Research Article

Derivative-Order Geometry of Holomorphic and Meromorphic Functions: Flow, Collapse, Spectral Periodicity, and Asymptotic Fingerprints

Mohammad Abu-Ghuwaleh[1]

1. Zarqa University, Az-Zarqa, Jordan

Let f be a nonpolynomial holomorphic or meromorphic function on a plane domain U, and define the derivative-ladder ratios

$q_n := \frac{f^{(n+1)}}{f^{(n)}}, n \geq 0.$

We treat $(q_n)_{n\geq 0}$ as a discrete geometry on the derivative-order axis. With the forward difference $\Delta q_n := q_{n+1} - q_n$, the ladder satisfies the exact flow law

$q_n' = q_n \Delta q_n,$

so order slope is the logarithmic derivative of the rung. The second difference

$K_n := \Delta^2 q_n$

also satisfies $K_n = (\log R_n)'$, where $R_n := q_{n+1}/q_n$, and therefore acts as an order-curvature defect invariant for the departure from the rigid exponential and affine-power regimes.

The paper develops three layers of theory. The first is exact structural calculus: inverse realization of compatible ladders, differential-polynomial identities for $f^{(n+m)}/f^{(n)}$, and local transport laws showing how zeros and poles of successive derivatives propagate through the ladder and how resonance fields record multiplicities. The second is exact rigidity: vanishing curvature at one rung yields local exponential or affine-power structure, and if $\Delta^r q_n \equiv 0$ for some finite r, then automatically $\Delta^2 q_n \equiv 0$. In particular, higher polynomial geometries on the derivative-order axis do not exist: every finite-order ladder collapses to the exponential model or the affine-power model. The third is asymptotic rigidity: periodic ladders are classified by constant-coefficient equations $f^{(p)} = \kappa f$, and for exponential-polynomial source models we prove an asymptotic order-dimension law

$q_n(z) = \lambda\left(1 + \frac{d}{n} + O(n^{-2})\right), \Delta q_n(z) = -\frac{d\lambda}{n^2} + O(n^{-3}), K_n(z) = \frac{2d\lambda}{n^3} + O(n^{-4}),$

doi.org/10.32388/VQBNBD

locally uniformly on compact sets, where d is the degree of the dominant polynomial prefactor. Thus the ladder asymptotically recovers hidden spectral multiplicity from its own curvature defect.
The fields $q_0 = f'/f$ and $q_1 = f''/f'$ are classical, belonging respectively to the logarithmic-derivative and pre-Schwarzian traditions. The novelty here lies in treating the full infinite ladder as a coupled meromorphic object with exact flow, inverse realization, singular fingerprints, finite-order collapse, periodic spectral closure, and asymptotic fingerprint laws.

Correspondence: papers@team.qeios.com — Qeios will forward to the authors

1. Introduction

Classical work on successive derivatives of holomorphic and meromorphic functions has focused mainly on zeros, growth, normality, and final sets. A line running from Pólya and Rådström through Boas–Reddy, Hellerstein–Williamson, Gethner, Bergweiler–Eremenko, and Meyrath–Müller studies where the zeros of $f^{(n)}$ move, how they accumulate, and what they reveal about the analytic character of f [1][2][3][4][5][6][7]. In a different direction, MacLane studied sequences of derivatives through normal families and universality [8]. Ratios of successive derivatives also appear in computational settings, for example in root-finding for polynomials [9]. On the neighboring side of geometric function theory, f'/f belongs to the logarithmic-derivative tradition [10], while f''/f' is the pre-Schwarzian derivative [11][12]; higher Schwarzian theories provide related differential-polynomial invariants built from finitely many derivatives of a single map [13][14]. More recently, Grätsch studied lifts of logarithmic derivatives for sequences of meromorphic functions [15].

The present paper isolates a different organizing principle. Instead of observing the derivative tower

$$f, f', f'', f^{(3)}, \dots$$

mainly through its zero sets or coefficients, we promote the adjacent ratios

$$q_n := \frac{f^{(n+1)}}{f^{(n)}}, n \geq 0,$$

to the primary object. The derivative index n is treated as a genuine discrete coordinate. This produces a derivative-order geometry: first differences in n become order slopes, second differences become order curvatures, and the whole ladder is governed by an exact nonlinear flow.

 doi.org/10.32388/VQBNBD

The foundational identity is elementary but unexpectedly rigid:

$$q_n' = q_n(q_{n+1} - q_n) = q_n \Delta q_n.$$

Thus the order slope is exactly the logarithmic derivative of the rung. Writing

$$R_n := \frac{q_{n+1}}{q_n}, K_n := \Delta^2 q_n,$$

one obtains

$$K_n = (\log R_n)'.$$

This makes K_n a natural order-curvature defect invariant. When $K_n \equiv 0$, the resonance quotient R_n is constant and one rung of the ladder becomes rigid. When K_n is merely small, the reciprocal field $1/q_n$ is forced to be close to an affine function, so the ladder is close to a local exponential or affine-power model. When the entire ladder has finite difference order in n, the universal flow immediately collapses it to difference degree at most one.

Two results control the paper. The first is the finite-order collapse theorem (Theorem 6.2), which shows that higher polynomial geometries on the derivative-order axis simply do not exist: if $\Delta^r q_n \equiv 0$ for some finite r, then already $\Delta^2 q_n \equiv 0$, hence the ladder is forced into the exponential or affine-power regime. The second is the asymptotic order-dimension law (Theorem 8.3), which shows that outside exact rigidity the scaled fields $q_n - \lambda$, Δq_n, and K_n still carry rigid information: in dominant exponential-polynomial families they recover the degree of the dominant polynomial amplitude.

