Research Article

On Ekeland Variational Principle and Its Applications Through Fuzzy Quasi Metric Spaces with Non-Archimedean t-norm

Riya Dutta[1], Prasun Kumar Nayak[1], Sujit Kumar De[1]

1. Department of Mathematics, Midnapore College – Autonomous, Midnapore, India

The aim of this article is to introduce the Ekeland variational principle (EVP) and some results in fuzzy quasi metric space (FQMS) under the non-Archimedean t-norms. In this article, the basic topological properties and a partial order relation are defined on FQMS. Utilizing the Brézis-Browder principle on a partially ordered set, we extend the EVP to FQMS as well. Moreover, we derive Takahashi's minimization theorem, which ensures the existence of a solution to an optimal problem without taking the help of compactness and convexity properties on the underlying space. Furthermore, we give an equivalence chain between these two theorems. Finally, two fixed point results, namely the Banach fixed point and the Caristi-Kirk fixed point theorems, are described extensively.

Corresponding author: Prasun Kumar Nayak, prasunkrnayak@gmail.com

1. Introduction

In 1972, Ekeland introduced an approximate minimizer of a bounded below and lower semi-continuous function on a complete metric space, named the Ekeland variational principle (EVP). The EVP, an enthralling theory, has some comprehensive applications in optimization theory, game theory, optimal control theory, non-linear analysis, and dynamical systems, etc. In 2010, Q. H. Ansari[1] developed several versions of EVP, Takahashi's minimization theorem (TMT), Banach contraction principle (BCP), and Caristi's fixed point theorem (CFPT) with some applications to equilibrium problems. Because of its wide research interest, several authors have introduced EVP in various directions. Alleche et al.[2] gave a new version of EVP for countable systems of equilibrium problems on a complete metric space, Khanh et al.[3] defined three types of Ekeland points and their existence based on an induction theorem in a partial

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metric space, Bao[4] derived an exact and approximate vectorial version of EVP based upon Dance-Heged üs-Medvegyv's fixed point theorem for a dynamical system on a complete metric space, Iqbal[5] presented a variational principle without assuming the completeness property and solved some minimization problems by taking a non-lower semi-continuous function in the metric space. On the other side, Cobzas[6] provided EVP on a complete quasi-metric space as an extension of the Brézis-Browder maximality principle, Ai-Homidan[7] gave a new version of Takahashi's minimization theorem with two different types of conditions in the setting of a complete quasi-metric space, and further, they constructed error bound solutions and weak sharp solutions for equilibrium problems. Recently, Zao et al. [8] extended Lin-Du's abstract maximal element principle to generalize EVP for essential distance in the environment of a quasi-ordered set. Furthermore, a broad extension of EVP involving set perturbations attracted many researchers to work in this direction. Some multi-objective optimization problems and vector variational inequality problems were analysed by Hai[9] based on EVP relating to set perturbations.

Moreover, many researchers have a lot of attraction to work on different fuzzy versions of EVP. In 1975, Kramosil and Michalek[10] introduced an idea of a fuzzy metric space, which indicates the uncertainty of distance functions. This idea was extended in 1994 by George and Veeramani[11]. They defined a fuzzy metric space in a different way, called the GV fuzzy metric space. Several research works on EVP have been done in various types of fuzzy metric spaces in different directions. In the setting of the GV fuzzy metric space, Abbasi et al.[12] extended EVP, Caristi's fixed point theorem, Takahashi's minimization theorem, and described an equivalence relation on EVP and TMT in 2016. The Caristi type mapping was developed by Martínez-Moreno et al.[13] in an Archimedean-type fuzzy metric space. Qiu et al.[14] also extended the above theorems in GV fuzzy metric spaces subsequently.

However, the variational principle and fixed point results have also been discussed in the setting of a fuzzy metric space[15]. The set-valued EVP has been established incorporating a set-valued map on a locally convex fuzzy metric space[16]. Some related works about EVP on a fuzzy metric space were presented by Pei-jun[15], in which the author has defined a fuzzy metric space as a quadruple (X, d, L, R) and discussed EVP on a α-level set. Removing the symmetric property, several authors have generalised the fuzzy metric as a fuzzy quasi metric space and have established multiple results. Gregori et al.[17] have generalised the KM-fuzzy metric space and the GV-fuzzy metric space to the KM-fuzzy quasi metric space and the GV-fuzzy quasi metric space respectively and claimed that every fuzzy quasi metric induces a quasi metrizable topology and vice-versa. Similarly, Romaguera[18] introduced the bicompletion and D-completion of fuzzy quasi metric spaces via quasi uniform isomorphism. At the same time, Romaguera et.al[19] constructed some contraction mappings to establish the existence and uniqueness of a fixed point result on a preordered complete fuzzy quasi metric space. Recently, an extension of EVP, TMT, and CFPT on a fuzzy quasi metric space under an Archimedean t-norm has been studied extensively[20].

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Author(s) with publication year	Use of t-norm	Metric structure	Use of EVP and TMT	Completeness and Fixed point results	Area of Application
Gregori and Mascarell, 2005	none	T_2 fuzzy quasi metric Space	none	bi-completeness of fuzzy quasi metric space	none
Mihet, 2010	none	fuzzy metric Space	none	fixed point theory using fuzzy contractive mappings in G-complete fuzzy metric spaces	the domain of words
Cobzas, 2011	none	T_1 Quasi Metric Space	EVP on quasi metric space	none	none
Romaguera, and Tirado, 2014	none	fuzzy quasi metric space	none	fixed point theory using a continuous non-decreasing self map	solution for the general recurrence equations
Al-Homidan, Ansari and Kassay, 2019	none	quasi metric space	TMT on quasi metric Space	none	error bounds and weak sharp solutions for equilibrium solutions
Wu and Tang, 2021	Archimedean t-norm	T_1 Fuzzy quasi metric space	EVP and TMT on fuzzy quasi metric space	CFPT on fuzzy quasi metric space	
This paper	non Archimedean t-norm	T_1 Fuzzy quasi metric space	EVP and TMT on fuzzy quasi	BCP and CFPT using fuzzy quasi version of EVP with a proper bounded below and	existence of solution of equilibrium points

Author(s) with publication year	Use of t-norm	Metric structure	Use of EVP and TMT	Completeness and Fixed point results	Area of Application
			metric space	lower semi continuous function	

Table 1. Major literature review over the related topic

From the above literature study, it is seen that none of the researchers has discussed EVP in the light of FQMS utilizing a non-Archimedean t-norm. Thus, in this study, we present EVP, TMT, BCP, CFPT, and related results in the setting of FQMS under the presence of non-Archimedean t-norms. This article has been organised as follows: section 2 includes the formation of FQMS from Quasi Metric Space and defines three types of Cauchy sequences, three types of convergences, and seven types of completeness properties on it. Section 3 contains the EVP of the Fuzzy Quasi version, section 4 develops Takahashi's minimization theorem and an equivalent chain between EVP and TMT. Section 5 represents two types of fixed point results, namely Banach contraction fixed point and Caristi's fixed point theorems. Finally, section 6 ends with the conclusion of the proposed study followed by the scope of future research.

2. Preliminaries

In this section, we introduce some basic definitions and properties over the fuzzy quasi metric spaces which will be used to develop the proposed study.

Definition 2.1. [21] *Let X be a non-empty set. A function $d_q : X \times X \to [0, \infty)$ is called quasi metric if the following properties hold for all $x, y, z \in X$:*

(M1): $d_q(x, x) = 0$

(M2): $d_q(x, y) = 0 \implies x = y$

(M3): $d_q(x, y) = d(y, x)$

(M4): $d_q(x, y) \leq d(x, z) + d(z, y)$.

Then the ordered pair (X, d) is called quasi metric space.

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Generally, a metric is defined by means of a distance function, but if the distance function itself assumes fuzzy flexibility, then the subject under study is a part of a fuzzy metric space ([22][23]).

