LONGITUDINAL DOPPLER FOR

OBSERVERS IN UNIFORM ACCELERATION

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ABSTRACT

Electromagnetic waves are emitted during uniform acceleration g, with an observer positioned at distance H from the source. If f_0 represents the frequency of the emitter, as detected at rest with the source, the observed frequencies are usually represented using the formula $f = f_0 (1+gH/c^2)$. This relation was derived from the classic Doppler effect formula, taking into consideration the light-time H/c between emission and absorption, just as in the stationary case. By using the relativistic relations properly, the exact light-time required to connect source and observer is determined; it is not the same as in the non-stationary case. Nonetheless, if g represents the proper acceleration, the frequency ratio remains consisitent, as in the classic case as $(1+gH/c^2)$, dependent on a peculiar compensation of two relativistic effects.

1. INTRODUCTION

Feynman [1], following the reasoning of Einstein [2], considered the EM waves emitted from an oscillator A situated at the front of a rocket towards B at its rear, while measuring an acceleration g.

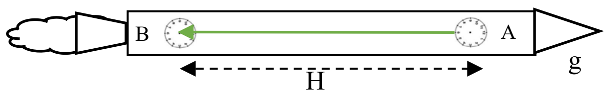

Fig.1 A rocket is in acceleration, the rear observer receives EM waves from the front one.

In the configuration, illustrated in the fig. 1, under a constant acceleration g, the position of B gets closer to the position of the emission of a pulse of light, emitted by A. The light-time is approximated as H/c [1],[2], which is the same as having a source and an observer at rest.

An observer in motion within the reference frame of the emitter experiences a longitudinal Doppler. If f_A represents the frequency of the emitted light at rest with the source, and the velocity v is much less than the speed of light (v<<c) [1], [2], then the approximation $f_B/f_A = 1 + v/c$ holds . By substituting $v=g*H/c$, into the previous equation, we obtain

$f_B/f_A \approx (1+gH/c^2)$, a result derived by Feynman [1], as a first order approximation of the relativistic Doppler effect :

$\frac{(1+\frac{v}{c})}{\sqrt{1-\frac{v^2}{c^2}}} \approx (1+gH/c^2)$. By considering the simplest case of a stationary source and observer in uniform acceleration, the light-time connecting source and observer will be calculated using the relativistic formulas. This calculation will lead to the derivation of a general result for the frequency shift of light for accelerated obsevers. Surprisingly, the formula maintains the same form as the classic one, with g representing the proper acceleration. The derivation of the relevant results will follow, including the case of the accelerated rocket. A short discussion will be presented afterwards.

2. LONGITUDINAL DOPPLER IN UNIFORM ACCELERATION

Two clocks C' and C'' are stationary at distance H , they are synchronized using Einstein's synchronization procedure. At time t_0 , observer B departs with a constant acceleration g, while simultaneously A emits a pulse of radiation. The speed v_1 of B in the inertial reference frame of the source is calculated at the moment of absorption t_1, as shown in the figure below. This calculation allows for the application of the tested formula of the Relativistic Doppler effect.

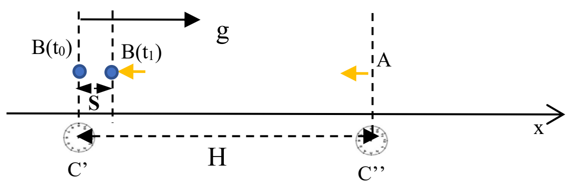

Fig.2 Emission by a stationary source and absorption of a pulse by an accelerated observer

According to relativistic dynamics [3] the displacement S of an accelerated body with proper acceleration g as a function of the elapsed time t, starting with $t_0=0$ is given by the formula

$$S = \frac{c^2}{g}\sqrt{1+\left(\frac{gt}{c}\right)^2} - \frac{c^2}{g} \qquad (\approx 1/2\ g\ t^2 \text{ in the classic case}) \qquad (1)$$

The ligh-time to reach B, is $c\ t_l = H - S$. (2)

The speed [3] $v_1 = \frac{gt_1}{\sqrt{1+\left(\frac{gt_1}{c}\right)^2}}$ ($\approx g\ t_l$ for $v_B \ll c$) (3)

is the value of the speed of B in the IRF of the source at the instant t_l of absorption.