This viewpoint is adjacent to the logarithmic derivative, the pre-Schwarzian derivative, and the literature on successive derivatives, but it does not reduce to any one of them. The novelty is not the isolated appearance of $q_0 = f'/f$ or $q_1 = f''/f'$. It is the fact that the full ladder

$$q_0, q_1, q_2, \dots$$

forms a coupled meromorphic object with an exact differential-difference law, an inverse realization theorem, a transport law for singularities, a rigidity theory, a periodic spectral theory, and a nontrivial asymptotic calculus.

The main results may be summarized as follows.

1. **Universal flow and order geometry.** The ladder satisfies $q_n' = q_n \Delta q_n$, so $\Delta q_n = q_n'/q_n$ and $K_n = (\log R_n)'$.

 doi.org/10.32388/VQBNBD

2. **Inverse realization.** Every compatible ladder satisfying the universal flow comes from an actual holomorphic or meromorphic function once the bottom rung is realized as a logarithmic derivative.
3. **Differential calculus and ODE closure.** Each quotient $f^{(n+m)}/f^{(n)}$ is a differential polynomial in a single rung q_n, which turns linear ODEs for $f^{(n)}$ into nonlinear equations for q_n.
4. **Singular fingerprints.** Zeros and poles of successive derivatives propagate explicitly under the ladder operator, and the local multiplicity of a zero or pole is recorded by the resonance field.
5. **Rigidity and quantitative near-rigidity.** Vanishing order curvature at one rung forces local exponential or affine-power structure; small order curvature forces quantitative proximity to the same rigid models.
6. **Finite-order collapse.** If $\Delta^r q_n \equiv 0$ for some finite r, then automatically $\Delta^2 q_n \equiv 0$. Hence every order-polynomial ladder collapses to the exponential model or the affine-power model.
7. **Periodic spectral classification.** Periodic ladders are exactly the ladders coming from constant-coefficient equations $f^{(p)} = \kappa f$, equivalently from finite packets of exponentials with frequencies among the pth roots of κ.
8. **Asymptotic fingerprints.** For exponential-polynomial source models the ladder has an explicit large-order expansion whose first three scales recover the dominant spectral root and the degree of its polynomial prefactor. In particular
$n\left(\frac{q_n}{\lambda} - 1\right), -\frac{n^2}{\lambda}\Delta q_n, \frac{n^3}{2\lambda}K_n$
all converge to the same integer d, a hidden spectral multiplicity carried by the dominant packet.

The last point is the main asymptotic layer of the paper. It shows that the order-curvature field is not merely an internal diagnostic. In natural external families—in particular, exponential-polynomial and dominant constant-coefficient ODE models—the scaled curvature defect remembers multiplicity data that is invisible from a single rung alone.

To avoid trivial terminal derivatives, we usually assume that f is nonpolynomial. Polynomial ladders fit naturally into the same formulas up to the last nonzero rung, and we indicate this truncated case when needed.

2. The derivative ladder and its order geometry

Throughout, $U \subset \mathbb{C}$ is a domain and $f \in \mathcal{M}(U)$ is a nonpolynomial meromorphic function. Then $f^{(n)} \not\equiv 0$ for every $n \geq 0$, so the quotients below are well-defined meromorphic functions.

doi.org/10.32388/VQBNBD

Definition 2.1. For $n \geq 0$ we define the nth derivative ratio

$$q_n := \frac{f^{(n+1)}}{f^{(n)}}$$

and the nth resonance quotient

$$R_n := \frac{f^{(n)} f^{(n+2)}}{\left(f^{(n+1)}\right)^2} = \frac{q_{n+1}}{q_n}.$$

The sequence $(q_n)_{n \geq 0}$ is the *derivative ladder* of f.

We also use the forward difference on the order axis:

$$\Delta q_n := q_{n+1} - q_n, \Delta^2 q_n := q_{n+2} - 2q_{n+1} + q_n.$$

Theorem 2.2 (Universal ladder flow). *For every $n \geq 0$,*

$$q_n' = q_n(q_{n+1} - q_n) = q_n \Delta q_n \tag{2.1}$$

as a meromorphic identity on U. Equivalently,

$$q_{n+1} = q_n + \frac{q_n'}{q_n} =: T(q_n), T(q) := q + \frac{q'}{q}, \tag{2.2}$$

and

$$R_n = 1 + \frac{q_n'}{q_n^2}. \tag{2.3}$$

Proof. On the open set where $f^{(n)} f^{(n+1)} \neq 0$ we compute

$$q_n' = \left(\frac{f^{(n+1)}}{f^{(n)}}\right)' = \frac{f^{(n+2)} f^{(n)} - \left(f^{(n+1)}\right)^2}{\left(f^{(n)}\right)^2}.$$

Since

$$\frac{f^{(n+2)} f^{(n)}}{\left(f^{(n)}\right)^2} = \frac{f^{(n+2)}}{f^{(n+1)}} \cdot \frac{f^{(n+1)}}{f^{(n)}} = q_{n+1} q_n,$$

we obtain

$$q_n' = q_n q_{n+1} - q_n^2 = q_n(q_{n+1} - q_n).$$

Both sides are meromorphic on U, so the identity extends to all of U. Rearranging gives (2.2), and dividing by q_n^2 gives (2.3). □

 doi.org/10.32388/VQBNBD

Remark 2.3. Equation (2.2) shows that the upward ladder is generated by iteration of a single nonlinear first-order operator:

$$q_n = T^n(q_0), n \geq 0.$$

Thus the ladder is not a free meromorphic sequence but the forward orbit of $q_0 = f'/f$ under T.

