Review of: "Representation of physical quantities: From scalars, vectors, tensors and spinors to multivectors"

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Potential competing interests: No potential competing interests to declare.

With no new results, this is not a research paper but lecture notes to provide a “comprehensive introduction to geometric algebra (GA).” As an enthusiast myself of this approach to physics, rather than the conventional scalars, vectors, and tensors generally adopted for over a century in physics (which may only loosely, since it is no algebra, be termed vector algebra), I believe this will be useful, especially to graduate students. One section, Sec. 1.1, is confusingly rendered and could be improved. The several works of Hestenes, who has been an evangelist advocating the wide adoption of GA and has written books for this purpose, are cited, but a few other crucial ones could be usefully included. I note that the latest references given date back to 2003 for journal articles and 2011 for books. However, there is more recent work on GA, including pedagogical articles. More detailed comments follow.

1. The end of the third paragraph on increasing groups using GA could give some specific references.

2. Sec. 1 is a standard introduction. Eq (2), with too many running equality signs and running into a right arrow in between, could be confusing: lhs is a vector, but the end shows a relation between scalar components. It might be better to split it into a sub-equation.

3. The paragraph with Eq (3) is also confusing in stating first that physical quantities such as velocity and force are polar vectors, but then bringing in rotation. A student would think of Coriolis or centripetal forces, the force of a magnetic field on a moving charge, etc., also equally as physical quantities. It might be better to reserve "vector" for similar behavior under rotations and to make the further distinction between polar and axial upon extending to behavior under reflection.

4. Sec.1.1 on tensors and dyadic elements is rather garbled. The end of p.2 and below Eq(2), components T_{ij} are themselves termed tensors of rank 2, and below Eq(4), D^{ij} is itself called a dyad, but the very next sentence says a dyad has magnitude and two associated directions. The D^{ij} and T_{ij} are nine scalar elements; it is the product of e_i and e_j with hats that carries the direction (not $u^i v^j$), just as in Eq(1), the boldface **w** on the LHS is a vector, and w_i are its components (scalar quantities). Boldface **w**, or **D** and **T**, has its own existence, irrespective of rotations or other transformations of some underlying triad axes that change only their components. Statements such as “all tensors are not dyads,” and that for purposes of behavior under rotation, two vectors u and v in Eq(4) have the decomposition into irreducible sets of rank 0, 1, and 2, scalar, vector, and rank-2 tensor, the question of 9 vs. 6 (itself decomposable as 1+5), could all be presented more clearly.

The entry in Wikipedia is also confusing about dyads and dyadics.

Next, in the last two paras of Sec.1.1, when the gradient operation is introduced as one (u) of the two vectors, while note is taken of the scalar invariant of the divergence of v, they go on to further gradients on dyads instead of taking note of the curl as the natural vector counterpart.

5. A middle para in Sec.1.2 correctly interprets T_{ij} in terms of stimulus j and response i. Stress-strain in elasticity provides a good example. See U. Fano and A. R. P. Rau, Symmetries in Quantum Physics (Academic, 1996), Sec. 1.1, and also further sections for other matters about tensors and dyadics.

6. Sec. 2 extends the 3D treatment to higher dimensions, elements of Grassmann algebra, wedge and multiple wedge products, the volume in nD space, k-blades. This is the meat of the paper. While some of the Havel papers are referenced, one that could have been referenced is the later Havel and Doran arXiv:0403136.

7. Sec.3.1 is a standard on quaternions, and 3.2 treats rotations with them, much better than the conventional vector rendition in physics. A particularly nice pedagogical one for both 2D and 3D that could be noted here is by Felix Klein: Elementary Mathematics from an Advanced Viewpoint: Arithmetic, Algebra, Analysis (Dover 2004). Sec. 3.3(ii) on spinors, quaternions, and the following has a nice discussion, including some of the history. Whereas Maxwell would have liked to use quaternions in his equations of electromagnetism (he wrote out components), his followers, “the Maxwellians,” such as Heaviside, and especially Gibbs in the US, advocated vector analysis and prevailed. Given over a century of this having seeped so deeply into current physics, Hestenes’s advocacy of replacing it by GA is unlikely, but it is good to spread GA more widely through papers such as the authors’ current one.

Sec.3.2 emphasizes that rotations should be associated with 2D planes, not axes, a feature not well appreciated by students. Indeed, this goes back to all angular coordinate descriptions in what are termed hyperspherical coordinates, where an nD vector is decomposed in successive steps with the length R and a polar angle (range 0 to pi) to the next (n-1)D, down to a final 2D plane with an azimuthal angle (range 0 to 2pi)*;* as in 3D with Rcos(theta) along the z-axis and Rsin(theta) in the x-y plane for further splitting into cos/sin phi components in that plane.

8. Sec. 4 on Clifford algebra goes hand in hand with the topics discussed for GA. Another compact pedagogical paper on this that could be noted is P.R. Girard, Eur. J. Phys. **5**, 25 (1984).

9. At the end of p.18, Pauli and Dirac matrices/algebra are briefly mentioned. A recent paper on GA has an extensive discussion with a dictionary of Pauli qubits, Dirac matrices, and complex quaternions, along with references to more recent GA work since that in references 50-53 cited by the authors: A. R. P. Rau, Symmetry **13**, 1732 (2021).