\beta = M = \frac{v}{c}. \tag{22}$$

where v is the particle moving velocity, relative to Lab frame, c is the wave speed. In fluid dynamics, we also call this ratio as Mach number.

The Lorentz transformation from moving particle frame to Lab frame reads:

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct_m \\ x_m \end{bmatrix}. \tag{23}$$

where, γ is the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{24}$$

x^m and t^m are measured distance and time interval in moving particle frame. t and x are measured distance and time interval in Lab frame.

From Eq. (23) we can get the spatial distance transformation in moving direction of x, from moving particle frame to Lab frame,

$$x = (\gamma\beta)(ct_m) + \gamma x_m. \tag{25}$$

Then a length measured in Lab frame is

$$x_0 = (x_2 - x_1) = \gamma\big(x_{m,2} - x_{m,2}\big) + \gamma v\big(t_{m,2} - t_{m,1}\big). \tag{26}$$

Since the two measurements made simultaneously in moving particle frame, $t_{m,2} = t_{m,1}$, thus:

$$x_0 = \gamma x_{mov} \quad or \quad x_{mov} = \frac{x_0}{\gamma} = \alpha x_0 \tag{27}$$

It seems the length is contracted in the direction of motion. The contraction factor is the reciprocal of Lorentz factor: $\alpha = 1/\gamma$. The particle mass is invariant, recalling the mass density definition,

$$\rho_{mov} = \frac{m_0}{x_{mov}} = \frac{m_0}{\frac{x_0}{\gamma}} = \gamma\left(\frac{m_0}{x_0}\right) = \gamma\rho_0\,. \tag{28}$$

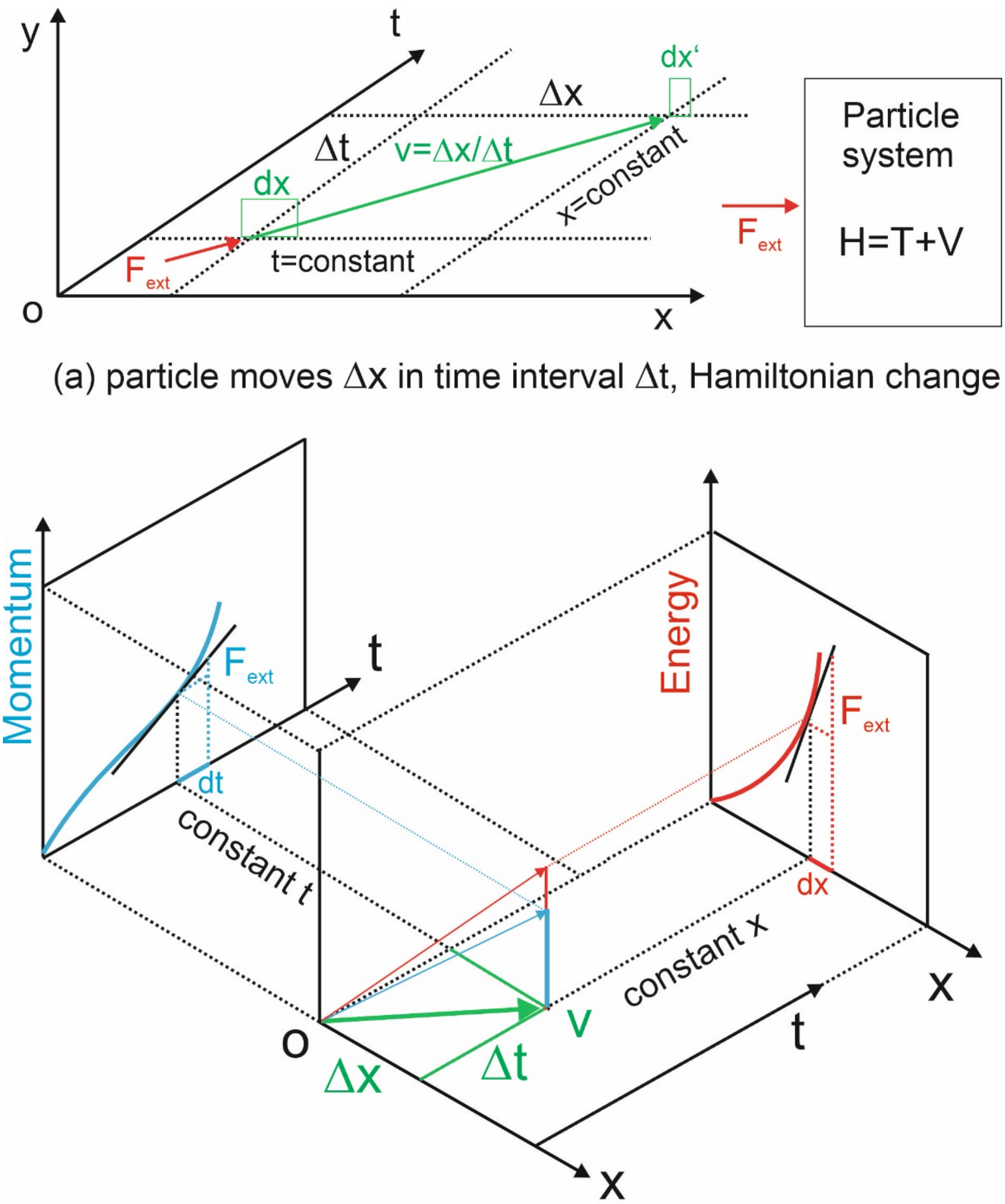
So, in the Lab frame, because of the relative motion of the particle, the length is contracted, which leads the fluid to be compressed, as a result, the mass density increase, and the length compression factor is the reciprocal of the Lorentz factor, therefore, the density increase factor is the Lorentz factor, γ. This compression and density increase effect is illustrated in Fig. 4 (b) and (c).