Definition 2.4. The *order curvature* of the ladder at rung n is

$$K_n := \Delta^2 q_n = q_{n+2} - 2q_{n+1} + q_n.$$

We say that the ladder is *order-flat at rung* n if $K_n \equiv 0$.

Proposition 2.5 (First and second order identities). *For every $n \geq 0$ the following meromorphic identities hold on U:*

$$\Delta q_n = \frac{q_n'}{q_n}, \tag{2.4}$$

$$R_n = 1 + \frac{\Delta q_n}{q_n}, \tag{2.5}$$

$$K_n = \frac{R_n'}{R_n}. \tag{2.6}$$

In particular, on any simply connected region where q_n is zero-free,

$$\Delta q_n = (\log q_n)', K_n = (\log R_n)'.$$

Proof. Identity (2.4) is just (2.1) divided by q_n. Identity (2.5) follows from

$$R_n = \frac{q_{n+1}}{q_n} = \frac{q_n + q_n'/q_n}{q_n}.$$

Finally,

$$\frac{R_n'}{R_n} = \frac{q_{n+1}'}{q_{n+1}} - \frac{q_n'}{q_n} = \Delta q_{n+1} - \Delta q_n = \Delta^2 q_n = K_n,$$

using (2.4) twice. □

Corollary 2.6. *For every fixed n, the ladder is order-flat at rung n if and only if R_n is constant on U.*

Proof. By (2.6), the identity $K_n \equiv 0$ is equivalent to $R_n'/R_n \equiv 0$, hence to R_n being constant. □

Remark 2.7 (Rungwise logarithmic and pre-Schwarzian fields). On any region where $f^{(n)}$ does not vanish,

$$q_n = (\log f^{(n)})'.$$

On any region where $f^{(n+1)}$ does not vanish, the next rung q_{n+1} is the pre-Schwarzian derivative of $f^{(n)}$:

doi.org/10.32388/VQBNBD

$$\frac{\left(f^{(n)}\right)''}{\left(f^{(n)}\right)'} = \frac{f^{(n+2)}}{f^{(n+1)}} = q_{n+1}.$$

Consequently, on such a region,

$$S_{f^{(n)}} = q'_{n+1} - \frac{1}{2} q^2_{n+1}.$$

Thus the derivative ladder packages, all at once, the logarithmic derivatives of the successive derivatives and the pre-Schwarzian fields of the successive rungs.

Proposition 2.8 (Telescoping products). *For every* $n, m \geq 0$*,*

$$\prod_{j=0}^{m-1} q_{n+j} = \frac{f^{(n+m)}}{f^{(n)}} \tag{2.7}$$

as a meromorphic identity, with the empty product interpreted as 1*.*

Proof. Repeated cancellation gives

$$\prod_{j=0}^{m-1} \frac{f^{(n+j+1)}}{f^{(n+j)}} = \frac{f^{(n+1)}}{f^{(n)}} \cdot \frac{f^{(n+2)}}{f^{(n+1)}} \cdots \frac{f^{(n+m)}}{f^{(n+m-1)}} = \frac{f^{(n+m)}}{f^{(n)}}.$$

□

Proposition 2.9 (Affine covariance). *Let* $a, c \in \mathbb{C} \setminus \{0\}$ *and* $b \in \mathbb{C}$*, and define* $F(w) := cf(aw+b)$*. Then*

$$q_n^F(w) = a q_n^f(aw+b), R_n^F(w) = R_n^f(aw+b), K_n^F(w) = a K_n^f(aw+b).$$

In particular, resonance quotients are invariant under affine reparametrization of the independent variable.

Proof. Since $F^{(n)}(w) = ca^n f^{(n)}(aw+b)$, we obtain

$$q_n^F(w) = \frac{ca^{n+1} f^{(n+1)}(aw+b)}{ca^n f^{(n)}(aw+b)} = a q_n^f(aw+b),$$

and hence

$$R_n^F(w) = \frac{q_{n+1}^F(w)}{q_n^F(w)} = R_n^f(aw+b).$$

Applying Δ^2 to the identity for q_n^F gives the formula for K_n^F. □

Proposition 2.10 (Local differential calculus of a rung). *Define differential polynomials recursively by*

$$P_0[q] = 1, P_{m+1}[q] = P'_m[q] + q P_m[q].$$

Then for every $n, m \geq 0$*,*

doi.org/10.32388/VQBNBD

$$\frac{f^{(n+m)}}{f^{(n)}} = P_m[q_n] \tag{2.8}$$

as a meromorphic identity on U. In particular,

$$P_1[q] = q, P_2[q] = q' + q^2, P_3[q] = q'' + 3qq' + q^3.$$

Proof. Fix n and write $g = f^{(n)}$. Then $q_n = g'/g$. For $m = 0$, $g/g = 1 = P_0[q_n]$. If $g^{(m)}/g = P_m[q_n]$, then

$$\frac{g^{(m+1)}}{g} = \left(\frac{g^{(m)}}{g}\right)' + \frac{g'}{g} \cdot \frac{g^{(m)}}{g} = P'_m[q_n] + q_n P_m[q_n] = P_{m+1}[q_n].$$

This proves (2.8) by induction. □

Proposition 2.11 (ODE closure of a rung). *Let g be a nonzero meromorphic function on U, and let $q = g'/g$. Suppose that g satisfies a linear differential equation*

$$g^{(r)} + a_{r-1}g^{(r-1)} + \cdots + a_1 g' + a_0 g = 0 \tag{2.9}$$

with meromorphic coefficients $a_0, \ldots, a_{r-1}$ on U. Then q satisfies the nonlinear differential equation

$$P_r[q] + a_{r-1}P_{r-1}[q] + \cdots + a_1 P_1[q] + a_0 = 0. \tag{2.10}$$

In particular, if a derivative $f^{(N)}$ satisfies (2.9), then the ladder rung q_N satisfies (2.10).