Definition 2.2. [24] *A binary operation* $* : [0,1]^2 \to [0,1]$ *is said to be a continuous* t*-norm if it satisfies the following conditions for all* $a, b, c, d \in [0,1]$:

i. $a * (b * c) = (a * b) * c$

ii. $a * 1 = a$

iii. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Moreover, the basic t*-norms; minimum, product, and Lukasiewicz continuous* t*-norms are defined by*

$$a *_m b = \min\{a, b\}; a \cdot b = ab; \text{ and } a *_L b = \max\{a + b - 1, 0\}$$

respectively.

Definition 2.3. *A structure which has a pair of non-zero elements, one of which is infinitesimal with respect to the other, is said to be non-Archimedean. It is easy to see that the* t*-norm "* $*_m$ *" is not Archimedean, while the other two* t*-norms are Archimedean.*

Definition 2.4. [20] *Let* X *be an arbitrary non-empty set,* $*$ *being a continuous* t*-norm and a mapping* $M_q : X^2 \times (0, \infty) \to (0,1]$ *be a fuzzy membership function. Then a 3-tuple* $(X, M_q, *)$ *is said to be a fuzzy quasi metric space (FQMS) if it satisfies the following conditions for all* $x, y, z \in X$ *and* $t > 0$ *:*

(FQMS 1): $M_q(x, y, t) > 0$

(FQMS 2): $M_q(x, y, t) = 1$ if and only if $x = y$

(FQMS 3): $M_q(x, y, s + t) \geq M_q(x, z, t) * M_q(z, y, s)$

(FQMS 4): $M_q(x, y, .) : (0, \infty) \to (0, 1]$ *is continuous*

(FQMS 5): $\lim_{t \to \infty} M_q(x, y, t) = 1.$

The function M_q is called the Fuzzy Quasi Metric (FQM), and it denotes the degree of closeness between x and y with respect to t.

The conjugate FQM $\bar{M}_q$, corresponding to each FQMS $(X, M_q, *)$, is defined as $\bar{M}_q(x, y, t) = M_q(y, x, t)$ for all $x, y \in X$ and $t > 0$.

Also, we define the mapping $M_q^s : X^2 \times (0, \infty) \to (0, 1]$ as

$M_q^s(x, y, t) = \min\{M_q(x, y, t), \bar{M}_q(x, y, t)\}, \forall x, y \in X$ and $t > 0$,

which is a fuzzy metric on X.

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In the rest of the article, we shall use the "minimum" t-norm ($*_m$) to express the triangle inequality, and we redefine FQMS as $(X, M_q, *_m)$.

Example 2.5. [25] *Let (X, d_q) be a quasi-metric space, let $\phi : (0,\infty) \to (0,\infty)$ be an increasing left-continuous function with $\phi(t+s) \geq \phi(t) + \phi(s)$ and let $g : [0,\infty) \to [0,1]$ be a decreasing left-continuous function such that $g(0) = 1$. Then $(X, M_q, *_m)$ is a fuzzy quasi metric space, where the fuzzy set $M_q : X \times X \times (0,\infty)$ is given for each $x, y \in X$ and $t \in (0,\infty)$ by*

$$M_q(x,y,t) = g\Big(\frac{d_q(x,y)}{\phi(t)}\Big)$$

Definition 2.6. *Let $(X, M_q, *_m)$ be a fuzzy quasi metric space. A set $A(\subseteq X)$ is called*

i. *left bounded (l-bounded) if and only if there exists $t > 0$ and $0 < r < 1$ such that $M_q(x,y,t) > 1 - r$ for all $x, y \in A$.*

ii. *right bounded (r-bounded) if and only if there exists $t > 0$ and $0 < r < 1$ such that $\bar{M}_q(x,y,t) > 1 - r$ for all $x, y \in A$.*

Definition 2.7. *Let us consider a FQMS $(X, M_q, *_m)$. For any given parameter of fuzziness $0 < \epsilon < 1$, center x and radius $a > 0$, we can define the left open ball (l-open ball) $B_l(x,a,\epsilon)$ and the left closed ball (l-closed ball) $B_l[x,a,\epsilon]$ respectively as:*

$B_l(x,a,\epsilon) = \{y \in X : M_q(x,y,a) > 1 - \epsilon\}$

and $B_l[x,a,\epsilon] = \{y \in X : M_q(x,y,a) \geq 1 - \epsilon\}$

and similarly, we may define the right open ball (r-open ball) $B_r(x,a,\epsilon)$ and the right closed ball (r-closed ball) $B_r[x,a,\epsilon]$ respectively as:

$B_r(x,a,\epsilon) = \{y \in X : \bar{M}_q(x,y,a) > 1 - \epsilon\}$

and $B_r[x,a,\epsilon] = \{y \in X : \bar{M}_q(x,y,a) \geq 1 - \epsilon\}$

Definition 2.8. *Let $(X, M_q, *_m)$ be a FQMS. A sequence $< x_n >_{n\in\mathbb{N}}$ is said to be*

i. *l-Cauchy if and only if for each $\epsilon \in (0,1), t > 0$ $\exists n_0 \in \mathbb{N}$ such that* $M_q(x_n, x_m, t) > 1 - \epsilon$ for any $m \geq n \geq n_0$.

ii. *r-Cauchy if and only if for each $\epsilon \in (0,1), t > 0$ $\exists n_0 \in \mathbb{N}$ such that* $M_q(x_m, x_n, t) > 1 - \epsilon$ for any $m \geq n \geq n_0$.

iii. *Cauchy if and only if for each $\epsilon \in (0,1), t > 0$ $\exists n_0 \in \mathbb{N}$ such that* $M_q^s(x_m, x_n, t) > 1 - \epsilon$ for any $n, m \geq n_0$.

Definition 2.9. *Let X be a non empty set. A sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in a FQMS $(X, M_q, *_m)$ is said to be*

i. *l-converges to $a \in X$, if and only if* $\lim_{n\to\infty} M_q(a, x_n, t) = 1$ for all $t > 0$, *i.e. for each* $\epsilon \in (0,1)$ and $t > 0$, $\exists n_0 \in \mathbb{N}$ *such that* $M_q(a, x_n, t) > 1 - \epsilon$ for all $n \geq n_0$.

ii. *r-converges to $a \in X$, if and only if* $\lim_{n\to\infty} M_q(x_n, a, t) = 1$ for all $t > 0$, *i.e. for each* $\epsilon \in (0,1)$ and $t > 0$, $\exists n_0 \in \mathbb{N}$ *such that* $M_q(x_n, a, t) > 1 - \epsilon$ for all $n \geq n_0$.

iii. *converges to $a \in X$, if and only if* $\lim_{n\to\infty} M_q^s(a, x_n, t) = 1$ for all $t > 0$, *i.e. for each* $\epsilon \in (0,1)$ and $t > 0$, $\exists n_0 \in \mathbb{N}$ *such that* $M_q^s(a, x_n, t) > 1 - \epsilon$ for all $n \geq n_0$.

Definition 2.10. *The FQMS $(X, M_q, *_m)$ is*

i. *ll-complete if every l-Cauchy sequence is l-converges to a point in X.*

Similarly we can define rr, lr, rl-completeness in FQMS.

ii. *l-complete (r-complete) if every l-Cauchy (r-Cauchy) sequence converges to a point in X.*

iii. *complete if every Cauchy sequence converges to a point in X.*

Example 2.11. *Let us consider $X = [1/2, 1)$ as a non-empty set. Consider the fuzzy quasi metric $(M_q, *_m)$ defined in example 2.5, and we define a quasi metric $d : X \times X \to (0, \infty)$ by*

$$\begin{aligned} d(x,y) = & \quad 0, \ x \leq y \\ = & \quad 1, \ x > y \end{aligned}$$

Then every l-Cauchy sequence in X is l-convergent. Therefore, $(X, M_q, *_m)$ is ll-complete fuzzy quasi metric space

Remark 2.12.

i. *Let $< x_n >_n$ be a sequence in a FQMS $(X, M_q, *_m)$. If $\langle x_n \rangle$ is l-convergent to x and r-convergent to y, then we get $x = y$.*

ii. *Let $< x_n >_n$ be a sequence in a FQMS $(X, M_q, *_m)$. The sequence $\langle x_n \rangle$ is l-convergent in (X, M_q), if every l-Cauchy sequence $\langle x_n \rangle$ in (X, M_q) has a l-convergent subsequence in (X, M_q).*

3. Ekeland Variational Principle in Complete Fuzzy Quasi Metric Space

We shall establish Ekeland's Variational Principle on complete FQMS using an extension theorem of the Brézis-Browder principle[6][26] on a partially ordered set. This theorem ensures that a partially ordered set has a minimal (dually maximal) element by choosing a strictly increasing function on it. So first, we recall the Brézis-Browder principle on an ordered set. Then we construct a partial order relation on X and then we apply the

Brézis-Browder principle to establish the EVP on FQMS.