3.3 Hamiltonian and Lagrangian Density in Lab Frame

Observed in the Lab frame, the particle has a velocity, thus it possesses momentum, the momentum along x-direction for compressible flow is

$$\vec{p} = \rho_{mov}\vec{v} = (\gamma\rho_0)\vec{v} = (\gamma\beta)(\rho_0 c). \tag{29}$$

Compared with Eq. (12), it seems the wave momentum increase a factor of $(\gamma\beta)$.



(a) particle moves Δx in time interval Δt, Hamiltonian change

(b) momentum-time and energy-x equivalence

Fig. 6. External force acts on system, projected onto momentum-time and energy-spatial coordinates.

If an external force acts on the system, the linear momentum will change. By definition, we have the following equation in inertial frame – the Lab frame,

$$\vec{F}_{external} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma\rho_0\vec{v}). \tag{30}$$

where $\vec{F}_{external}$ represents volumetric external force density and $(\gamma\rho_0\vec{v})$ is the volumetric linear momentum density.

This external force density obeys other useful identities (surroundings do work on the system, thereby the energy contents within the system will increase), such that

$$\vec{F}_{external} = \frac{d\mathcal{H}}{dx}. \tag{31}$$

where $\mathcal{H}$ is the total volumetric energy density inside the system (including kinetic and potential energy density, we call it as Hamiltonian density).

Comparison Eq. (30) with (31), we have the following relation:

$$\frac{d\vec{p}}{dt} = \frac{d\mathcal{H}}{dx} = \vec{F}_{external}. \tag{32}$$

Thus, the infinitesimal energy change of the system is

$$d\mathcal{H} = \frac{d\vec{p}}{dt}dx. \tag{33}$$

Substituting Eq. (29) into Eq. (33), using the chain rule, we have

$$d\mathcal{H} = (\gamma\rho_0 v)dv + v^2 d(\gamma\rho_0). \tag{34}$$

If the fluid is incompressible ($c = \infty, \beta = 0\ and\ \gamma = 1$), the first term in the LHS is just the infinitesimal change of the kinetic energy in Newtonian mechanics, relative to some inertial coordinate system, for a system whose mass density, ρ_0 , keeps constant during the motion.

$$dT_0 = (\rho_0 v)dv \quad and \quad T_0 = \frac{1}{2}(\rho_0 v^2) \tag{35}$$

The second term of the LHS of Eq. (34) represents the increase of the volumetric energy density, due to the mass density increase effect. This effect (mass and energy gaining) seems to transfer mass and energy into the system due to the relative motion, similar to a variable-mass system.

Using the product rule, the second term of Eq. (34) can be re-written as:

$$v^2 d(\gamma\rho_0) = d(\gamma\rho_0 v^2) - 2(\gamma\rho_0 v)dv. \tag{36}$$

Thus, Eq. (34) can be expressed as:

$$d\mathcal{H} = (\gamma\rho_0 v)dv + [d(\gamma\rho_0 v^2) - 2(\gamma\rho_0 v)dv]. \tag{37}$$

Integrating along the velocity direction (integrating along the velocity direction implies both time coordinate and position coordinate change simultaneously, see Fig. 6 (a)), with the initial value of $\mathcal{H} = \rho_0 c^2$ at v=0, we have

$$\mathcal{H} = -\alpha\rho_0 c^2 + [\gamma\rho_0 v^2 + 2\alpha\rho_0 c^2] = \alpha\rho_0 c^2 + \gamma\rho_0 v^2. \tag{38}$$

