



Some mathematical issues regarding a new approach towards quantum foundation

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Abstract

In this article, the weakest possible theorem giving a foundation behind the Hilbert space formalism of quantum theory is stated. The necessary postulates are formulated, and the mathematics is spelled out in detail. It is argued that, from this approach, a general epistemic interpretation of quantum mechanics is natural. Some applications to the Bell experiment and to decision theory are briefly discussed. The article represents the conclusion of a series of articles and books on quantum foundations.

Keywords: Accessible theoretical variables; Bell experiment; decision theory; epistemic interpretation; inaccessible theoretical variables; postulates; quantum foundation.

1 Introduction

In a number of articles, the newest ones being Helland (2024a,b,c), this author has proposed a completely new foundation of quantum theory, a foundation based upon theoretical variables, which in a given context may be attached to an observer or to a group of communicating observers. These variables can be accessible or inaccessible. An example of a (maximal) accessible variable may be the positions $\mathbf{q} = (q_1, \dots, q_n)$ of n independent particles; another example may be their momenta $\mathbf{p} = (p_1, \dots, p_n)$. A typical inaccessible variable may be the vector (\mathbf{q}, \mathbf{p}) .

By assuming the existence of two different maximal accessible variables - in Niels Bohr's terminology two complementary variables - and making some additional assumptions, it is shown in the above articles that essentially the whole Hilbert space apparatus results. The purpose of the present article is to look closer at the additional assumptions. It turns out that these can be considerably weakened.

What we do have to assume, is that the two variables, called θ and η in the general theory, may be seen as functions of some basic inaccessible variable ϕ , and that groups act on both ϕ and θ .

In Helland (2024a) it was assumed that the two actual variables were related ($\eta(\phi) = \theta(k\phi)$ for some transformation k on the space Ω_ϕ , the range of ϕ). It is shown here that we only have to

assume: 1) The spaces Ω_θ and Ω_η have the same category; 2) There is a group M acting on Ω_ϕ . In the example above, we can let M be the multivariate Weil-Heisenberg group.

We also have to assume that there is a transitive group G acting on Ω_θ , and that a left-invariant measure μ with respect to G is given. In the above example, we can just let G be the translation group.

In Helland (2024a) it was assumed that a multivariate representation $U(\cdot)$ with certain properties existed. In the present paper, we show how such a simple representation can be constructed. This assumption is simply not necessary.

From this, the weakest possible version of my main theorem is given as Theorem 1 below. The conclusion of the theorem is that every accessible variable has a symmetric operator in $\mathcal{H} = L^2(\Omega_\psi, \nu)$ attached to it, where $\psi = (\theta, \eta)$, and ν is an invariant measure on Ω_ψ induced by the invariant measure μ on Ω_θ . This is the starting point for much of the Hilbert space apparatus. Apart from the above symmetry assumption, the essential assumption is only the existence of two complementary theoretical variables in the given context.

Note that there are no macroscopic assumptions here. Thus, this derivation also gives a foundation for what Khrennikov (2010, 2023) calls quantum-like models. These models have links to several scientific disciplines. The link to quantum decision theory will be discussed elsewhere. Links to relativity theory and quantum field theory are discussed in Helland (2023c) and in Helland and Parthasarathy (2024).

I will concentrate on the derivation of the Hilbert space apparatus and related derivations in this article. Assumptions that lead to the Born rule for probabilities are discussed, and the derivation is proved, in Helland (2024b). Derivation of the Schrödinger equation from a few postulates has been given for instance by Klein (2010).

Finally, it is a basic setting in that the theoretical variables are attached to an observer or to a group of communicating observers. A natural assumption in addition is that they are connected to the mind(s) of this/these observer(s). This leads to an epistemic interpretation of quantum theory, an interpretation that contains QBism (see for instance Fuchs et al., 2013) as a special case. Quantum theory is seen as a theory of our knowledge about the real world, not directly about the real world.

The plan of this article is as follows: In Section 2 the theory is outlined in its weakest possible version. In Section 3 some consequences, consequences to an understanding of the Bell theorem and to a new theory of decisions, are briefly discussed. In Section 4 some final remarks are given.