Let $(Z, \leq)$ *be a partially ordered set. For* $x \in Z$, *put* $S_+(x) = \{z \in Z : x \leq z\}$ and $S_-(x) = \{z \in Z : z \leq x\}$. *Here, the notation* $x < y$ imples $(x \leq y) \wedge (x \neq y)$ *and for the dual formulation, we just reverse the order of* x and y.

Lemma 3.1. [27] *Let* $(Z, \leq)$ *be a partially ordered set.*

i. *Suppose that* $\psi : Z \to \mathbb{R}$ *is a function satisfying the conditions:*

1. *the function* ψ *is strictly increasing;*
2. *for each* $x \in Z,\ \psi(S_-(x))$ *is bounded below;*
3. *for any decreasing sequence* $\langle x_n \rangle$ *in* Z *there exists* $y \in Z$ *such that* $y \leq x_n, n \in \mathbb{N}$.

   Then for each $x \in Z$ *there exists a minimal element* $z \in Z$ *such that* $z \leq x$.

ii. *Dually, let* $\phi : Z \to \mathbb{R}$ *be a function satisfying the conditions:*

1. *the function* ϕ *is strictly increasing;*
2. *for each* $x \in Z,\ \phi(S_+(x))$ *is bounded above;*
3. *for any increasing sequence* $\langle x_n \rangle$ *in* Z *there exists* $y \in Z$ *such that* $x_n \leq y, n \in \mathbb{N}$.

   Then for each $x \in Z$ *there exists a maximal element* $z \in Z$ *such that* $x \leq z$.

Theorem 3.2. *Let* $(X, M_q, *_m)$ *be a FQMS and a function* $\mathscr{F} : X \to \mathbb{R}$ *on* X. *Define a relation on* X *by*

$x \leq y \Leftrightarrow \frac{t}{1+t}\mathscr{F}(x) + 1 - M_q(x, y, t) \leq \frac{t}{1+t}\mathscr{F}(y)$.

Then the relation "$\leq$" *is a partial order.*

Proof. ***Reflexive****: It is obvious that* $x \leq x$ *holds,*

Anti-symmetric*: Let* $x \leq y$ *and* $y \leq x$ *hold. Then*

$\frac{t}{1+t}\mathscr{F}(x) + 1 - M_q(x, y, t/2) \leq \frac{t}{1+t}\mathscr{F}(y)$ *and* $\frac{t}{1+t}\mathscr{F}(y) + 1 - M_q(y, x, t/2) \leq \frac{t}{1+t}\mathscr{F}(x)$

hold respectively. From these two relations, we get, $\{M_q(x, y, t/2) + M_q(y, x, t/2)\} \leq 2 \implies M_q(x, y, t/2) = M_q(y, x, t/2) = 1 \implies x = y$.

Transitive*: Let* $x \leq y$ *and* $y \leq z$ *hold. Then*

$\frac{t}{1+t}\mathscr{F}(x) + 1 - M_q(x, y, t/2) \leq \frac{t}{1+t}\mathscr{F}(y)$ *and* $\frac{t}{1+t}\mathscr{F}(y) + 1 - M_q(y, z, t/2) \leq \frac{t}{1+t}\mathscr{F}(z)$

hold respectively. Now,

$$\begin{aligned}\frac{t}{1+t}\mathscr{F}(x)+1-M_q(x,z,t) \leq& \quad \frac{t}{1+t}\mathscr{F}(x)+1-\{M_q(x,y,t/2)*M_q(y,z,t/2)\}\\ \leq& \quad \frac{t}{1+t}\mathscr{F}(x)+1-\min\{M_q(x,y,t/2),M_q(y,z,t/2)\}\\ \leq& \quad \frac{t}{1+t}\mathscr{F}(x)+1-M_q(x,y,t/2)+1-M_q(y,z,t/2)\\ \leq& \quad \frac{t}{1+t}\mathscr{F}(y)+1-M_q(y,z,t/2)\\ \leq& \quad \frac{t}{1+t}\mathscr{F}(z)\end{aligned}$$

Thus $x \leq z$ holds. Hence, the proof. □

Theorem 3.3. *Let $(X, M_q, *_m)$ be a FQMS and consider a function $\mathscr{F} : X \to \mathbb{R}$ on X. Consider the partial order relation given in theorem 3.2,*

$x \leq y \Leftrightarrow M_q(x,y,t) \geq 1-(\frac{t}{1+t})[\mathscr{F}(y)-\mathscr{F}(x)]$

i. *If X is a ll-complete FQMS and $\mathscr{F} : X \to \mathbb{R}$ is bounded below and lower semi-continuous on X, then every element x of X is minored by a minimal element z in X.*

ii. *If X is a rr-complete FQMS and $\mathscr{F} : X \to \mathbb{R}$ is bounded above and upper semi-continuous on X, then every element x of X is majored by a maximal element z in X.*

Proof.

i. *From the definition of FQMS, we know that if $x \neq y$, then $M_q(x,y,t) < 1$. Consequently,*

$x < y \iff (x \leq y) \wedge (x \neq y) \implies 1 > M_q(x,y,t) \geq 1-(\frac{t}{1+t})[\mathscr{F}(y)-\mathscr{F}(x)]$.

This shows that $\mathscr{F}$ is strictly increasing; therefore, condition (a)(lemma 3.1,(i)) holds.

Since $\mathscr{F}$ is bounded below, therefore (b)(lemma 3.1,(i)) holds.

Now we consider a decreasing sequence $\langle x_n \rangle \in \mathbb{N}$ in X. Then $\mathscr{F}(x_{n+1}) \leq \mathscr{F}(x_n), \forall n \in \mathbb{N}$. Since $\mathscr{F}$ is bounded below, then $\langle \mathscr{F}(x_n) \rangle$ has an infimum, say b and the sequence is convergent. Consequently, it is Cauchy, so that, for a given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$\mathscr{F}(x_{n+p}) - \mathscr{F}(x_n) < \epsilon \text{ for all } n \geq n_0 \text{ and } p \in \mathbb{N}$.

This implies that $M_q(x_n, x_{n+p}, t) \geq 1-(\frac{t}{1+t})\epsilon > 1-\epsilon$.

This claims that $< x_n >$ is a l-Cauchy sequence. Since X is ll-complete, then from the definition 2.10, it l-converges to some point $y \in X$, i.e. $\lim_{t\to 0} M_q(y, x_n, t) = 1$.

Again, since $x_{n+k} \leq x_n \implies M_q(x_{n+k}, x_n, t) \geq 1-(\frac{t}{1+t})[\mathscr{F}(x_n)-\mathscr{F}(x_{n+k})]$.