It can be seen that the “variable-mass” term can be split into two parts: $(\gamma\rho_0 v^2)$ and $(2\alpha\rho_0 c^2)$. The Hamiltonian can be written explicitly as:

$$\mathcal{H} = \rho_0 c^2 \left(\sqrt{1-\beta^2}\right) + \rho_0 v^2 \left(\frac{1}{\sqrt{1-\beta^2}}\right). \tag{39}$$

The first term of LHS can be interpreted as "potential energy density",

$$V(v, c, t) = \alpha\rho_0 c^2 = \alpha p_0, \tag{40}$$

and the second term as "kinetic energy density",

$$T(v, c, t) = \gamma\rho_0 v^2 = (\gamma\beta\rho_0)cv = (\gamma\beta^2)(\rho_0 c^2) = (\gamma\beta^2)p_0. \tag{41}$$

Compared with Eq. (13), it seems the wave energy increase a factor of $(\gamma\beta^2)$.

Thus, Hamiltonian density can be written more concisely as:

$$\mathcal{H}(v, c, t) = V + T. \tag{42}$$

Both "potential" and "kinetic" energy are functions of flow velocity and wave propagation speed, thus, strictly speaking, both terms are inseparable. The "kinetic" energy will increase if the flow velocity increases, while the "potential" will decrease if the flow velocity increases. Furthermore, because of the finite wave propagation speed inside the factor γ and α, any change of the "potential" and "kinetic" energy cannot be felt instantaneously by other particles in the field, there always have a time lag, more or less. This is different from classical Newtonian mechanics.

It can be seen, that when velocity approaches zero, the "kinetic" energy becomes zero, while the "potential" energy will increase to Eq. (8), namely the volumetric energy density, $p_0 = \rho_0 c^2$, in CoM coordinate. It is nothing else, just the stagnation pressure in CoM coordinate, in fluid statics, it is also called the hydrostatic pressure.

On the other hand, when the velocity approaches the wave speed, the "potential" energy decreases to zero, while the "kinetic" energy becomes to infinitely great since the volume is compressed to an infinitely small layer so that the mass density grows infinitely great, see Fig. 7.

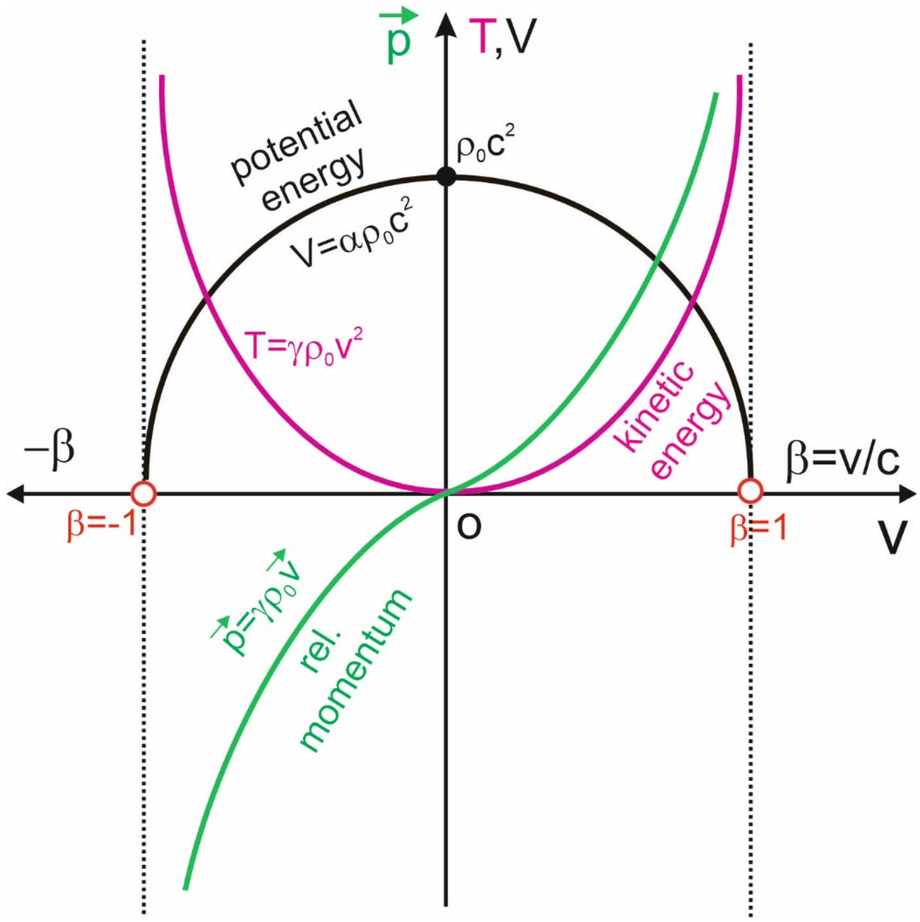


Fig. 7 Generalized momentum, kinetic and potential energy density as a function of relative velocity in subsonic flow regime. Sonic condition ($\beta = 1$) is not defined, where Lorentz factor $\gamma \to \infty$.