2 The main theorems

I repeat that my main notion is that of theoretical variables, which can be almost anything. The theoretical variables can be accessible or inaccessible. From a mathematical point of view, I only

assume that if λ is a theoretical variable and θ is a fixed function of λ , then θ is a theoretical variable. And if λ is accessible, then θ is accessible.

For physical modelling I assume a fixed context, and that an observer or a group of communicating observers in this context has/have a set of theoretical variables associated with him/them. In the case of a group of observers, their communication should be related to these variables. Then my first postulate is as follows:

Postulate 1

There is an inaccessible variable ϕ such that all accessible variables can be seen as functions of ϕ . There is a group M acting on Ω_ϕ .

In simple physical examples, such a ϕ can easily be constructed. As a general statement covering all possible situations, Postulate 1 can also be given a religious interpretation, see Helland (2022d, 2023d).

One possible option is to replace Postulate 1 with some assumptions in category theory; see the arXiv version of Helland (2024a). Category theory in the foundation of quantum mechanics has also been considered by others, for instance, Coecke and Papette (2009) or Döring and Isham (2008). This option will not be considered further in the present article.

My main theorems will refer to a situation where we have two different maximal accessible theoretical variables, which I, following Niels Bohr, will call two complementary variables. I will show that the whole Hilbert space apparatus follows under weak conditions from the assumption that we have two such maximal accessible variables. The term ‘maximal’ means roughly that the variable cannot be extended and still be accessible. To be precise, I need to define a partial ordering among the variables.

Definition 1

Say that $\theta \leq \lambda$ if $\theta = f(\lambda)$ for some function f .

This is a partial ordering among all theoretical variables, and also a partial ordering among the accessible ones. Note that ϕ from Postulate 1 is an upper bound in the accessible case. We will say that θ is a maximal accessible variable if it is maximal with respect to this partial ordering. By Zorn’s Lemma, which is equivalent to the Axiom of Choice, maximal variables always exist. For those who do not believe in Zorn’s Lemma, we add an additional postulate.

Postulate 2

For every accessible variable ξ there is a maximal accessible variable θ such that $\xi \leq \theta$.

In order to achieve a meaningful theory, we also need some symmetry assumptions. One such is given by the existence of a group M acting upon ϕ (Postulate 1). Another assumption is given

by the existence of a group G acting upon θ . In concrete examples, these groups can be easily constructed.

Postulate 3

To a given accessible theoretical variable θ there is a group G acting upon θ , and there is a left-invariant measure μ with respect to G on Ω_θ , the range space of θ . The group G is transitive and has a trivial isotropy group.

Conditions for the existence of an invariant measure are discussed in Helland (2021). Note that if an invariant measure is supposed to act on every single theoretical variable, there is consequently an invariant measure on every set of theoretical variables.

Finally, in this article, I will assume for two complementary variables θ and η :

Postulate 4

The range space Ω_θ is either finite and has the same number of values as Ω_η , or, more generally, Ω_θ and Ω_η have the same category.

(In terms of category theory, this means that Ω_θ and Ω_η are objects, and that there is a morphism from Ω_θ to Ω_η , and another morphism from Ω_η to Ω_θ . More intuitively, it means that there is a bijective function connecting Ω_θ and Ω_η .)

This is all we need for our first results, a Proposition and a basic Theorem.

Proposition 1

Assume that the basic inaccessible variable ϕ satisfies Postulate 1, and that two given accessible variables θ and η satisfy Postulate 4. Then θ and η are either in one-to-one correspondence, or the following holds: There exists an accessible variable ξ which is a bijective function of η , a transformation k in Ω_ϕ , and a function f acting on Ω_ϕ such that $\theta = f(\phi)$ and $\xi = f(k\phi)$.

In many applications, it turns out that this will hold with ξ equal to η . In that case we say that θ and η are *related*: $\theta = f(\phi)$ and $\eta = f(k\phi)$ for some k .

Proof

The finite-dimensional case was treated in Section 7.2 of Helland (2024a), so I will here look at the more general case. Choose a function f such that $\theta = f(\phi)$, and fix $\phi = \phi_1$. Let ϕ_2 be any point in Ω_ϕ such that $\eta(\phi_1) = f(\phi_2)$. Such a ϕ_2 must exist, since $\{\eta(\phi)\}$ has the same category as $\{\theta(\phi)\} = \{f(\phi)\}$.