Now,

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$$
\begin{aligned}
M_q(y,x_n,t) > & \quad \min\{M_q(y,x_{n+k},t-s), M_q(x_{n+k},x_n,s)\}\\
\geq & \quad M_q(x_{n+k},x_n,t)\\
\geq & \quad 1-(\frac{t}{1+t})[\mathscr{F}(x_n)-\mathscr{F}(x_{n+k})]\\
\geq & \quad 1-(\frac{t}{1+t})[\mathscr{F}(x_n)-\liminf \mathscr{F}(x_{n+k})]\\
\geq & \quad 1-(\frac{t}{1+t})[\mathscr{F}(x_n)-\mathscr{F}(y)]
\end{aligned}
$$

which shows that $y \leq x_n$ for all $n \in \mathbb{N}$.

ii. *To prove the second assertion, we can apply the first assertion on* $(X, \leq_{\bar{M}_q,-\mathscr{F}})$. *Then we get*

$$
\begin{aligned}
x \leq_{M_q,\mathscr{F}} y \iff & \quad M_q(x,y,t) \geq 1-(\frac{t}{1+t})[\mathscr{F}(y)-\mathscr{F}(x)]\\
\iff & \quad \bar{M}(y,x,t) \geq 1-(\frac{t}{1+t})[-\mathscr{F}(x)-(-\mathscr{F}(y))]\\
\iff & \quad y \leq_{\bar{M}_q,-\mathscr{F}} x
\end{aligned}
$$

for all $x,y \in X$. *The space* $(X, \bar{M}_q)$ *and the function* $-\mathscr{F}$ *satisfy all the conditions of the first assertion of this theorem. So, for every* $x \in X$ *there exists a minimal element* z *in* $(X, \leq_{\bar{M}_q,-\mathscr{F}})$, *i.e.* z *is the maximal element in* $(X, \leq_{M_q,\mathscr{F}})$ *and* $x \leq_{M_q,\mathscr{F}} z$.

□

Theorem 3.4. *(Ekland Variational Principle)*

Suppose that $(X, M_q, *_m)$ *is a* ll*-Complete FQMS and a mapping* $\mathscr{F}: X \to \mathbb{R} \cup \{+\infty\}$ *is a proper bounded below and lower semi-continuous function. Given any* $\epsilon \in (0,1)$, *let* $\tilde{x} \in X$ *be such that*

$$
\mathscr{F}(\tilde{x}) \leq \inf \mathscr{F}(X) + \epsilon. \tag{1}
$$

Then for every $\lambda \in (0, 1+t)$, *there exists an* $\bar{x} = \bar{x}(\epsilon, \lambda) \in X$ *such that*

$(a)(\frac{t}{1+t})\mathscr{F}(\bar{x}) + 1 - \frac{\epsilon}{\lambda} M_q(\bar{x}, \tilde{x}, t) \leq (\frac{t}{1+t})\mathscr{F}(\tilde{x})$

$(b) M_q(\bar{x}, \tilde{x}, t) \geq \lambda(1 - (\frac{t}{1+t}))$

$(c) \forall x \in X \diagdown \{\bar{x}\}, (\frac{t}{1+t})\mathscr{F}(x) + 1 - \frac{\epsilon}{\lambda} M_q(x, \bar{x}, t) > (\frac{t}{1+t})\mathscr{F}(\bar{x})$.

Proof. Consider a set $Y = \{y \in X : (\frac{t}{1+t})\mathscr{F}(y) \leq 1 - \frac{\epsilon}{\lambda} M_q(\tilde{x}, y, t) + (\frac{t}{1+t})\mathscr{F}(\tilde{x})\}$

The set Y *is non-empty as* $\tilde{x} \in Y$. *Now we have to show that* Y *is a closed subset of* X. *Suppose* $< y_n >_{n \in \mathbb{N}}$ *is a sequence in* Y *and that it is* l*-convergent to some* y *in* X *i.e.* $\lim_{n \to \infty} M_q(y, y_n, t) = 1$ *for all* $t > 0$. *Then we have*

$$
\begin{aligned}
(\frac{t}{1+t})\mathscr{F}(y_n) \leq & \quad 1 - \frac{\epsilon}{\lambda} M_q(\tilde{x}, y_n, t) + (\frac{t}{1+t})\mathscr{F}(\tilde{x})\\
\leq & \quad 1 - \frac{\epsilon}{\lambda} \min\{M_q(\tilde{x}, y, s), M_q(y, y_n, t-s)\} + (\frac{t}{1+t})\mathscr{F}(x_\epsilon)
\end{aligned}
$$

for all $n \in \mathbb{N}$. *Since* $\mathscr{F}$ *is lower semi-continuous, then we get*

$(\frac{t}{1+t})\mathscr{F}(y) \leq \frac{t}{1+t} \lim_{n \to \infty} \mathscr{F}(y_n) \leq 1 - \frac{\epsilon}{\lambda} M_q(\tilde{x}, y, t) + (\frac{t}{1+t})\mathscr{F}(\tilde{x})$.

This shows that $y \in Y$.

Since X is ll-Complete FQMS and Y is a non-empty closed subset of X, then Y is also ll-Complete FQMS, i.e., every l-Cauchy sequence in Y is l-convergent to some $y \in Y$.

Now we consider an equivalent FQM $M_q(x,y,t) = \frac{\epsilon}{\lambda} M_q(x,y,t), x, y \in Y, t > 0$. Defining an order relation $\leq$ on Y by

$$\begin{aligned} x \leq y \Longleftrightarrow \qquad & M_q(x,y,t) \geq 1 - (\frac{t}{1+t})[\mathscr{F}(y) - \mathscr{F}(x)] \\ \Longleftrightarrow \qquad & \frac{\epsilon}{\lambda} M_q(x,y,t) \geq 1 - (\frac{t}{1+t})[\mathscr{F}(y) - \mathscr{F}(x)] \end{aligned}$$

for all $x, y \in Y$, it follows that all the hypotheses of theorem 3.3 are satisfied by this FQMS $(Y, M_q, *_m)$ and with $\varphi = \mathscr{F}|_Y$. Consequently, there exists a minimal element $\bar{x} \in Y$ such that $\bar{x} \leq \tilde{x}$. Since

$$\begin{aligned} \bar{x} \leq \tilde{x} \Longleftrightarrow \qquad & \frac{\epsilon}{\lambda} M_q(\bar{x},\tilde{x},t) \geq 1 - (\frac{t}{1+t})[\mathscr{F}(\tilde{x}) - \mathscr{F}(\bar{x})] \\ \Longleftrightarrow \qquad & (\frac{t}{1+t})\mathscr{F}(\bar{x}) + 1 - \frac{\epsilon}{\lambda} M_q(\bar{x},\tilde{x},t) \leq (\frac{t}{1+t})\mathscr{F}(\tilde{x}) \end{aligned}$$

It follows that $\bar{x}$ satisfies the condition (a).

By equation (1) and (a),

$$\begin{aligned} (\frac{t}{1+t})\mathscr{F}(\bar{x}) + 1 - \frac{\epsilon}{\lambda} M_q(\bar{x},\tilde{x},t) \leq \qquad & (\frac{t}{1+t})\mathscr{F}(\tilde{x}) \\ \leq \qquad & (\frac{t}{1+t})[\inf \mathscr{F}(X) + \epsilon] \\ \leq \qquad & (\frac{t}{1+t})[\mathscr{F}(\bar{x}) + \epsilon] \end{aligned}$$

implies, $M_q(\bar{x},\tilde{x},t) \geq \lambda \frac{1-(\frac{t}{1+t})\epsilon}{\epsilon} \geq \lambda(1 - (\frac{t}{1+t}))$, showing (b) holds too.