Proof. Divide (2.9) by g and invoke Proposition 2.10. □

Remark 2.12 (Spectral closure). If $f^{(N+r)} = \kappa f^{(N)}$ for some constant $\kappa \in \mathbb{C}$, then Proposition 2.11 yields the scalar closure law

$$P_r[q_N] = \kappa.$$

The periodic ladders classified in Section 7 are exactly the global order-axis manifestation of this closure when $N = 0$ and the same spectral relation recurs along the whole ladder.

3. Inverse realization of compatible ladders

The universal ladder equation is not only necessary. Up to the realization of the bottom rung as a logarithmic derivative, it is also sufficient.

Theorem 3.1 (Inverse realization theorem). *Let $V \subset \mathbb{C}$ be simply connected, and let $(q_n)_{n\geq 0}$ be meromorphic functions on V satisfying*

$$q'_n = q_n(q_{n+1} - q_n), n \geq 0. \tag{3.1}$$

Assume that there exists a nonzero meromorphic function g_0 on V such that

$$\frac{g_0'}{g_0} = q_0. \tag{3.2}$$

Define recursively

$$g_{n+1} := q_n g_n, n \geq 0.$$

Then

$$g_n' = g_{n+1} \qquad (n \geq 0),$$

so $g_n = g_0^{(n)}$ *for all* n. *In particular, with* $f := g_0$ *we have*

$$q_n = \frac{f^{(n+1)}}{f^{(n)}} \qquad (n \geq 0).$$

Proof. Equation (3.2) gives

$$g_0' = q_0 g_0 = g_1.$$

Assume inductively that $g_n' = g_{n+1}$. Then

$$g_{n+1}' = (q_n g_n)' = q_n' g_n + q_n g_n'.$$

Using (3.1) and the induction hypothesis,

$$g_{n+1}' = q_n(q_{n+1} - q_n) g_n + q_n g_{n+1}.$$

Since $g_{n+1} = q_n g_n$, the right-hand side simplifies to

$$q_n q_{n+1} g_n = q_{n+1} g_{n+1} = g_{n+2}.$$

Thus $g_{n+1}' = g_{n+2}$, completing the induction. □

Corollary 3.2 (Holomorphic realization). *Let* $V \subset \mathbb{C}$ *be simply connected, and let* $(q_n)_{n \geq 0}$ *be holomorphic functions on* V *satisfying (3.1). Then there exists a zero-free holomorphic function* f *on* V *such that*

$$q_n = \frac{f^{(n+1)}}{f^{(n)}} \qquad (n \geq 0).$$

Moreover, f *is unique up to multiplication by a nonzero constant.*

Proof. Since q_0 is holomorphic on simply connected V, there is a holomorphic primitive Q with $Q' = q_0$. Set $f = e^Q$. Then $f'/f = q_0$, so Theorem 3.1 applies. Uniqueness up to multiplication by a nonzero constant follows from equality of logarithmic derivatives. □

Remark 3.3 (Meromorphic integrability of the bottom rung). In the meromorphic setting, the existence of g_0 in Theorem 3.1 is not automatic. On a simply connected domain, the natural condition is that q_0 itself be a logarithmic derivative of a meromorphic function; equivalently, the residues of q_0 at its poles must all be integers. Thus the compatibility equations control the ladder above the bottom rung, while residue arithmetic at the bottom rung controls global meromorphic realizability.

4. Local singular transport

The operator $T(q) = q + q'/q$ has a transparent local effect on poles and zeros. This is the local transport mechanism behind derivative ladders.

Proposition 4.1 (Transport under the ladder operator). *Let q be meromorphic near $a \in \mathbb{C}$.*

i. *If q has a simple pole at a with residue $\rho \neq 0$, then $T(q)$ also has a simple pole at a and* $Res(T(q), a) = \rho - 1.$

1. *If q has a zero of order $m \geq 1$ at a, then $T(q)$ has a simple pole at a with residue m.*

Proof. If $q(z) = \rho(z-a)^{-1} + h(z)$ with h holomorphic, then

$$q'(z) = -\rho(z-a)^{-2} + O(1), \frac{q'(z)}{q(z)} = -\frac{1}{z-a} + O(1),$$

so

$$T(q)(z) = q(z) + \frac{q'(z)}{q(z)} = \frac{\rho - 1}{z-a} + O(1).$$

This proves (i).

If $q(z) = (z-a)^m u(z)$ with $u(a) \neq 0$, then

$$\frac{q'(z)}{q(z)} = \frac{m}{z-a} + \frac{u'(z)}{u(z)},$$

whence

$$T(q)(z) = q(z) + \frac{q'(z)}{q(z)} = \frac{m}{z-a} + O(1).$$

This proves (ii). □

For actual ladders, poles of q_n record the local order of $f^{(n)}$.