The group M acting upon Ω_ϕ need not be transitive. The points ϕ_1 and ϕ_2 either lie on the same orbit of M or on different orbits. In the first case, there exists a $k \in M$ such that $\phi_2 = k\phi_1$, so that $\eta(\phi_1) = f(k\phi_1)$. In the second case, let a be a function of ϕ which characterizes the orbits. Then

there exists a ϕ_3 on the orbit containing ϕ_1 such that $\phi_2 = a(\phi_3)$, and by definition $\phi_3 = k\phi_1$ for some $k \in M$. In this case we get $\eta(\phi_1) = f(\phi_2) = f(a(\phi_3)) = f(a(k\phi_1))$. Define $\xi(\phi_1) = f(k\phi_1)$. Then $\eta(\phi_1) = f(a(f^{-1}(\xi(\phi_1))))$, which means that η is a bijective function of ξ . The inverse f^{-1} is well defined, since, by $\xi(\phi_1) = f(k\phi_1)$, and letting ϕ_1 vary, the range of ξ has the same category as the range of η .

Since this holds for every $\phi_1 \in \Omega_\phi$, the Proposition is proved.

□

The Theorem, which is a refinement of the basic Theorem 4 in Helland (2024a), runs as follows

Theorem 1

Assume Postulate 1 and Postulate 2. Let θ and η be two maximal accessible variables satisfying Postulate 4 that are not in one-to-one correspondence, and assume that they are real-valued or real vectors. Let θ satisfy Postulate 3. Then there exists a Hilbert space \mathcal{H} , and to every accessible variable ζ there exists a unique symmetric operator A^ζ in \mathcal{H} .

Proof

By Proposition 1 there exists a maximal variable ξ such that ξ and θ are related. Then it follows from Theorem 4 of Helland (2024a) (see also Theorem 1 of Helland, 2022a) that there exists a Hilbert space \mathcal{H} such that every accessible variable is associated with a unique symmetric operator in \mathcal{H} if the following condition holds:

There exists a unitary multi-dimensional representation $U(\cdot)$ of G such that for some $|\theta_0\rangle$ the coherent states $U(g)|\theta_0\rangle$ are in one-to-one correspondence with the values of $g \in G$ and hence with the values of θ .

I will now define a simple representation $U(\cdot)$ on the Hilbert space $L^2(\Omega_\theta, \mu)$ satisfying this condition..

Proposition 2

For $f \in L^2(\Omega_\theta, \mu)$ and $g \in G$, define $U(g)f(\theta) = f(g^{-1}\theta) = h(\theta)$. Then the above condition holds.

Proof of Proposition 2.

Lemma 1

$h \in L^2(\Omega_\theta, \mu)$.

Proof

By the invariance, $\int_{\Omega_\theta} |f(g^{-1}\theta)|^2 d\mu = \int_{\Omega_\theta} |f(\theta)|^2 d\mu < \infty$. □

Lemma 2

The mapping $g \rightarrow U(g)$ is a homomorphism.

Proof

$$U(g_1)U(g_2)f(\theta) = U(g_1)f(g_2^{-1}\theta) = f(g_2^{-1}g_1^{-1}\theta) = f((g_1g_2)^{-1}\theta).$$

$$U(g^{-1})f(\theta) = f(g\theta) = U(g)^{-1}f(\theta).$$

□

Lemma 3

$U(g)$ is unitary for every $g \in G$.

Proof

$$\int_{\Omega_\theta} (f_1(\theta)^* U(g)^\dagger) f_2(\theta) d\mu = \int_{\Omega_\theta} f_1(\theta)^* (U(g)^{-1} f_2(\theta)) d\mu.$$

□

Lemma 4

Choose $f_0 \in L^2(\Omega_\theta, \mu)$ such that $f_0(\cdot)$ is a bijective function of θ . Then there is a one-to-one correspondence between $g \in G$ and the coherent functions $f_g(\theta) = U(g)f_0(\theta) = f_0(g^{-1}\theta)$.