Now, if we consider a $x \in Y \diagdown \{\bar{x}\}$. By the minimality of $\bar{x}$, the inequality $x \leq \bar{x}$ does not hold, so that

$$(\frac{t}{1+t})\mathscr{F}(x) + 1 - \frac{\epsilon}{\lambda} M_q(x,\bar{x},t) > (\frac{t}{1+t})\mathscr{F}(\bar{x})$$

which shows that (c) is satisfied for such an x. Now, if $x \in X \diagdown Y$, then

$$(\frac{t}{1+t})\mathscr{F}(x) > 1 - \frac{\epsilon}{\lambda} M_q(\tilde{x},x,t) + (\frac{t}{1+t})\mathscr{F}(\tilde{x}).$$

Now, if possible, let condition (c) not hold, then we have

$$\begin{aligned} (\frac{t}{1+t})\mathscr{F}(\bar{x}) \geq \qquad & (\frac{t}{1+t})\mathscr{F}(x) + 1 - \frac{\epsilon}{\lambda} M_q(x,\bar{x},s) \\ > \qquad & (\frac{t}{1+t})\mathscr{F}(\tilde{x}) + 1 - \frac{\epsilon}{\lambda} M_q(\tilde{x},x,t) + 1 - \frac{\epsilon}{\lambda} M_q(x,\bar{x},t) \\ > \qquad & \frac{t}{1+t}\mathscr{F}(\tilde{x}) + 1 - \frac{\epsilon}{\lambda} \min\{M_q(\tilde{x},x,t-s) M_q(x,\bar{x},s)\}, \text{ for } 0 < s < t \\ > \qquad & (\frac{t}{1+t})\mathscr{F}(\tilde{x}) + 1 - \frac{\epsilon}{\lambda} M_q(\tilde{x},\bar{x},t) \end{aligned}$$

which contradicts the fact that $\bar{x} \in Y$. *Consequently, the condition* (c) *holds for* $x \in X \setminus Y$, *as well as* $x \in X \setminus \{\bar{x}\}$. *Hence the proof.* $\square$

Corollary 3.5. *Suppose that* $(X, M_q, *_m)$ *is a* rr*-Complete FQMS and a mapping* $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ *is a proper bounded above and upper semi-continuous function. Given any* $\epsilon \in (0, 1)$, *let* $\tilde{x} \in X$ *be such that*

$$\mathscr{F}(\tilde{x}) \leq \inf \mathscr{F}(X) + \epsilon$$

Then for every $\lambda \in (0, 1 + t)$, *there exists an* $\bar{x} = \bar{x}(\epsilon, \lambda) \in X$ *such that*

$(a) (\frac{t}{1+t})\mathscr{F}(\tilde{x}) + 1 - \frac{\epsilon}{\lambda} M_q(\tilde{x}, \bar{x}, t) \leq (\frac{t}{1+t})\mathscr{F}(\bar{x})$

$(b) M_q(\tilde{x}, \bar{x}, t) \geq \lambda(1 - (\frac{t}{1+t}))$

$(c) \forall x \in X \setminus \{\bar{x}\}, (\frac{t}{1+t})\mathscr{F}(\bar{x}) + 1 - \frac{\epsilon}{\lambda} M_q(\bar{x}, x, t) > (\frac{t}{1+t})\mathscr{F}(x)$.

Theorem 3.6. (Ekeland Variational Principle-weak form): *Suppose that* $(X, M_q, *_m)$ *is a* ll*-Complete FQMS and a mapping* $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ *is a proper bounded below and lower semi-continuous function. Given any* $\epsilon \in (0, 1)$, *let* $\tilde{x} \in X$ *be such that*

$$\mathscr{F}(\tilde{x}) \leq \inf \mathscr{F}(X) + \epsilon.$$

Then there exists an $\bar{x} = \bar{x}(\epsilon, \lambda) \in X$ *such that*

$$\frac{t}{1+t}\mathscr{F}(\bar{x}) + 1 - \epsilon M_q(\bar{x}, \tilde{x}, t) \leq \frac{t}{1+t}\mathscr{F}(\tilde{x}) \tag{2}$$

Proof. It can be easily proved by putting $\lambda = 1$ *in theorem 3.4.* $\square$

Next, we are to show that the validity of the weak version of the Ekeland variational principle ensures the completeness of the space.

Example 3.7. *Let us consider* $X = [0, 1)$ *as a non-empty set and* M_q *as a fuzzy quasi metric defined in example 2.5. Then* $(X, M_q, *_m)$ *is* ll*-complete FQMS. Now consider* $\mathscr{F} : X \to \mathbb{R}$ *defined by*

$$\begin{aligned} \mathscr{F}(x) = & \quad x^2, x \neq 0 \\ = & \quad 0, \text{ otherwise} \end{aligned}$$

which is lower semi-continuous at 0 *and bounded below. Then there exists a point* $\bar{x} (= 0) \in X$ *which satisfies the EVP.*

Let $(X, M_q, *_m)$ *be FQMS and* $\mathbb{F} : X \times X \to \mathbb{R}$ *be a bifunction. If there exists* $\bar{x} \in X$ *such that* $\mathbb{F}(\bar{x}, y) \geq 0$ *for all* $y \in X$, *then* $\bar{x}$ *is called the solution of the equilibrium problem (EP)*[7].

Theorem 3.8. *(Equilibrium version of EVP)*

Let $(X, M_q, *_m)$ *be* ll*-complete FQMS and* $\mathbb{F} : X \times X \to \mathbb{R}$ *be a bifunction. Assume that there exists a proper* *bounded below and lower semi-continuous function* $\mathscr{F}: X \to \mathbb{R} \cup \{+\infty\}$ *such that*

$$\mathbb{F}(x, y) \geq \mathscr{F}(y) - \mathscr{F}(x) \text{ for all } x, y \in X \tag{3}$$

Then for any given $\epsilon \in (0, 1)$, *let* $\hat{x} \in X$ *be such that* $\inf \mathbb{F}(\hat{x}, x) > -\infty$ *and for every* $\lambda > 1, t > 0$, *there exists an* $\bar{x} = \bar{x}(\epsilon, \lambda) \in X$ *such that*

a. $\frac{t}{1+t}\mathbb{F}(\bar{x}, \tilde{x}) + \frac{\epsilon}{\lambda} M_q(\bar{x}, \tilde{x}, t) \geq 1$

b. $\forall x \in X \setminus \{\bar{x}\},\ \frac{\epsilon}{\lambda} M_q(x, \bar{x}, t) - \frac{t}{1+t}\mathbb{F}(\bar{x}, x) < 1$

Theorem 3.9. (Converse of EVP): *Let* $(X, M_q, *_m)$ *be a FQMS. If for every lower semi-continuous function* $\mathscr{F}: X \to \mathbb{R}$ *and for every* $\epsilon > 0$ *there exists* $y_\epsilon \in X$ *such that*

$\forall x \in X,\ (\frac{t}{1+t})\mathscr{F}(y_\epsilon) + 1 - \epsilon M_q(y_\epsilon, x, t) \leq (\frac{t}{1+t})\mathscr{F}(x)$

then the FQMS X *is* ll*-complete.*

Proof. Suppose that $\langle x_n \rangle$ *is a* l*-Cauchy sequence in* X*. Consider a well-defined function* $\mathscr{F}: X \to \mathbb{R}$, *given by*

$$\mathscr{F}(x) = 1 - \limsup_n M_q(x, x_n, t).$$

First, we shall show that the function $\mathscr{F}$ *is lower semi-continuous. Let* $x \in X$ *be fixed and* $x' \in X$ *arbitrary. Then, as per definition,*

$$\begin{aligned} M_q(x, x_n, t) \geq & \quad \min\{M_q(x, x', t - s), M_q(x', x_n, s)\} \\ \geq & \quad \min\{1 - \epsilon, M_(x', x_n, s)\} \\ \geq & \quad M_q(x', x_n, s) - \epsilon \\ \implies 1 - M_q(x', x_n, t) \geq & \quad 1 - M_q(x, x_n, s) - \epsilon \end{aligned}$$

holds for every $n \in \mathbb{N}$, *which yields*

$$F(x') \geq F(x) - \epsilon$$

implying the lower semi-continuity of the function $\mathscr{F}$ *at the point* x.

Now, we have to prove that $\lim_{n \to \infty} \mathscr{F}(x_n) = 0$.

We have for every $\epsilon > 0$, *there exists* $n_\epsilon \in \mathbb{N}$ *such that*

$$M_q(x_n, x_{n+k}, t) > 1 - \epsilon, \forall n \geq n_\epsilon, \forall k \in \mathbb{N}.$$

Now, $\mathscr{F}(x_n) = 1 - \lim_n \sup M_q(x_n, x_{n+k}, t) < \epsilon \implies \lim_n \mathscr{F}(x_n) = 0$.