From (1) and (2) the following equation is obtained $\left(\frac{H}{c}+\frac{c}{g}- t_1\right) = \frac{c}{g}\sqrt{1+(\frac{gt_1}{c})^2}$ *(see appendix part 1)*

the elapsed time is found by solving in t_l, we find:

$$t_l = \frac{H}{c}\frac{\left(1+\frac{gH}{2c^2}\right)}{1+\frac{gH}{c^2}} = \frac{H}{c} - \frac{gH^2}{2c^3} + \frac{g^2H^3}{2c^5} + \ldots \approx \frac{H}{c} \quad (gH \ll c^2) \qquad (4)$$

(see Appendix Part 2)

This is the exact light-time between the emission by the stationary source and the absorption by the accelerated observer, as measured by a stationary clock located at that point. In the limit of small gH, the interval is well approximated by H/c.

The difference of light-time compared to the classical case is the following,

considering the power series in Eq.(4): $\Delta t_{var} = t_l - \frac{H}{c} \approx -\frac{gH^2}{2c^3}$

That represents the variation of the time taken by light (light-time) to connect the source with an observer experiencing constant acceleration g. As expected, when the observer is moving towards the source (approaching the source) the exact light-time is shorter than H/c, $\Delta t_{var} < 0$. Conversely, if the acceleration has a negative value (observer departing from the source), the time interval becomes longer, with $\Delta t_{var} > 0$.

By replacing t_l from Eq(4), in the Eq(3) we have *(see Appendix Part 3)*

$$v_l = \frac{\frac{gH}{c}\left(1+\frac{gH}{2c^2}\right)}{\frac{1}{2}\left(\frac{gH}{c^2}\right)^2+\frac{gH}{c^2}+1} = \frac{gH}{c} - \frac{(gH)^2}{2c^3} + \frac{(gH)^4}{4c^7} + \cdots \approx \frac{gH}{c} \quad (gH \ll c^2) \qquad (5)$$

The final speed, approximated to the second order in gH, is $v_l \approx \frac{gH}{c} - \frac{(gH)^2}{2c^3} = \frac{gH}{c}(1 - \frac{gH}{2c^2})$

If the direction of the speed v is towards the emitter (when $g > 0$), the classic speed $V_{classic}$ is greater than v (all positive). Consequently, the observer's speed in the event of the absorption is a slightly lower than the calculated classic value. However, if the observer departs from the source (illustrated as $v<0$) the magnitude of the speed increases more than in the classic case. This results in light taking more time to reach the observer compared to the classic scenario.

The Relativistic Doppler Effect formula gives $f_B/f_A = \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$, considering $v=v_l$

The numerator $(1+ v_l /c)$ is given by $\frac{\left(1+\frac{gH}{c^2}\right)^2}{\frac{1}{2}\left(\frac{gH}{c^2}\right)^2+\frac{gH}{c^2}+1}$ *(see appendix part 4)* (6)

The denominator $1/\sqrt{(1-v_l^2/c^2)}$ is $\frac{\frac{1}{2}\left(\frac{gH}{c^2}\right)^2+\frac{gH}{c^2}+1}{\left(1+\frac{gH}{c^2}\right)}$, (7)

By multiplying (6) and (7) the final result is

$f_B/f_A = (1+gH/c^2)$ (8)

It’s surprising that, the frequency ratio given by of Eq.(8), is identical to the classic formula and is exact, with g representing the proper acceleration of the observer. Additionally the frequency shift $\Delta f / f = gH/c^2$, often found in electrodynamics, is exact.

When heading towards the light-beam (g>0), the light-time gets shorter than H/c, hence the speed reached is reduced in comparison to v=gH/c: $v_1 \approx \frac{gH}{c}(1 - \frac{gH}{2c^2}) < gH/c$.
In the Eq.(6) the classic value $(1+gH/c^2)$ gets multiplied by the quantity $(1+gH/c^2) / [1+gH/c^2+1/2(gH/c^2)^2] \approx 1 - (gH/c^2)^2/2$, representing the higher order contribution. This effect causes a decrease in the blueshift detected by the observer due to the lower final speed at the moment of detection.
The decrease mentioned above is exactly counterbalanced by the relativistic effects in Eq.(7), which are the reciprocal of the previous term (a factor not considered in the classic configuration). This occurs because the period of the absorber becomes longer due to its velocity in the frame of the source. As a result, the incoming radiation is observed with a greater blueshift compared to the classic case. This leads to the conclusion that distinct relativistic effects precisely cancelled eachother out. The original formula of the Doppler ratio in accelerated motion, derived classically, turns out to be exact, at least for a stationary emitter, when considering the proper acceleration of the observer.