If we define Eq. (29) as the generalized momentum, Hamiltonian can be expressed by the generalized momentum as

$$\mathcal{H}(\vec{p}, c, t) = \frac{(\gamma\rho_0 v)^2}{\gamma\rho_0} + \alpha\rho_0 c^2 = \frac{\vec{p}^2}{\gamma\rho_0} + \left(\frac{c}{\gamma^2\beta}\right)\vec{p}. \tag{43}$$

From the relationship between Lagrangian and Hamiltonian:

$$\mathcal{H} = \vec{p} \cdot \vec{v} - \mathcal{L} \quad or \ \mathcal{L} = \vec{p} \cdot \vec{v} - \mathcal{H}. \tag{44}$$

and substituting Eq. (29) into Eq. (44), using the result of Eq. (38), we can get:

$$\mathcal{L} = -V = -\alpha\rho_0 c^2 = -\frac{1}{\gamma\beta^2}(\rho_0 v^2). \tag{45}$$

When velocity approaches to zero, the Lagrangian density will approach to the negative potential energy density in CoM frame: where the kinetic energy approaches to zero [2].

$$\mathcal{L} = -\rho_0 c^2 = -p_0. \tag{46}$$

It can be proved, that the derivative of Lagrangian density with respect to velocity gives out the generalized momentum density of Eq. (29).

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\vec{v}} = (\gamma\rho_0\vec{v}). \tag{47}$$

With the definition of Eq. (22), $\beta = v/c$, we can re-write Eq. (38) as

$$\mathcal{H} = \gamma\rho_0 v^2 + \alpha\rho_0 c^2 = \gamma\rho_0 c^2 = \gamma p_0. \tag{48}$$

where we use the following identity:

$$\gamma\beta^2 + \alpha = \gamma\beta^2 + \frac{1}{\gamma} = \gamma. \tag{49}$$

The total energy density, observed in the Lab frame (kinetic plus potential) can also be expressed as another pure potential energy density, as if the observer is just co-moving with the particle (so that no kinetic energy, only potential energy), but multiplying a factor of $\gamma \geq 1$, since the compression effect of the mass density. That means if an observer is just co-moving with the particle, he will observe another stagnation pressure, it will change to (γp_0). As illustrated in Fig. 8.