Proof

$f_0(g_1^{-1}\theta) = f_0(g_2^{-1}\theta)$ implies $f_0(g^{-1}\theta) = f_0(\theta)$ for $g = g_1^{-1}g_2$. Since f_0 is bijective, it follows that $g = e$. □

Theorem 1 follows now from Theorem 4 of Helland (2024a) and Proposition 2. □

This result, together with the other results of Helland (2024a,b,c) now gives a very simple alternative foundation of quantum theory. Note that this is a purely mathematical theory, and it can be interpreted in different directions. In an ordinary physical setting, it is natural to interpret the accessible theoretical variables as ordinary physical variables, but also connected to the mind of an observer or to the joint minds of a communicating group of observers. This gives a general epistemic interpretation of quantum theory, an interpretation which has QBism as a special case. Quantum theory is then a theory of an observer's or a group of observers' knowledge about the real world, not a theory directly about the real world.

In this article, I will not go into detail with the results of the articles mentioned above, but one mathematical result deserves to be mentioned.

Theorem 2

If θ and η are related through a transformation k of Ω_ϕ , then there exists a unitary operator $S(k)$ such that $A^\eta = S(k)^\dagger A^\theta S(k)$.

Proof

See Theorem 5 of Helland (2024a). \square

I should also mention again the consequences for accessible theoretical variables that take a discrete set of values.

- Every accessible variable has a symmetric operator associated with it.
- The set of eigenvalues of an operator is equal to the possible values of the variable.
- An accessible variable is maximal if and only if all eigenvalues are simple.
- The eigenvectors can, in the maximal case, be interpreted in terms of a question together with an answer. Specifically, it means in a context with several variables, a chosen variable θ may be associated with a question ‘What is the value of θ ?’ or ‘What will θ be if we measure it?’, and a specific eigenvector of A^θ , corresponding to the eigenvalue u may be identified with the answer ‘ $\theta = u$ ’.
- In the general case, eigenspaces have the same interpretation.
- The operators of related variables are connected by a unitary similarity transform.

It should also be mentioned that, in addition to the Postulates of this article and of Helland (2024b,c), a final postulate is needed to compute probabilities of independent events. A version of such a postulate is

Postulate 5

If the probability of an event E_1 is computed by a probability amplitude z_1 from the Born rule in the Hilbert space \mathcal{H}_1 , the probability of an event E_2 is computed by a probability amplitude z_2 from the Born rule in the Hilbert space \mathcal{H}_2 , and these two events are independent, then the probability of the event $E_1 \cap E_2$ can be computed from the probability amplitude $z_1 z_2$, associated with the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

This postulate can be motivated by its relation to classical probability theory: If $P(E_1) = |z_1|^2$ and $P(E_2) = |z_2|^2$, then

$$P(E_1 \cap E_2) = P(E_1)P(E_2) = |z_1|^2 |z_2|^2 = |z_1 z_2|^2.$$

In Theorem 1 it was concluded that the relevant operators were symmetric. This is a simple property: $\langle u|Av \rangle = \langle Au|v \rangle$ for all $|u\rangle, |v\rangle$ in the domain of A . To be precise, we need operators

corresponding to the two maximal accessible variables θ and η are *self-adjoint*. Then the spectral theorem is valid (Hall, 2013), and it can be used to define operators corresponding to other accessible variables.

Look at the cases $A = A^\theta$ and $A = A^\eta$. Then we can recall the formulae (9) and (10) in Helland (2024a):

$$A^\theta = \int f_\theta(n) |v_n\rangle \langle v_n| v(dn), \quad (1)$$

$$A^\eta = \int f_\eta(n) |v_n\rangle \langle v_n| v(dn), \quad (2)$$

where $n \in N$, a group acting on $\psi = (\theta, \eta)$, v is a left-invariant measure on Ω_ψ , $f_\theta(n) = \theta$, $f_\eta(n) = \eta$, $|v_n\rangle = W(n)|v_0\rangle$ with $W(\cdot)$ being an irreducible representation of N , and $\int |v_n\rangle \langle v_n| v(dn) = I$.