Again, from the given condition $(\frac{t}{1+t})\mathscr{F}(y) + 1 - \epsilon M_q(y, x_n, t) \leq (\frac{t}{1+t})\mathscr{F}(x_n)$, *taking* $\lim_n \sup$ *of both sides, we get*

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$$
\begin{aligned}
& (\frac{t}{1+t})\mathscr{F}(y) + 1 - \epsilon[1-\mathscr{F}(y)] < 0 \\
\implies \quad & (\epsilon + \frac{t}{1+t})\mathscr{F}(y) < \epsilon - 1 \\
\implies \quad & \mathscr{F}(y) < \frac{\epsilon - 1}{\epsilon + (\frac{t}{1+t})} < \epsilon
\end{aligned}
$$

This gives $\lim_n \sup M_q(y, x_n, t) > 1 - \epsilon$*, which implies that* $\langle x_n \rangle$ *is* l*-convergent. Hence* X *is* ll*-complete.* □

Corollary 3.10. *Let* (X, M_q) *be a FQMS. If for every upper semi-continuous function* $\mathscr{F} : X \to \mathbb{R}$ *and for every* $\epsilon > 0$ *there exists* $y_\epsilon \in X$ *such that*

$$
(\frac{t}{1+t})\mathscr{F}(x) + 1 - \epsilon M_q(x, y_\epsilon, t) \leq (\frac{t}{1+t})\mathscr{F}(y_\epsilon), \ \forall x \in X
$$

then the FQMS X *is* rr*-complete.*

4. Applications on Optimization Theory

Theorem 4.1. (Takahashi's minimization theorem): *Let* $(X, M_q, *_m)$ *be a* ll*-complete FQMS and a mapping* $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper bounded below and lower semi-continuous function. Assume that there exists* $\rho > 0$ *and for each* $\hat{x} \in X$ *with* $\inf_{x \in X} \mathscr{F}(x) < \mathscr{F}(\hat{x})$*, there exists* $z \in X (z \neq \hat{x})$ *such that*

$$
\rho M_q(z, \hat{x}, t) \geq 1 - \frac{t}{1+t}[\mathscr{F}(\hat{x}) - \mathscr{F}(z)],
$$

then there exists $\bar{x} \in X$ *such that* $\mathscr{F}(\bar{x}) = \inf_{x \in X} \mathscr{F}(x)$*.*

Proof. *On the contrary, suppose* $\inf_{x \in X} \mathscr{F}(x) < \mathscr{F}(y)$ *for all* $y \in X$*, and let* $\hat{x} \in dom(\mathscr{F})$*. We define inductively a sequence* $\langle x_n \rangle$ *in* X *starting with* $x_1 = \hat{x}$*. Suppose that* $x_n \in X$ *is known. Put*

$$
S_{n+1} = \{x \in X : \frac{t}{1+t}\mathscr{F}(x_n) \geq \frac{t}{1+t}\mathscr{F}(x) + 1 - \rho M_q(x, x_n, t)\},
$$

and choose $x_{n+1} \in S_{n+1}$ *such that*

$$
\mathscr{F}(x_{n+1}) \leq \frac{1}{2}\{\inf_{x \in S_{n+1}} \mathscr{F}(x) + \mathscr{F}(x_n)\}.
$$

Now we have to verify that the definition of x_{n+1} *is correct. To do this, let us first show that* $\mathscr{F}(x_n) > \inf_{x \in S_{n+1}} \mathscr{F}(x)$*. Suppose that* $\mathscr{F}(x_n) = \inf_{x \in S_{n+1}} \mathscr{F}(x)$*. Then, by hypothesis,* $\mathscr{F}(x_n) > \inf_{x \in X} \mathscr{F}(x)$ *such that, by the given condition, there exists* $y \in S_{n+1} \setminus \{x_n\}$*, yielding a contradiction*

$$
\begin{aligned}
\frac{t}{1+t}\mathscr{F}(y) \leq \quad & \frac{t}{1+t}\mathscr{F}(x_n) - [1 - \rho M_q(y, x_n, t)] \\
\leq \quad & \frac{t}{1+t}\mathscr{F}(x_n) \\
\implies \mathscr{F}(y) \leq \quad & \mathscr{F}(x_n) = \inf_{x \in S_{n+1}} \mathscr{F}(x)
\end{aligned}
$$

which contradicts $y \in S_{n+1} \setminus \{x_n\}$. *Consequently,* $\mathscr{F}(x_n) > \inf_{x \in S_{n1}}$ *and* $\mathscr{F}(x_{n+1}) < \mathscr{F}(x_n)$.

Therefore, we may claim that $\langle x_n \rangle$ *is a* l*-Cauchy sequence. Since* $x_{n+1} \in S_{n+1}$ *for all* $n \in \mathbb{N}$*, then we have*

$$\rho M_q(x_{j+1}, x_j, t) \geq 1 - \frac{t}{1+t}[\mathscr{F}(x_{j+1}) - \mathscr{F}(x_j)]; \text{ for all} j \in \mathbb{N} \tag{4}$$

If $n > m$ *then, using equation (4), we obtain*

$$\begin{aligned} 1 - \rho M_q(x_m, x_n, t) &\leq 1 - \min\{\rho M_q(x_m, x_{m+1}, t_1), \ldots, \rho M_q(x_{n-1}, x_n, t_{n-m})\} \\ &\leq 1 - \rho M_q(x_m, x_{m-1}, t_1), \ldots, 1 - \rho M_q(x_{n+1}, x_n, t_{m-n}) \\ &\leq \frac{t}{1+t} \sum_{j=m-1}^{n} [\mathscr{F}(x_{j+1}) - \mathscr{F}(x_j)] \\ &\leq \frac{t}{1+t}[\mathscr{F}(x_m) - \mathscr{F}(x_n)] \qquad (5) \end{aligned}$$

Since the sequence $\langle \mathscr{F}(x_n) \rangle$ *is decreasing and the function* $\mathscr{F}$ *is bounded below, so* $\langle \mathscr{F}(x_n) \rangle$ *is convergent in* $\mathbb{R}$ *and hence it is Cauchy. Now, given* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$|\mathscr{F}(x_m) - \mathscr{F}(x_n)| < \frac{\epsilon}{\frac{t}{1+t}}$*; for all* $m, n \geq \mathbb{N}$

Then, by equation (5),

$1 - \rho M_q(x_m, x_n, t) \leq \frac{t}{1+t}[\mathscr{F}(x_m) - \mathscr{F}(x_n)] < \epsilon$*; for all* $n > m > N$*,*

which shows that the sequence $\langle x_n \rangle$ *is* l*-Cauchy.*

Since $(X, M_q, *_m)$ *is* ll*-complete, then* $\exists\, \tilde{x} \in X$ *such that* x_n $l\text{-convergent to } \tilde{x}$. *Since* $\mathscr{F}$ *is lower semi-continuous, then*

$$\lim_{n \to \infty} M_q(x_m, x_n, t) \leq M_q(\tilde{x}, x_n, t).$$

By taking the limit as $m \to \infty$ *in equation (5) and using the lower semi-continuity of* $\mathscr{F}$*, we obtain*

$$\begin{aligned} \rho M_q(\tilde{x}, x_n, t) &\geq \lim_{n \to \infty} \rho M_q(x_m, x_n, t) \\ &\geq 1 - \frac{t}{1+t}[\mathscr{F}(x_n) - \mathscr{F}(x_m)] \\ &\geq 1 - \frac{t}{1+t}[\mathscr{F}(x_n) - \mathscr{F}(\tilde{x})] \qquad (6) \end{aligned}$$

On the other hand, by the given condition, there exists $z \in X$ *such that* $z \neq \tilde{x}$ *and we get*

$$\rho M_q(z, \tilde{x}, t) \geq 1 - \frac{t}{1+t}[\mathscr{F}(\tilde{x}) - \mathscr{F}(z)] \tag{8}$$

From the definition of FQMS, we have

$$\begin{aligned} M_q(z, x_n, s) &\geq \min\{M_q(z, \tilde{x}, t), M_q(\tilde{x}, x_n, s-t), \text{ for } 0 < t < s \\ &\geq M_q(z, \tilde{x}, t) \end{aligned}$$

From equations (6), (7), and (8), we obtain

$$\begin{aligned}\frac{t}{1+t}\mathscr{F}(z) \leq & \quad \frac{t}{1+t}\mathscr{F}(\tilde{x}) - [1-\rho M_q(z,\tilde{x},t)] \\ \leq & \quad \frac{t}{1+t}\mathscr{F}(x_n) - [1-\rho M_q(z,x_n,s)]\end{aligned}$$

Consequently, $z \in S_{n+1}$ *for all* $n \in \mathbb{N}$.