The obtained formulas, rely exclusively on the relativity of time and the experimentally tested relativistic Doppler effect. Now let’s explore how to apply these formulas to the configuration of the accelerated rocket.

3. THE CASE OF THE ACCELERATED ROCKET

In the context of the accelerated rocket, the emission also takes place within a non-inertial frame. It’s important to highlight that the choice of an initial speed $v_0=0$ is a deliberate decision to prevent v_0 from appearing in the formula for the frequency shift. This maintains the independence of the formula from the intial speed v_0. If the frequency shift measured internally were to depend on the initial speed, it could potentially allow one to deduce their speed from within the system solely by measuring the frequency shift. This scenario would only require the knowledge of the the acceleration g and distance H, without referring to external observations or internal clock readings. From a relativistic standpoint, as discussed in [4], if the observer at the rear detects an acceleration g, the front clock within the rocket will register a slightly different acceleration. Accoriding to Special Relativity, the two clocks will exhibit different proper accelerations due to these factors.

It was reported by Feynman [1], starting from the formula $(1+ v/c) /\sqrt{(1- v^2/c^2)}$ “*Assuming that the acceleration and the length of the ship are small enough that this velocity is much smaller than c, we can neglect the term in v^2/c^2*”. He derived the result as already mentioned considering that the first order approximation is (1+ v/c) where v<<c, hence his first order approximation of the Relativistic Doppler effect $(1+ v/c) /\sqrt{(1- v^2/c^2)} \approx (1+gH/c^2)$.
However, the outcome of the detailed calculations results in $(1+ v/c) /\sqrt{(1- v^2/c^2)} = (1+gH/c^2)$. This implies that the classic formula is exact when g represents the proper acceleration. Consequently, it cannot be derived from an higher-order, more accurate formula.
The longitudinal Doppler effect in accelerated motion, as derived classically, is indeed exact, and cannot be derived as a first order approximation from a higher order formula. While the difference between the light-times in inertial motion and acceleration experience a slight adjustment, its first order approximation is $\Delta t_{var} \approx -\frac{1}{2} gH^2/c^3 = -\frac{1}{2} vH/c^2$ where v=gH/c.

4. CONCLUSIONS

The light-time required to connect an observer undergoing uniform accelerated motion with constant acceleration g and an emitter, initially separated by H, is $t_1 = \frac{H}{c}\frac{\left(1+\frac{gH}{2c^2}\right)}{1+\frac{gH}{c^2}}$. While the light-time at rest remains H/c, the difference with t_1 is approximately $|\Delta t_{var}| \approx |\frac{1}{2} H/c \; gH/c^2|$, it is a tiny fraction of the light-time at rest, $|\Delta t_{var}|/(H/c) \approx |\frac{1}{2} gH/c^2|$.

To some surprise, the Doppler effect, with a source at frequency f_0 ,set at adistance H from an observer measuring acceleration g, is exactly the ratio $f/f_0 = (1+gH/c^2)$ or $\Delta f / f_0 = gH/c^2$. It is the same as in the classic case, with g the proper acceleration. That formula is valid also in an accelerated rocket and was inappropriately considered by Feynman derivable as a first order approximation from the Relativistic Doppler formula.

In the scenario of blueshift, when the observer is moving towards the source, the observer's oscillator would registers, only on the account of its speed, the incoming radiation with a reduced blueshift compared to the classic case. Conversely, relativistic effects lead to an increase in the proper period of the observer's oscillator, resulting in a more pronounced blueshift in the detected incoming radiation.

What's particularly intriguing is the observation that these two effects are reciprocals of each other. As a result, they effectively cancel each other out, resulting in a net effect that maintains the consistency of the observed blueshift.