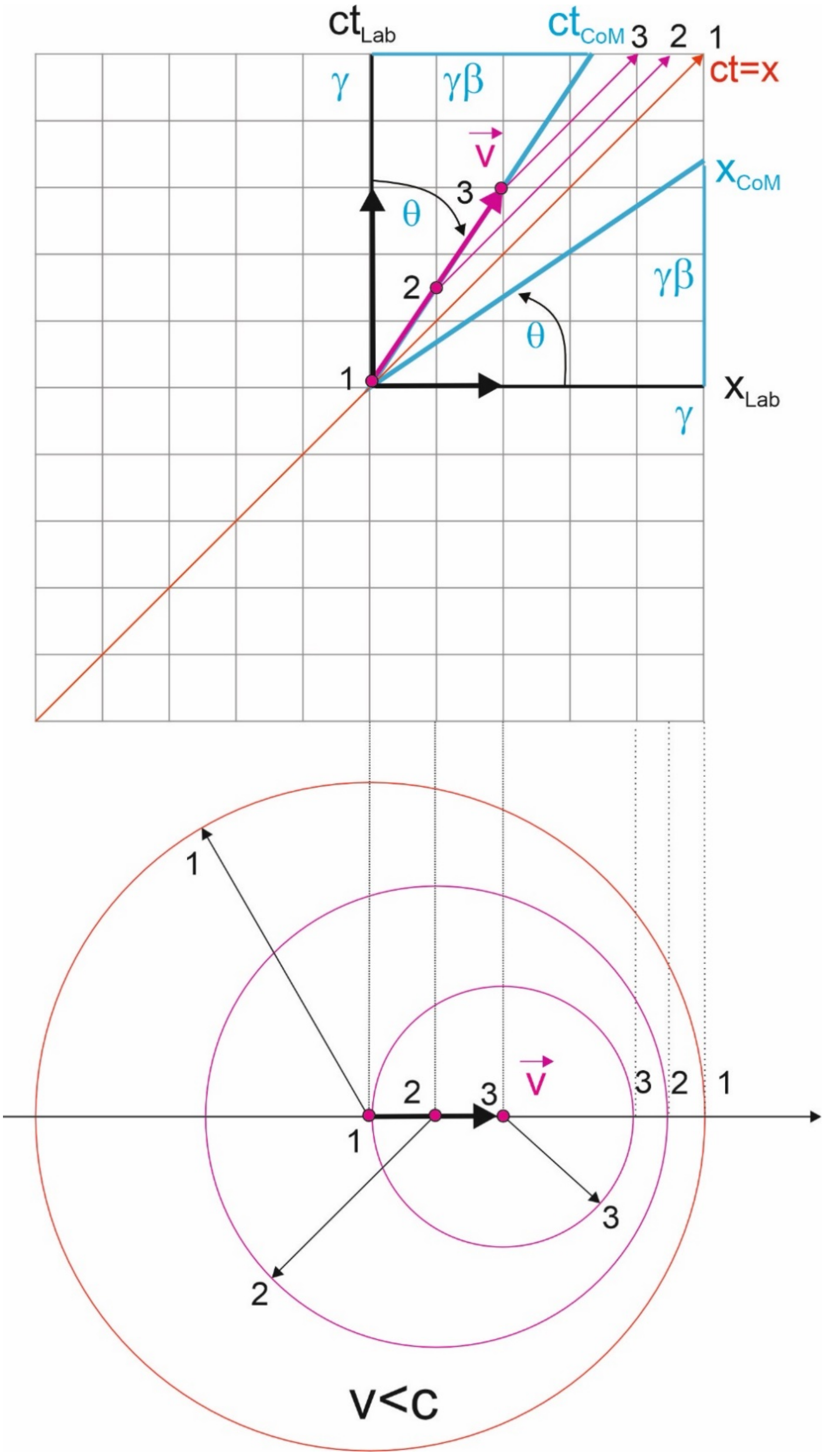


Fig. 8 An observer is just co-moving with the particle in the Lab frame. The kinetic energy is zero in view of the particle, while the stagnation pressure will increase.

If the kinetic energy, potential energy and Hamiltonian are normalized by the potential energy in CoM frame of $p_0 = \rho_0 c^2$, they are reads:

$$\begin{cases} \frac{T}{\rho_0 c^2} = \gamma\beta^2 \\ \frac{V}{\rho_0 c^2} = \alpha \\ \frac{\mathcal{H}}{\rho_0 c^2} = \gamma \end{cases} . \tag{50}$$

3.4 Supersonic Flow Regime

The Lorentz factor is not defined by $v = c$. However, when the flow velocity is bigger than the wave propagation speed, the flow goes into the supersonic flow regime, and both Lorentz factor, γ, and expansion factor, α, become thereby imaginary numbers. Under these circumstances, the Hamiltonian of Eq. (48) can be expressed as:

$$\frac{\mathcal{H}}{\rho_0 c^2} = \frac{1}{i\left(\sqrt{\beta^2 - 1}\right)} = -\delta i. \tag{51}$$

and the potential and kinetic energy of Eq. (40) and (41) can be now expressed as:

$$\frac{V}{\rho_0 c^2} = \pi i. \tag{52}$$

$$\frac{T}{\rho_0 c^2} = -\delta\beta^2 i. \tag{53}$$

where

$$\delta = \frac{1}{\sqrt{\beta^2 - 1}} \; and \;\; \pi = \sqrt{\beta^2 - 1}\,. \tag{54}$$