Now, since

$$2\mathscr{F}(x_{n+1}) - \mathscr{F}(x_n) \leq \inf_{\tilde{x}\in S_{n+1}} \mathscr{F}(\tilde{x}) \leq \mathscr{F}(z). \tag{9}$$

Then, as per equations (7) and (9), we have

$$\begin{aligned}\frac{t}{1+t}\mathscr{F}(z) < & \quad \frac{t}{1+t}\mathscr{F}(z) + [1-\rho M_q(z,\tilde{x},t)] \\ \leq & \quad \frac{t}{1+t}\mathscr{F}(\tilde{x}) \\ \leq & \quad \frac{t}{1+t}\lim_{n\to\infty}\mathscr{F}(x_n) \\ = & \quad \frac{t}{1+t}\lim_{n\to\infty}\{2\mathscr{F}(x_{n+1}) - \mathscr{F}(x_n)\} \\ \leq & \quad \frac{t}{1+t}\mathscr{F}(z)\end{aligned}$$

which is a contradiction. Therefore, there exists $\bar{x} \in x$ *such that* $\mathscr{F}(\bar{x}) = \inf_{x\in X} \mathscr{F}(x)$. *Hence the theorem.* □

Corollary 4.2. *Let* $(X, M_q, *_m)$ *be a* rr*-complete FQMS and* $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ *be an upper semi-continuous function, proper and bounded above. Assume that there exists* $\rho > 0$ *and for each* $\hat{x} \in X$ *with* $\inf_{x\in X} \mathscr{F}(x) < \mathscr{F}(\hat{x})$, *there exists* $z \in X(z \neq \hat{x})$ *such that*

$$M_q(\hat{x}, z, t) \geq 1 - \frac{t}{1+t}[\mathscr{F}(z) - \mathscr{F}(\hat{x})],$$

then there exists $\bar{x} \in X$ *such that* $\mathscr{F}(\bar{x}) = \inf_{x\in X} \mathscr{F}(x)$.

Remark 4.3. *Theorems 3.4 and 4.1 are equivalent.*

Proof. First, we prove theorem 3.4 by using theorem 4.1. Assume that theorem 4.1 and all the hypotheses of theorem 3.4 hold. Let $\hat{x} \in X$, *consider a set* $Y = \{y \in X : (\frac{t}{1+t})\mathscr{F}(y) + 1 - \frac{\epsilon}{\lambda} M_q(y, \hat{x}, t) \leq (\frac{t}{1+t})\mathscr{F}(\hat{x})\}$. Y *is non-empty as* $\hat{x} \in Y$ *and* Y *is closed (see the proof of theorem 3.4), hence the statement (a) in theorem 3.4 holds.*

Now, for each $z \in Y$, *we get*

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$$\begin{aligned}(\frac{t}{1+t})\mathscr{F}(z)+1-M_q(z,\hat{x},t) \leq & \quad (\frac{t}{1+t})\mathscr{F}(\hat{x}) \\ \leq & \quad (\frac{t}{1+t})[\inf\mathscr{F}(X)+\epsilon] \\ \leq & \quad (\frac{t}{1+t})[\mathscr{F}(z)+\epsilon]\end{aligned}$$

implies $M_q(z,\hat{x},t) \geq \lambda(1-\frac{t}{1+t})$. Hence, the statement (b) in theorem 3.4 holds. If possible, let the statement (c) in theorem 3.4 not be true; therefore, there exists $y \in X (y \neq z)$ such that

$$\frac{t}{1+t}\mathscr{F}(y)+1-\frac{\epsilon}{\lambda}M_q(y,z,t) \leq \frac{t}{1+t}\mathscr{F}(z).$$

Now, from the definition of FQMS, we have,

$$\begin{aligned}1-\frac{\epsilon}{\lambda}M_q(y,\hat{x},t) \leq & \quad 1-\frac{\epsilon}{\lambda}\min\{M_q(y,z,t-s),M_q(z,\hat{x},s)\} \\ \leq & \quad 1-\frac{\epsilon}{\lambda}M_q(y,z,t-s)+1-\frac{\epsilon}{\lambda}M_q(z,\hat{x},s) \\ \leq & \quad \frac{t}{1+t}[\mathscr{F}(z)-\mathscr{F}(y)+\mathscr{F}(\hat{x})-\mathscr{F}(z)] \\ \leq & \quad \frac{t}{1+t}[\mathscr{F}(\hat{x})-\mathscr{F}(y)]\end{aligned}$$

Therefore, $y \in Y$. Then, by theorem 4.1, there exists $\bar{x} \in X$ such that $\mathscr{F}(\bar{x}) = \inf_{x\in Y}\mathscr{F}(x)$, which contradicts the fact that there exists $y_0 \in Y$ with $\mathscr{F}(y_0) < \mathscr{F}(\bar{x})$.

Hence, the statement (c) in theorem 3.4 is true.

Conversely, we have to prove theorem 4.1 by using theorem 3.4. Let theorem 3.4 hold and consider all the hypotheses of theorem 4.1. Put $\lambda = 1$ and $\epsilon = \rho$ in the statement (c) of theorem 3.4, then for each $\bar{x} \in X$ we have,

$$\frac{t}{1+t}\mathscr{F}(x)+1-\rho M_q(x,\bar{x},t) > \frac{t}{1+t}\mathscr{F}(\bar{x}), \text{ with } x \neq \bar{x} \tag{10}$$

If possible, let $\mathscr{F}(\bar{x}) > \inf_{x\in X}\mathscr{F}(x)$. By the hypothesis of theorem 4.1, there exists $z \in X, z \neq \bar{x}$ such that we get the following inequality,

$$\frac{t}{1+t}\mathscr{F}(z)+1-\rho M_q(z,\bar{x},t) \leq \frac{t}{1+t}\mathscr{F}(\bar{x}). \tag{11}$$

which contradicts equation (10) for $\rho = \frac{\epsilon}{\lambda}$. Hence, $\mathscr{F}(\bar{x}) = \inf_{x\in X}\mathscr{F}(x)$. $\square$

Theorem 4.4. *(Equilibrium version of TMT)*

Let $(X, M_q, *_m)$ be a ll-complete FQMS and $\mathbb{F} : X \times X \to \mathbb{R}$ be a bifunction. Assume that there exists a proper bounded below and lower semi-continuous function $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ such that

$$\mathbb{F}(x,y) \geq \mathscr{F}(y)-\mathscr{F}(x) \text{ for all } x,y \in X. \tag{12}$$

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Assume that there exists $\rho > 0$ and for each $\hat{x} \in X$ with $\inf_{x \in X} \mathscr{F}(x) < \mathscr{F}(\hat{x})$, there exists $z \in X (z \neq \hat{x})$ such that

$$\frac{t}{1+t}\mathbb{F}(z,\hat{x}) + \rho M_q(z,\hat{x},t) \geq 1$$

then there exists $\bar{x} \in X$ such that $\mathbb{F}(\bar{x}, y) \geq 0$ for all $y \in X$.