Theorem 4.2 (Exact local order transport). *Let $f \in \mathcal{M}(U)$ be nonpolynomial, fix $N \geq 0$, and let $a \in U$.*

 doi.org/10.32388/VQBNBD

a. *Suppose that* $ord_a f^{(N)} = m \geq 1$. *Then for every* $0 \leq j \leq m-1$,

$ord_a f^{(N+j)} = m - j, q_{N+j}(z) = \frac{m-j}{z-a} + O(1), R_{N+j}(a) = \frac{m-j-1}{m-j}$.

In particular, q_{N+m} *is holomorphic at* a.

b. *Suppose that* $ord_a f^{(N)} = -p \leq -1$. *Then for every* $j \geq 0$,

$ord_a f^{(N+j)} = -(p+j), q_{N+j}(z) = -\frac{p+j}{z-a} + O(1), R_{N+j}(a) = \frac{p+j+1}{p+j}$.

Proof. For (a), write

$$f^{(N)}(z) = (z-a)^m u(z), u(a) \neq 0,$$

with u holomorphic. Differentiating shows inductively that for $0 \leq j \leq m$,

$$f^{(N+j)}(z) = (z-a)^{m-j} u_j(z), u_j(a) \neq 0,$$

so $ord_a f^{(N+j)} = m - j$. Therefore

$$q_{N+j}(z) = \frac{f^{(N+j+1)}(z)}{f^{(N+j)}(z)} = \frac{m-j}{z-a} + O(1),$$

and then

$$R_{N+j}(a) = \lim_{z \to a} \frac{q_{N+j+1}(z)}{q_{N+j}(z)} = \frac{m-j-1}{m-j}.$$

When $j = m$, the order is 0, so q_{N+m} is holomorphic at a.

For (b), write

$$f^{(N)}(z) = (z-a)^{-p} u(z), u(a) \neq 0,$$

with u holomorphic. Differentiating shows inductively that for every $j \geq 0$,

$$f^{(N+j)}(z) = (z-a)^{-(p+j)} v_j(z), v_j(a) \neq 0,$$

so $ord_a f^{(N+j)} = -(p+j)$. Hence

$$q_{N+j}(z) = \frac{f^{(N+j+1)}(z)}{f^{(N+j)}(z)} = -\frac{p+j}{z-a} + O(1),$$

and consequently

$$R_{N+j}(a) = \frac{p+j+1}{p+j}.$$

□

doi.org/10.32388/VQBNBD

Corollary 4.3 (Residue interpretation). *Every pole residue of every rung q_n is an integer. More precisely, if $a \in U$ is a zero or pole of $f^{(n)}$, then*

$$Res(q_n, a) = ord_a f^{(n)}.$$

Proof. This is immediate from $q_n = (f^{(n)})'/f^{(n)}$. □

Remark 4.4. Theorem 4.2 shows that the order data of successive derivatives move along the ladder at unit speed. A zero of multiplicity m creates a pole staircase with residues $m, m-1, \ldots, 1$ before regularization. A pole of order p creates an infinite pole staircase with residues $-p, -(p+1), -(p+2), \ldots$.

5. Rigidity and curvature defect at one rung

We now turn to rigidity at one rung.

Theorem 5.1 (Single-rung rigidity). *Let $V \subset \mathbb{C}$ be simply connected, let $g \in \mathcal{M}(V)$ be nonconstant, and suppose that*

$$\frac{gg''}{(g')^2} \equiv c \tag{5.1}$$

on V for some constant $c \in \mathbb{C}$. Then exactly one of the following holds.

i. *If $c = 1$, then*

   $g(z) = Ae^{\lambda z}$

   for some constants $A \neq 0$ and $\lambda \in \mathbb{C}$.

ii. *If $c \neq 1$, then on every simply connected component of $V \setminus \{z_0\}$, $g(z) = A(\alpha z + \beta)^{1/(1-c)}$*

   for suitable constants $A \neq 0$ and $(\alpha, \beta) \neq (0, 0)$, after choosing a branch of the power.

Proof. Set $q = g'/g$. Then (5.1) is exactly $q_1/q_0 = c$ for the ladder of g, hence by Theorem 2.2,

$$q' = q(cq - q) = (c-1)q^2.$$

If $c = 1$, then $q' = 0$, so $q = \lambda$ is constant and $g'/g = \lambda$. Hence $g = Ae^{\lambda z}$.

If $c \neq 1$, then

$$\left(\frac{1}{q}\right)' = 1 - c,$$

so locally

doi.org/10.32388/VQBNBD

$$q(z) = \frac{1}{a - (c-1)z}$$

for a constant $a \in \mathbb{C}$. Integrating $g'/g = q$ gives

$$g(z) = A(a - (c-1)z)^{1/(1-c)},$$

which is the stated affine-power form after renaming the affine factor. □

Corollary 5.2 (Order-flatness at one rung). *Fix $N \geq 0$. On a simply connected domain $V \subset U$, the following are equivalent:*

i. $K_N \equiv 0$ *on* V.