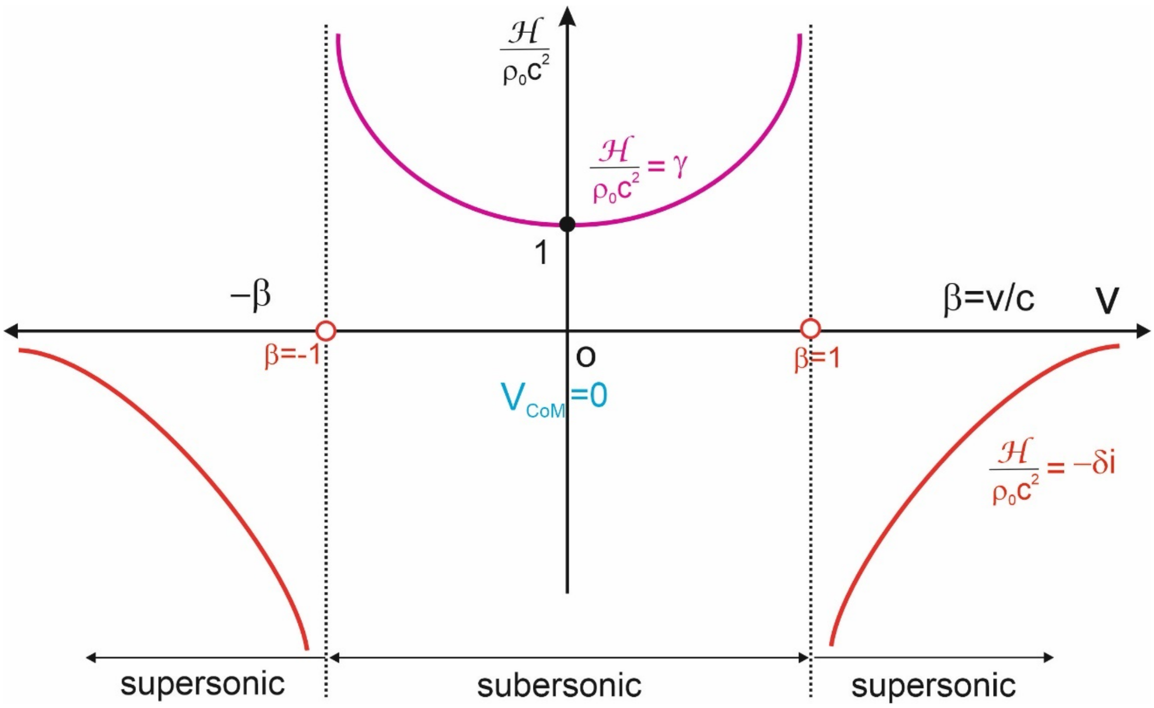


Fig. 9 normalized Hamiltonian as a function of relative velocity in subsonic and supersonic flow regimes

Fig. 9 illustrates the normalized Hamiltonian as a function of relative velocity in subsonic and supersonic flow regimes. The normalized kinetic and potential energy are given in Fig. 10.