Theorem 4.5. (Converse of Takahashi's Minimization Theorem):

Let $(X, M_q, *_m)$ be FQMS and $\mathscr{F} : X \to \mathbb{R} \cup \{\infty\}$ be a bounded below, lower semi-continuous function. If for each $\tilde{x} \in X$ with $\inf_{x \in X} \mathscr{F}(x) < \mathscr{F}(\tilde{x})$,there exists $z \in X (z \neq \tilde{x})$ such that the following inequality holds:

$$\rho M_q(z,\tilde{x},t) \geq 1 - \frac{t}{1+t}[\mathscr{F}(\tilde{x}) - \mathscr{F}(z)].$$

Therefore there exists $\bar{x} \in X$ such that $\mathscr{F}(\bar{x}) = \inf_{x \in X} \mathscr{F}(x)$, then (X, M_q) is ll-complete FQMS.

Proof. Let $\langle x_n \rangle$ be a l-Cauchy sequence in X and consider the function $\mathscr{F} : X \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathscr{F}(x) = 1 - \limsup_{n} M_q(x, x_n, t).$$

Then Theorem 3.9 shows that $\lim_{n \to \infty} \mathscr{F}(x_n) = 0$. This implies $\inf_{x \in X} \mathscr{F}(x) = 0$.

Let us consider $\tilde{x} \in X$ with $\inf_{x \in X} \mathscr{F}(x) = 0 < \mathscr{F}(\tilde{x})$, then there exists a $n \in \mathbb{N}$ such that $\mathscr{F}(\tilde{x}) \leq \frac{1}{2}\mathscr{F}(x_n)$ and $1 - M_q(\tilde{x}, x_n, t) \leq \frac{t}{2(1+t)}\mathscr{F}(x_n)$. Therefore, for $x_n \neq \tilde{x}$, the condition of this theorem is represented by (for $\rho = 1$),

$$\frac{t}{1+t}\mathscr{F}(\tilde{x}) + 1 - M_q(\tilde{x}, x_n, t) \leq \mathscr{F}(x_n).$$

Thus, there exists $\bar{x} \in X$ such that $\mathscr{F}(\bar{x}) = \inf_{x \in X} \mathscr{F}(x) = 0$.

This implies $\mathscr{F}(\bar{x}) = 0 \implies \lim_{n \to \infty} M_q(\bar{x}, x_n, t) = 1$. Therefore, $\langle x_n \rangle$ is l-convergent to $\bar{x}$. Hence, (X, M_q) is ll-complete. □

5. Applications on Fixed Point Theory

By using the notion of the fuzzy metric space in the sense of Kramosil et al.[10], George and Veeramani[11] proved the Banach contraction principle (BCP) in a fuzzy metric space. However, Cobzas[6] established another type of fixed point result, named the Caristi-Kirk Fixed Point Theorem, by using EVP in the setting of a quasi metric space. Here, we shall prove these two fixed point results on the basis of FQMS by using EVP (Theorem 3.4).

Theorem 5.1. (Banach Contraction Theorem): *Let X be a ll-Complete FQMS and $\mathcal{T} : X \to X$ be a contraction mapping[28] satisfying*

$$M_q(\mathcal{T}x, \mathcal{T}y, \kappa t) \geq M_q(x, y, t), \text{ for all } x, y \in X, 0 < \kappa < 1, \tag{13}$$

then T has a unique fixed point in X.

Proof. Consider the function $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathscr{F}(x) = M_q(x, \mathcal{T}(x), t) \text{ for all } x \in X.$$

Then, as per definition, $\mathscr{F}$ is bounded below and lower semi-continuous on X. Now we choose ϵ $(0 < \epsilon < \lambda)$ such that, as per theorem 3.4, there exists $z \in X$ satisfying

$$\frac{\kappa t}{1+\kappa t}\mathscr{F}(z) + 1 - \frac{\epsilon}{\lambda}M_q(z, x, \kappa t) \leq \frac{\kappa t}{1+\kappa t}\mathscr{F}(x).$$

Now, putting $z = \mathcal{T}(x)$ in the above, we have

$$\begin{aligned}
& & \frac{\kappa t}{1+\kappa t}\mathscr{F}(\mathcal{T}x) + 1 - \frac{\epsilon}{\lambda}M_q(\mathcal{T}x, x, \kappa t) \leq \frac{\kappa t}{1+\kappa t}\mathscr{F}(x) \\
\iff & & \frac{\kappa t}{1+\kappa t}M_q(\mathcal{T}x, \mathcal{T}\mathcal{T}x, \kappa t) + 1 - \frac{\epsilon}{\lambda}M_q(\mathcal{T}x, x, t) \leq \frac{\kappa t}{1+\kappa t}M_q(x, \mathcal{T}x, t) \\
\iff & & \frac{\kappa t}{1+\kappa t}M_q(x, \mathcal{T}x, t) + 1 - \frac{\epsilon}{\lambda}M_q(\mathcal{T}x, x, t) \leq \frac{\kappa t}{1+\kappa t}M_q(x, \mathcal{T}x, t) \\
\iff & & \frac{\epsilon}{\lambda}M_q(\mathcal{T}x, x, t) \geq 1 \\
\iff & & M_q(\mathcal{T}x, x, t) = 1 \iff \mathcal{T}x = x
\end{aligned}$$

Therefore, $\mathcal{T}$ has a fixed point. Now we are to show that this fixed point is unique. If possible, there exists another fixed point $y(\neq x) \in X$ such that $\mathcal{T}y = y$.

$$\begin{aligned}
1 \geq M_q(x, y, t) = & \quad M_q(\mathcal{T}x, \mathcal{T}y, t) \\
\geq & \quad M_q(x, y, \frac{t}{k}) = M_q(\mathcal{T}x, \mathcal{T}y, \frac{t}{k}) \\
\geq & \quad M_q(x, y, \frac{t}{k^2}) = \ldots \\
\geq & \quad M_q(x, y, \frac{t}{k^n}) \to 1, \text{ as } n \to \infty
\end{aligned}$$

This implies $x = y$. Hence, the proof. $\square$

Theorem 5.2. (Caristi-Kirk Fixed Point Theorem): *Let $(X, M_q, *_m)$ be ll-complete FQMS. Consider a bounded below, lower semi-continuous function $\mathscr{F} : X \to \mathbb{R}$ and a fuzzy function $f : X \to X$ satisfy the following condition*

$$\frac{t}{1+t}\mathscr{F}(x) + 1 - M_q(f(x), x, t) \leq \frac{t}{1+t}\mathscr{F}(f(x))$$

then f has a fixed point in X.

Proof. We define an order relation on X for $x, y \in X$ as

$x \leq y \implies M_q(x, y, t) \geq 1 - (\frac{t}{1+t})[\mathscr{F}(y) - \mathscr{F}(x)]$

Then the hypothesis of the theorem shows that

$$f(x) \leq x \text{ for all } x \in X.$$

Now, from theorem 3.3, we can say that there exists a minimal element $z \in X$. *Then, from equation (14), we have* $f(z) \leq z$, *so* $f(z) = z$ *as* z *is the minimal element. Hence, the theorem.* $\square$

6. Conclusion

In this paper, the Ekeland variational principle is developed by using the Brézis-Browder principle on a partial order set over FQMS under a non-Archimedean t-norm. The existence of a solution to the optimization problem in the sense of Takahashi's minimization theorem has been established without compactness and convexity assumptions. Also, an equivalence relation between these two theorems and two types of equilibrium solutions is established here. Based on EVP, the Banach fixed point theorem and the Caristi-Kirk fixed point theorem are employed in this FQMS.

Moreover, these results can further be developed in several optimization theories, game theory, differential equations, and non-linear analysis, etc., in the setting of a fuzzy quasi-metric space. Indeed, this approach can be extended to some other fuzzy environments such as lock fuzzy set, dense fuzzy set, cloudy fuzzy set, fuzzy reasoning, and hesitant fuzzy set as well.

Statements and Declarations

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Author Contributions

All the authors have equally contributed to the final manuscript.

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