REFERENCES

[1] R. Feynman "The Feynman Lectures on Physics", Vol. 2 Ch. 42 par 6, (1965)

[2] A. Einstein "On the Influence of Gravitation on the Propagation of Light", Annalen der Physik, 35, pp. 898-908. (1911)

[3] C. Semay "Observer with a constant proper acceleration" Eu. J. Phys Vol 27, number 5, pp.1157 (2006)

[4] S. Boughn "The case of the identically accelerated twins" Am. J. Phys. 57 (9) pp. 791 (1989)

APPENDIX

1) **From the first equation** $c\,t = H - S$, replace S with the Eq.(1)
$ct = H - (c^2/g\ \sqrt{(1+(g\,t/c)^2)}\ -c^2/g)$;
$c\,t - H - c^2/g = -(c^2/g\ \sqrt{(1+(g\,t/c)^2)})$;
$(H/c + c/g - t) = c/g\ \sqrt{(1+(gt/c)^2)}$

2) **Find t_1 by solving the previous equation**,
by setting $k=c/g$,
$(H/c + k - t_1) = k\ \sqrt{(1+(t_1/k)^2)}$ by solving in t_1 (with Wolfram Alpha)
$t_1 = H/c\ (H/c+2\,c/g)\ /\ 2(H/c + c/g)$, by multiplying and dividing by (g/c)
$t_1 = [H/c\ (g/c)\ (H/c+2\,c/g)]\ /\ [2(H/c + c/g)(g/c)] = [H/c\ (gH/c^2 +2)]/[2(gH/c^2+1)]$;
$t_1 = H/c\ (1+gH/2c^2)/(1+ gH/c^2)$;

3) **Calculate $v = gt/\sqrt{(1+(gt/c)^2)}$ replacing t_1 (as function g,H,c)**
Let's replace $t = t_1$ as found in the previous, set $k= gH/c^2$; (with Wolfram Alpha)

$\mathbf{(1+(g\,t/c)^2)} = 1+ g^2/c^2\ (H/c\ (1+k/2)/(1+ k))^2 =1+ g^2/c^2\ H^2/c^2\ (1+k/2)^2/(1+ k)^2 = 1+ [(gH/c^2)^2\ (1+k/2)^2/(1+ k)^2]$;
$1+ [k^2\ (1+k/2)^2/(1+ k)^2] =[(1+ k)^2 +k^2\ (1+k/2)^2]/(1+ k)^2 =1/2\ [(k^2 +2k+2)\ /\ (k+1)]^2$;

$\mathbf{\sqrt{(1+(g\,t/c)^2)}} = (k^2 +2k+2)\ /2(k+1)$; $1/\sqrt{(1+(g\,t/c)^2)} = 2(k+1)/\ (k^2 +2k+2)$
$v = g\ \ t\ [2(k+1)/\ (k^2 +2k+2)] = gH/c\ [(1+k/2)/(1+ k)]\ [2(k+1)/\ (k^2 +2k+2)] = 2\ gH/c\ (1+k/2)/\ (k^2 +2k+2)$
replacing back $k= gH/c^2$,
$v= gH/c\ (1+gH/2c^2)/(1/2(gH/c^2)^2 +gH/c^2+1)$

4) **Find $f_B/f_A= (1+ v/c)\ /\sqrt{(1-v^2/c^2)}$, replacing v (as function g,H,c)**

Setting $k= gH/c^2$, from the previous calculations considering v (instead of v1)

$\mathbf{v/c} = 2\ gH/c^2\ (1+k/2)/\ (k^2 +2k+2) = 2\ k\ (1+k/2)/\ (k^2 +2k+2)$

from the previous
$\mathbf{(1+ v/c)} = 1 + 2\ k\ (1+k/2)/\ (k^2 +2k+2) = [(k^2 +2k+2) +2\ k\ (1+k/2)]/\ (k^2 +2k+2) = (2k^2 +4k+2)/\ (k^2 +2k+2) =$
$= 2(k^2 +2k+1)/(\ k^2 + 2k+2)$
And also
$\mathbf{(1-\ v^2/c^2)} = 1- [2k(1+k/2)/\ (k^2 +2k+2)]^2 = [(k^2 +2k+2)^2 - (2k(1+k/2))^2]/\ (k^2 +2k+2)^2 =$
$[(k^2 +2k+2)^2 - (2k+k^2)^2]/\ (k^2 +2k+2)^2 = 4(k+1)^2\ /(k^2 +2k+2)^2$
hence
$\mathbf{\sqrt{(1-\ v^2/c^2)}} = 2(k+1)\ /(k^2 +2k+2)$
Final result
$\mathbf{(1+ v/c)\ /\sqrt{(1-\ v^2/c^2)}} = 2(k^2 +2k+1)/\ [2(k+1)] = (k+1) = (1+ gH/c^2)$