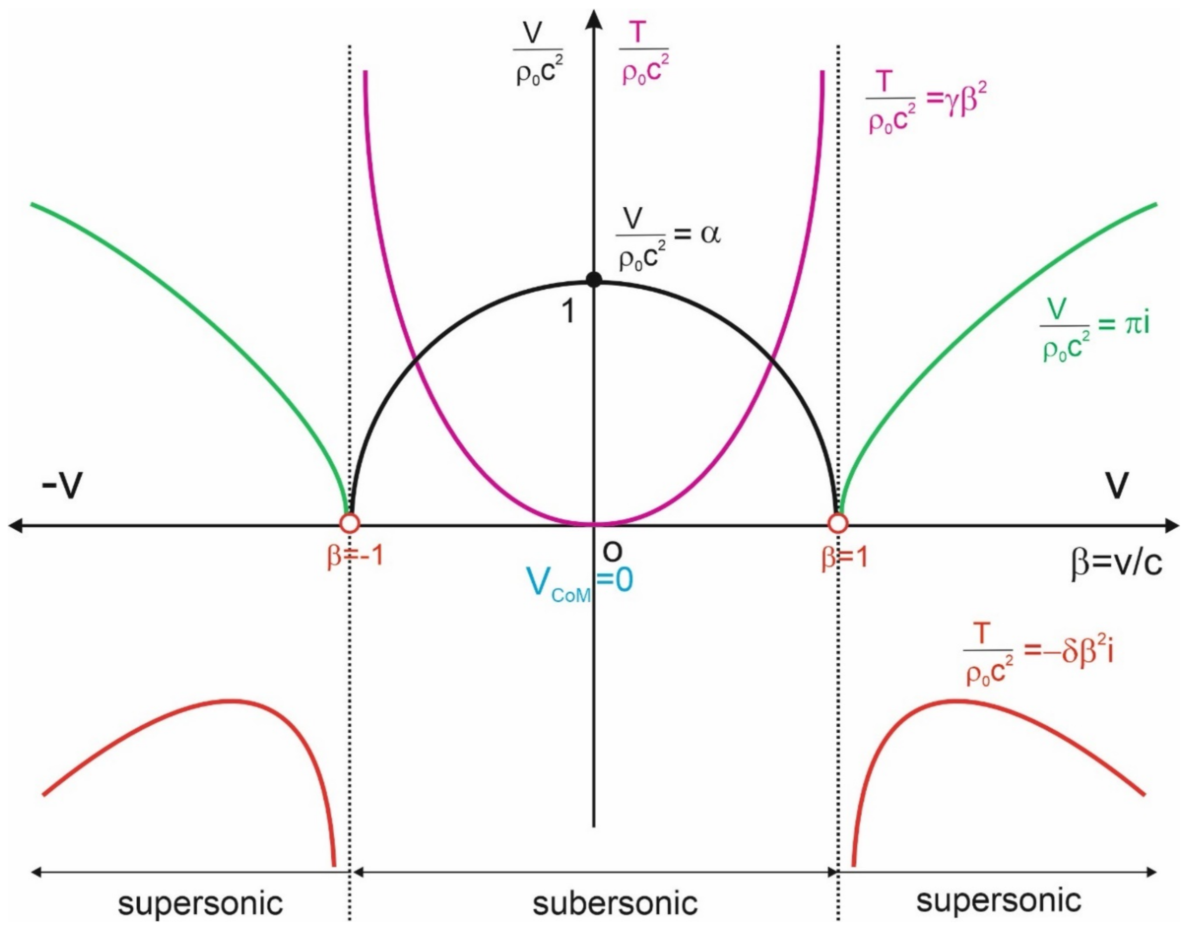


Fig. 10. normalized kinetic and potential energy as a function of relative velocity in subsonic and supersonic flow regimes

3.5 Comparison with Incompressible Flow

Different from the compressible fluid, where both "kinetic" and "potential" energy density are not only a function of velocity but also a function of wave propagation speed, see Eq. (38), (40) and (41).

For incompressible flow, the density is not a function of the velocity and wave speed, it is just a function of spatial position in the flow field, (in fact, for the incompressible fluid model, the wave propagation speed is infinitely greater, thereby Lorentz factor $\gamma = 1$, thus, no relative compression effect for incompressible fluid).

$$\rho = \rho_0(\vec{r}, t). \tag{55}$$

Thus, the kinetic energy density degenerates to the classical expression, as is expressed by Eq. (35), it has nothing to do with wave propagation speed, just like the classical definition in Newtonian mechanics.

$$T = \frac{1}{2}(\rho_0 v^2) = \frac{\vec{p}^2}{2\rho_0} \tag{56}$$

where the momentum reads:

$$\vec{p} = \rho_0 \vec{v} \tag{57}$$

Though the potential energy (pressure) varies from point to point in the field, the interaction energies (forces) between particles have no time lag, it depends neither on particle velocity (thereby the momentum), nor wave propagation velocity,

Thus, both kinetic energy and potential energy density in the fluid field depend merely on the position of $\vec{r}$. Under this circumstance, the Lagrangian and Hamiltonian density for incompressible fluid read:

$$\mathcal{L}(\vec{r}, \vec{v}, t) = \frac{1}{2}(\rho_0 \vec{v}^2) - p(\vec{r}, t) \tag{58}$$

$$\mathcal{H}(\vec{r}, \vec{p}, t) = \vec{v} \cdot \frac{\partial \mathcal{L}}{\partial \vec{v}} - \mathcal{L} = \frac{\vec{p}^2}{2\rho_0} + p(\vec{r}, t) \tag{59}$$

4. Euler Coordinates

In fluid dynamics, in contrast to the “material” Lagrangian point of view, we usually introduce a coordinate frame of a stationary observer who is looking at the particle motion from the outside – Eulerian coordinate. In the Eulerian description the coordinates are time t and a spatial vector, where the spatial vector does not label the position of a “material” particle, but rather that of a geometrical point in spatial space, it does change with time.

In fact, for the field, each geometrical point in spatial space can serve as a “stationary observer”, namely an inertial frame, since it has no relative motion to Lab frame. In other words, the total geometrical point set in spatial space constitutes a grid of inertial frames.

Accordingly, the physical quantities in the Eulerian specification are described by fields on space-time. Such a fundamental field is the velocity field, it is defined such that it gives the value of the Lagrangian velocity of a material particle just passing through a spatial position point (inertial frame) at time t. Another important field is the Hamiltonian density field, which is variant from point to point in space.

In classical mechanics, the Hamiltonian equation of motion for the generalized momenta reads (the rate of change of momentum is equal to the negative gradient of energy for a closed system):

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \qquad (60)$$

where p_i and q_i are the generalized momenta and the generalized coordinates. For the sake of simplification and illustration, we consider here only one degree of freedom (DoF).