1. R_N *is constant on* V.
2. $f^{(N)}$ *has one of the two forms described in Theorem 5.1.*

Proof. The equivalence of (i) and (ii) is Corollary 2.6. If (ii) holds, apply Theorem 5.1 to $g = f^{(N)}$. Conversely, if $f^{(N)}$ has one of those forms, a direct computation shows that R_N is constant. □

Proposition 5.3 (Second-order ODE interface). *Let $V \subset U$ be a domain on which f and f' do not vanish, and let $a, b \in \mathcal{M}(V)$. Then the following are equivalent on V:*

f satisfies

$$f'' = af' + bf.$$

The bottom rung $q_0 = f'/f$ satisfies the Riccati equation

$$q_0' + q_0^2 = aq_0 + b.$$

The next rung is given by

$$q_1 = a + \frac{b}{q_0}.$$

Proof. Dividing $f'' = af' + bf$ by f gives

$$\frac{f''}{f} = a\frac{f'}{f} + b.$$

Since $f''/f = q_0' + q_0^2$, this is exactly (ii). Dividing instead by f' gives

$$q_1 = \frac{f''}{f'} = a + b\frac{f}{f'} = a + \frac{b}{q_0},$$

doi.org/10.32388/VQBNBD

which is (iii). Conversely, either (ii) or (iii) recovers $f'' = af' + bf$ after multiplication by the appropriate denominator. □

Proposition 5.4 (Exact reciprocal linearization). *Fix $N \geq 0$ and let $V \subset U$ be a domain on which $f^{(N)}$ and $f^{(N+1)}$ do not vanish. If $R_N \equiv c$ is constant on V, then*

$$\left(\frac{1}{q_N}\right)' = 1 - c,$$

so

$$\frac{1}{q_N(z)} = (1-c)z + \beta$$

for some constant $\beta \in \mathbb{C}$. Consequently q_N is constant when $c = 1$, and reciprocal-affine when $c \neq 1$.

Proof. Since $q_N' = (R_N - 1)q_N^2$, we have

$$\left(\frac{1}{q_N}\right)' = -\frac{q_N'}{q_N^2} = 1 - R_N = 1 - c.$$

Integrating on each component of V gives the stated affine formula for $1/q_N$. □

Theorem 5.5 (Quantitative near-flatness and local linearization). *Fix $N \geq 0$, let $V \subset U$ be convex, and assume that*

$$f^{(N)} f^{(N+1)} f^{(N+2)} \neq 0$$

on V. Fix $z_0 \in V$ and set $c := R_N(z_0)$. If

$$\sup_{z \in V} |K_N(z)| \leq \varepsilon,$$

then for a suitable holomorphic branch of $\log R_N$ *on V one has*

$$\left|\log \frac{R_N(z)}{c}\right| \leq \varepsilon |z - z_0| \qquad (z \in V). \tag{5.2}$$

Consequently,

$$|R_N(z) - c| \leq |c| \left(e^{\varepsilon|z-z_0|} - 1\right) \qquad (z \in V), \tag{5.3}$$

and

$$\left|\frac{1}{q_N(z)} - \left(\frac{1}{q_N(z_0)} + (1-c)(z - z_0)\right)\right| \leq |z - z_0||c| \left(e^{\varepsilon|z-z_0|} - 1\right) \qquad (z \in V). \tag{5.4}$$

 doi.org/10.32388/VQBNBD

In particular, small order curvature forces the reciprocal rung $1/q_N$ to be close to an affine map, hence forces q_N to be close to a local exponential model when $c \approx 1$ and to a local affine-power model when $c \not\approx 1$.

Proof. Because $f^{(N)} f^{(N+1)} f^{(N+2)} \neq 0$ on V, the functions q_N, q_{N+1}, and $R_N = q_{N+1}/q_N$ are holomorphic and zero-free on V. Since V is convex, it is simply connected, so a holomorphic branch of $\log R_N$ exists on V. Using Proposition 2.5,

$$(\log R_N)' = \frac{R_N'}{R_N} = K_N.$$

Integrating along the line segment from z_0 to z yields

$$\log \frac{R_N(z)}{c} = \int_{z_0}^{z} K_N(\zeta) d\zeta,$$

which implies (5.2). Estimate (5.3) follows immediately.

Next,

$$\left(\frac{1}{q_N}\right)' = 1 - R_N$$

by the proof of Proposition 5.4. Therefore,

$$\frac{1}{q_N(z)} - \frac{1}{q_N(z_0)} = \int_{z_0}^{z} (1 - R_N(\zeta)) d\zeta.$$

Subtracting $(1-c)(z-z_0)$ and estimating along the segment gives

$$\left| \frac{1}{q_N(z)} - \left(\frac{1}{q_N(z_0)} + (1-c)(z-z_0) \right) \right| \leq |z - z_0| \sup_{\zeta \in [z_0, z]} |R_N(\zeta) - c|,$$

which together with (5.3) yields (5.4). □

Remark 5.6. Theorem 5.5 clarifies the geometric role of K_N. Exact vanishing $K_N \equiv 0$ is equivalent to exact rigidity by Corollary 5.2; small K_N gives quantitative proximity to the same rigid models. In this precise sense, order curvature is a defect invariant measuring the failure of rung N to lie in the exponential or affine-power regime.

6. Finite-order collapse and global rigidity

We now study the derivative ladder as a function of the order variable n. In this section, “finite order” refers to finite difference order in n, not to the classical growth order of an entire or meromorphic function.

doi.org/10.32388/VQBNBD

Definition 6.1. The ladder $(q_n)_{n\geq 0}$ is *order-polynomial of discrete degree at most d* if

$$\Delta^{d+1} q_n \equiv 0 \qquad (n \geq 0).$$

Equivalently, for each fixed z, the map $n \mapsto q_n(z)$ is a polynomial in n of degree at most d.