The domain D of A^θ is the set $|u\rangle \in \mathcal{H}$ where the integral $A^\theta |u\rangle$ converges. The domain D^\dagger of its adjoint is the set $|u\rangle \in \mathcal{H} = L^2(\Omega_\psi, v)$ such that the functional $\langle u | A^\theta \cdot \rangle$ is bounded, and for $|u\rangle \in D^\dagger$ the adjoint is defined by the requirement that $A^{\theta\dagger} |u\rangle$ is the unique vector $|w\rangle$ such that $\langle w | v \rangle = \langle u | A^\theta v \rangle$ for all $|v\rangle \in D$.

On investigating the self-adjointness of the symmetric operator A^θ , the main work lies in proving that $D^\dagger = D$. Let $|u\rangle \in D^\dagger$. Then $A^\theta |u\rangle$ must be defined, so we always have that $D^\dagger \subseteq D$. Let then $|u\rangle \in D$. Then by the symmetry also the integral $\langle w | = \langle u | A^\theta$ converges. The problem is to find conditions such that $|v\rangle \rightarrow \langle w | v \rangle$ defines a bounded functional. This means that there exists a constant C such that $|\langle w | v \rangle| = |\langle v | A^\theta u \rangle| \leq C \| |v\rangle \|$ for all $|v\rangle \in D$. A sufficient condition for this, given (1), is

Postulate 6

The integral $\int |f_\theta(n)| \langle u | v_n \rangle \langle v_n | u \rangle v(dn)$ converges for every $|u\rangle \in D$.

Note that for $|u\rangle \in D$, the corresponding integral without absolute values converges.

Proposition 3

If Postulate 6 holds, then A^θ is self-adjoint. A corresponding condition holds for A^η .

Proof

By the Cauchy-Schwarz inequality

$$|\langle v | A^\theta u \rangle|^2 \leq \int |f_\theta(n)| \langle u | v_n \rangle \langle v_n | u \rangle v(dn) \int |f_\theta(n)| \langle v | v_n \rangle \langle v_n | v \rangle v(dn),$$

and the last two integrals are finite when $|u\rangle \in D$ and $|v\rangle \in D$. Without loss of generality assume $\| |u\rangle \| = \| |v\rangle \| = 1$. \square

The spectral theorem implies:

Theorem 3

For maximal variables θ and η that are not bijective functions of each other, the corresponding operators A^θ and A^η do not commute.

Proof

I will prove this for the case of discrete-valued variables. The spectral theorem then gives

$$A^\theta = \sum_j \lambda_j \mathbf{v}_j \mathbf{v}_j^\dagger,$$

$$A^\eta = \sum_i \mu_i \mathbf{u}_i \mathbf{u}_i^\dagger.$$

Since θ is maximal, all the eigenvalues are different, so A^θ determines uniquely the set of eigenvectors $\{\mathbf{v}_j\}$ up to phase factors. Similarly, $\{\mathbf{u}_i\}$ is uniquely determined by A^η . The two sets of eigenvectors satisfy $\sum_j \mathbf{v}_j \mathbf{v}_j^\dagger = I$ and $\sum_i \mathbf{u}_i \mathbf{u}_i^\dagger = I$. These two sets of eigenvectors cannot be identical, for in this case θ , taking the values λ_j , and η , taking the values μ_i would be bijective functions of each other. But when at least one \mathbf{v}_j differs from the set of vectors $\{\mathbf{u}_i\}$, it follows from the formulae above that A^θ and A^η do not commute. \square

3 A mathematical consequence and an interpretation

This section has been included to illustrate an important aspect of my theory: From purely mathematical postulates and theorems, and by just adding a natural interpretation, the following conclusions are derived: 1) general psychological statements having universal validity; 2) an explanation of a physical phenomenon which has been verified empirically, but which otherwise seems to be difficult to understand. The theme of this section has been treated in previous articles, and the mathematical proofs are deferred to these articles. However, the above aspect of the theory has not been clearly stressed before.

The property of being related ($\eta(\phi) = \theta(k\phi)$ for some k), is an important relation between two maximal accessible theoretical variables θ and η . By Theorem 2, if θ and η are related, there is a unitary similarity transformation between the corresponding operators. This theorem has an inverse for finite-dimensional variables.