Substituting the generalized momenta of Eq. (29) into Eq. (60), we have

$$\frac{d\vec{p}}{dt} + \frac{\partial \mathcal{H}}{\partial x} = 0. \tag{61}$$

The material derivative in Eulerian description of a continuum field reads

$$\frac{d\vec{p}}{dt} = \left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x}\right)(\gamma \rho_0 v). \tag{62}$$

The first term of the LHS is the partial derivative of momentum density with respect to time (along x=constant line) in the field, and the second term is the partial derivative of momentum with respect to spatial coordinate of x (along t=constant line), and dot products the particle velocity vector, see Fig. 6. Since particle in an infinitesimal time interval of ∂t travels an infinitesimal distance of ∂x in the field relative to the Eulerian coordinate.

In field, the derivative of Hamiltonian with respect to position can be written as:

$$\frac{\partial \mathcal{H}}{\partial x} = \frac{\partial \mathcal{H}}{\partial v} \cdot \frac{\partial v}{\partial x}. \tag{63}$$

As argued by the derivation of Eq. (38), the derivative of Hamiltonian along the velocity direction implies both time coordinate and position coordinate change simultaneously in the field.

As an approximation, we assume the speed of wave propagation speed is locally constant in the vicinity of a point in the field.

Substituting the Eq. (38) into Eq. (63), with a bit algebra manipulation, we can get

$$\frac{\partial \mathcal{H}}{\partial x} = \left(\rho_0 \gamma^3 \frac{v^3}{c^2} + \gamma \rho_0 v\right) \frac{\partial v}{\partial x}. \tag{64}$$

Substituting Eq. (62) and (64) into Eq. (61), we have

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x}\right)(\gamma \rho_0 v) + \left(\rho_0 \gamma^3 \frac{v^3}{c^2} + \gamma \rho_0 v\right) \frac{\partial v}{\partial x} = 0. \tag{65}$$

Taking out a common factor of $(\gamma \rho_0 v)$ and rewrite the last term:

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x}\right)(\gamma \rho_0 v) + (\gamma \rho_0 v)(\gamma^2 \beta^2 + 1) \frac{\partial v}{\partial x} = 0. \tag{66}$$

Recalling the compressed density definition of Eq. (28), Eq. (66) can be written as:

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x}\right)(\rho v) + \left[(\gamma^2 \beta^2 + 1) \frac{\partial v}{\partial x}\right](\rho v) = 0. \tag{67}$$

It can also be re-written as:

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x}\right)\vec{p} + \left[(\gamma^2 \beta^2 + 1) \frac{\partial v}{\partial x}\right]\vec{p} = 0. \tag{68}$$

Obviously, the solution for this equation is just the relative momentum of the Eq. (29).

It should be noticed again; that this equation is not defined at $\beta = v/c = 1$, where the Lorentz factor is infinitely great.

5. Summary

Quite different from the incompressible fluid flow, where the wave propagation speed is infinitely great or not defined. This paper focuses on the Hamiltonian and dynamics of the compressible fluid flow with a finite wave propagation speed. It explores the concept and implications of choosing the center-of-linear-momentum coordinate frame. The initial system is assumed to be an infinitely dilute no-interacting particle system and thus no potential energy in this system. It reveals the equivalence and relationship between potential energy, pressure, and volumetric energy density (Hamiltonian) in the CoM frame. The equivalence between mass density and volumetric potential energy density in the CoM frame is emphasized, with the Hamiltonian representing the total internal energy in a closed reversible adiabatic system. The flow dynamics of the compressible fluid are generally described in the Lab frame – a pseudo inertial frame. The system has a motion relative to the Lab frame. This relative motion causes the length contraction and thereby density increase in the context of motion description and frame selection because the wave propagation speed is frame-independent. The mass density increase factor is the Lorentz factor. The Hamiltonian density was given out in the Lab frame. Furthermore, it discusses the Hamiltonian density concisely represented as the sum of potential and kinetic energy, both dependent on flow velocity and wave propagation speed, with a focus on the inseparability of these energy terms due to finite wave propagation speed. In the supersonic flow regime, both the Lorentz factor and expansion factor become imaginary numbers, leading to unique expressions for the Hamiltonian, kinetic energy, and potential energy. In the comparison with the incompressible flow, it is highlighted that for incompressible fluids, the compressibility is zero, leading to instantaneous propagation of disturbances at an infinitely great wave speed in the whole field without any time lag. This results in the potential energy density in the fluid field depending solely on position, with no wave function to describe dynamic behavior. At last, the conservation of momentum in the Euler coordinate system is derived. It is highlighted that for compressible fluids, there exists a singularity, namely when the flow velocity equals the wave speed, where the mass and energy densities are compressed to an infinitely great value.

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