Theorem 6.2 (Finite-order collapse on the derivative-order axis). *Assume that there exists $r \geq 1$ such that*

$$\Delta^r q_n \equiv 0 \qquad (n \geq 0). \tag{6.1}$$

Then in fact $\Delta^2 q_n \equiv 0$ for every n, and exactly one of the following alternatives holds.

i. ***Exponential case.*** *There exists $\lambda \in \mathbb{C}$ such that*

$q_n \equiv \lambda \qquad (n \geq 0),$

and therefore $f(z) = Ce^{\lambda z}$ for some $C \neq 0$.

ii. ***Affine-power case.*** *There exist $A \in \mathbb{C} \setminus \mathbf{Z}_{\geq 0}$ and $z_0 \in \mathbb{C}$ such that*

$q_n(z) = \frac{A-n}{z-z_0}, R_n(z) = \frac{A-n-1}{A-n}, K_n \equiv 0,$

and on every simply connected component of $U \setminus \{z_0\}$,

$f(z) = C(z - z_0)^A$

after choosing a branch.

If one allows truncated ladders, the same affine-power formula with $A \in \mathbf{Z}_{\geq 0}$ yields the polynomial case up to the terminal nonzero derivative.

Proof. Because $\Delta^r q_n \equiv 0$, the Newton expansion for forward differences gives

$$q_n = \sum_{k=0}^{r-1} \binom{n}{k} \Delta^k q_0.$$

Hence q_n is a polynomial in n of degree at most $r - 1$, with meromorphic coefficients. Write

$$q_n = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_0,$$

where $d \geq 0$ is maximal and $a_d \not\equiv 0$.

Suppose first that $d \geq 2$. Then

$$\Delta q_n = d a_d n^{d-1} + O(n^{d-2}),$$

so

$$q_n \Delta q_n = d a_d^2 n^{2d-1} + O(n^{2d-2}).$$

 doi.org/10.32388/VQBNBD

On the other hand,

$$q_n' = a_d' n^d + O(n^{d-1}).$$

This contradicts the universal identity $q_n' = q_n \Delta q_n$ when $d \geq 2$. Therefore $d \leq 1$.

If $d = 0$, then $q_n = q_0$ for every n, so (2.1) gives $q_0' = 0$. Thus $q_0 = \lambda$ is constant and $f'/f = \lambda$, yielding $f(z) = Ce^{\lambda z}$.

Assume now that $d = 1$. Write

$$q_n = an + b$$

with meromorphic functions a, b on U. Since $\Delta q_n = a$, the universal equation becomes

$$a'n + b' = a(an + b).$$

Comparing coefficients of n gives

$$a' = a^2, b' = ab. \tag{6.2}$$

If $a \equiv 0$, we are back in the exponential case. Otherwise, as a meromorphic identity,

$$\left(\frac{1}{a}\right)' = -\frac{a'}{a^2} = -1.$$

Hence $1/a = -z + z_0$ for some constant $z_0 \in \mathbb{C}$, and therefore

$$a(z) = -\frac{1}{z - z_0}.$$

Also,

$$\left(\frac{b}{a}\right)' = \frac{b'a - ba'}{a^2} = 0,$$

so b/a is constant. Writing $b/a = -A$ yields

$$b(z) = \frac{A}{z - z_0}, q_n(z) = \frac{A - n}{z - z_0}.$$

Integrating $q_0 = f'/f = A/(z - z_0)$ on simply connected components of $U \setminus \{z_0\}$ gives

$$f(z) = C(z - z_0)^A.$$

Since f is assumed nonpolynomial, $A \notin \mathbf{Z}_{\geq 0}$. Finally, an affine function of n has vanishing second forward difference, so $K_n = \Delta^2 q_n \equiv 0$ for all n. □

doi.org/10.32388/VQBNBD

Corollary 6.3 (Equivalent forms of full rigidity). *For a nonpolynomial meromorphic function f, the following are equivalent:*

i. $\Delta^r q_n \equiv 0$ *for some* $r \geq 1$ *and every* $n \geq 0$.

ii. $\Delta^2 q_n \equiv 0$ *for every* $n \geq 0$.

iii. q_n *is affine in the order variable:*

$q_n = a(z)n + b(z)$

for meromorphic functions a, b.

1. *Every resonance quotient* R_n *is independent of* z.
2. *Either*

   $q_n \equiv \lambda, R_n \equiv 1, f(z) = Ce^{\lambda z}$,

   or

   $q_n(z) = \frac{A-n}{z-z_0}, R_n(z) = \frac{A-n-1}{A-n}, f(z) = C(z-z_0)^A$

   on simply connected components of $U \setminus \{z_0\}$.

Proof. The implication (i)⇒(v) is Theorem 6.2. Trivially (ii)⇒(i). The equivalence (ii)⇔(iii) is the elementary characterization of affine sequences by vanishing second difference. By Proposition 2.5, (ii) is equivalent to $R_n'/R_n \equiv 0$ for every n, which is exactly (iv). Finally, the formulas in (v) visibly imply (ii). □

Corollary 6.4 (Eventual finite-order collapse). *Fix* $N \geq 0$. *If there exists* $r \geq 1$ *such that*

$$\Delta^r q_n \equiv 0 \qquad (n \geq N),$$

then $f^{(N)}$ *is locally exponential or affine-power. Equivalently, on the tail starting at* N, *the ladder collapses to one of the two rigid models of Theorem 6.2.*

Proof. Apply Theorem 6.2 to the function $g = f^{(N)}$, whose derivative ladder is $(q_{N+n})_{n\geq 0}$. □

Corollary 6.5 (Entire rigidity under finite-order collapse). *Let* f *be a nonpolynomial entire function. If* $\Delta^r q_n \equiv 0$ *for some* $r \geq 1$, *then*

$$f(z) = Ce^{\lambda z}$$

for suitable constants $C \neq 0$ *and* $\lambda \in \mathbb{C}$.