Theorem 4

Consider two maximal accessible finite-dimensional theoretical variables θ and η . If there is a transformation k in Ω_ϕ and a unitary transformation $W(k)$ such that $A^\eta = W(k)^\dagger A^\theta W(k)$, then θ and η are related.

Proof

Since, by Postulate 1, θ and η are functions of ϕ , the transformation k induces a transformation s on $\psi = (\theta, \eta)$. The theorem then follows from the Lemma of Section 3 in Helland (2023b). \square

In my basic theory, I have assumed that the theoretical variables are associated with an observer or with a communicating group of observers in a given context. Concentrate here on the first case, and call the observer O . Then we have

Theorem 5

Assume that two finite-dimensional related maximal accessible θ and η are associated with O in some fixed context. Then O cannot in the same context be associated with another maximal accessible variable λ which is related to θ , but not related to η .

Proof

This is a consequence of Theorem 4. See the proof of Theorem 1 in Helland (2023b). \square

The assumption that the variables are connected to the same context is crucial. In my interpretation of quantum theory, I connect the variables to the mind of an observer or to the joint minds of a communicating group of observers. Then we can consider maximal observations done in some fixed context, which also means some fixed time.

Corollary 1

Assume that O has two related finite-dimensional maximal accessible variables θ and η in his mind at some fixed time t . Then he cannot simultaneously have in his mind another maximal accessible variable that is related to θ , but not related to η .

It is crucial here that time is fixed. By letting time vary, O is able to think of many variables, also unrelated ones.

In Helland (2022b, 2023b) this conclusion is applied to the observer Charlie, which observes the results of Alice and Bob in the famous Bell experiment. It is concluded that from this statement it is possible to understand that, in practice, noting that Charlie can be any observer, the violation of the CHSH inequality in practice can be understood. Note that my conclusion here is not directly a consequence of quantum mechanics, but of a series of mathematical theorems, building upon the above 4 postulates.

Another application is to decision theory. Let O be faced with deciding among a finite number of actions a_1, \dots, a_r . Define the decision variable θ as equal to j if the action a_j is chosen ($j = 1, \dots, r$). Say that θ is accessible, and that the decision is accessible, if the decision can be carried out by O . The variable θ is called maximally accessible if the decision just can be carried out. Note that θ is finite-valued, so the theory of this article applies.

Corollary 2

Assume that O at some fixed time t has in his mind two related maximal decisions. Then he is not able to, at the same time think of another decision, which is related to the first of the two decisions, but not related to the second one.

Note that here, O can be any person. According to my theory, we all have this limitation in our minds. The conclusion can also be generalized to the decisions made by a communicating group of individuals.

This observation can also be used, together with the other results of this article, to give a new foundation for quantum decision theory, which I plan to discuss elsewhere.

4 Some final remarks

As discussed in Helland (2024a,c), there have been proposed other possible foundations of quantum theory, and my approach should be compared to these. I will claim that the postulates stated here, and also in Helland (2024b,c) are simpler than most proposals in the literature, but detailed arguments behind such a claim are beyond the scope of this article, which has concentrated on the rather simple mathematics behind my approach.

A related approach, based on much more mathematics, is presented in Dutailly (2018). That article also begins with variables and derives the Hilbert space formulation from them. I argue that the topological assumptions made by Dutailly are not strictly necessary.

A limitation of my approach is that I do not assume the full validity of the superposition principle. I limit the concept of state vectors to Hilbert space vectors which are eigenvectors of some physically meaningful operators. These can be identified by questions of the form ‘What is θ ? /What will θ be if we measure it?’ for some accessible variable θ , together with sharp answers of the form $\theta = u$. For some such questions, answers of the type ‘We don’t know’ are allowed. Thinking in this way, give simple explanations for paradoxes like Schrödinger’s cat and Wigner’s friend, see Helland (2023c).

The postulates of this article generalize and at the same time simplify the postulates of Helland (2024c), where the symmetry conditions and the question of when a symmetric operator was self-adjoint, were not taken into account. On the other hand, in Helland (2024c) conditions for the validity of the Born formula were discussed. A more thorough discussion of the Born formula is given in Helland (2024b).

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