Proof. By Theorem 6.2, only the exponential or affine-power cases are possible. The affine-power model $C(z-z_0)^A$ is entire only when $A \in \mathbf{Z}_{\geq 0}$, in which case it is a polynomial, contrary to the hypothesis. Hence only the exponential case remains. □

7. Periodic ladders and spectral models

Constant resonance is one rigidity mechanism. Periodicity in the order variable is another, and it admits a complete classification.

Theorem 7.1 (Periodic ladder classification). *Let $f \in \mathcal{M}(U)$ be nonpolynomial and fix $p \in \mathbb{N}$. Then the following are equivalent:*

i. $q_{n+p} = q_n$ *for every* $n \geq 0$.

ii. $q_p = q_0$.

iii. *There exists a constant* $\kappa \in \mathbb{C}$ *such that*

$$f^{(p)} = \kappa f \tag{7.1}$$

on U.

In this situation,

$$\kappa = \prod_{j=0}^{p-1} q_j,$$

and the product is constant.

Proof. The implication (i)⇒(ii) is immediate. For (ii)⇒(i), use $q_{n+1} = T(q_n)$ to obtain

$$q_{n+p} = T^n(q_p) = T^n(q_0) = q_n.$$

Thus (i) and (ii) are equivalent.

Assume (ii) and set

$$K := \prod_{j=0}^{p-1} q_j.$$

By Proposition 2.8,

$$K = \frac{f^{(p)}}{f}.$$

Moreover,

$$\frac{K'}{K} = \sum_{j=0}^{p-1} \frac{q_j'}{q_j} = \sum_{j=0}^{p-1} (q_{j+1} - q_j) = q_p - q_0 = 0,$$

using Proposition 2.5. Hence K is constant, say $K = \kappa$, and so $f^{(p)} = \kappa f$. This proves (ii)⇒(iii).

doi.org/10.32388/VQBNBD

Finally, if (iii) holds, then $f^{(n+p)} = \kappa f^{(n)}$ for every $n \geq 0$, so

$$q_{n+p} = \frac{f^{(n+p+1)}}{f^{(n+p)}} = \frac{\kappa f^{(n+1)}}{\kappa f^{(n)}} = q_n.$$

Thus (iii)⇒(i). □

Corollary 7.2 (Eventual periodicity). *Fix $N \geq 0$ and $p \geq 1$. Then the following are equivalent:*

i. $q_{n+p} = q_n$ *for every* $n \geq N$.

ii. *There exists* $\kappa \in \mathbb{C}$ *such that*

$$f^{(N+p)} = \kappa f^{(N)}.$$

Proof. Apply Theorem 7.1 to $g = f^{(N)}$. □

Corollary 7.3 (Period two and a Riccati equation). *For a nonpolynomial meromorphic function f, the condition $q_2 = q_0$ is equivalent to the existence of a constant $\kappa \in \mathbb{C}$ such that*

$$f'' = \kappa f.$$

Equivalently,

$$q_0' + q_0^2 = \kappa. \tag{7.2}$$

In this case,

$$q_1 = \frac{\kappa}{q_0}, q_{2m} = q_0, q_{2m+1} = q_1.$$

Moreover, if $s^2 = \kappa$, then every local nonconstant period-two bottom rung is of the form

$$q_0(z) = s\,\frac{Ae^{sz} - Be^{-sz}}{Ae^{sz} + Be^{-sz}}$$

for constants $A, B \in \mathbb{C}$, not both zero, on simply connected regions avoiding the zeros of the denominator.

Proof. The equivalence $q_2 = q_0 \Leftrightarrow f'' = \kappa f$ is the case $p = 2$ of Theorem 7.1. Since

$$q_0' + q_0^2 = \frac{f''}{f},$$

we obtain (7.2). Also,

$$q_0 q_1 = \frac{f''}{f} = \kappa,$$

so $q_1 = \kappa / q_0$, and the parity relations follow from period two.

doi.org/10.32388/VQBNBD

If $f'' = \kappa f$ and $s^2 = \kappa$, then locally

$$f(z) = Ae^{sz} + Be^{-sz},$$

whence

$$q_0 = \frac{f'}{f} = s\,\frac{Ae^{sz} - Be^{-sz}}{Ae^{sz} + Be^{-sz}}.$$

Conversely, any such q_0 is the logarithmic derivative of a local solution of $f'' = \kappa f$ and therefore generates a period-two ladder. □

Corollary 7.4 (Spectral representation of periodic ladders). *Assume that U is simply connected and that $q_{n+p} = q_n$ for every $n \geq 0$. Then there exists $\kappa \in \mathbb{C}$ such that*

$$f^{(p)} = \kappa f,$$

and f is the restriction to U of an entire exponential polynomial of the form

$$F(z) = \sum_{\omega^p = \kappa} c_\omega e^{\omega z},$$

where the sum ranges over the distinct pth roots of κ and the coefficients $c_\omega \in \mathbb{C}$ are constants.

Proof. By Theorem 7.1, periodicity of the ladder is equivalent to $f^{(p)} = \kappa f$ for some constant κ. The corresponding constant-coefficient ODE has characteristic equation $x^p - \kappa = 0$ with distinct roots ω satisfying $\omega^p = \kappa$, so every local solution is the restriction of a global entire function of the displayed form. □

8. Asymptotic fingerprints and spectral dominance

Sections 5–7 classify ladders with exact curvature collapse or exact closure in the order variable. The present section moves beyond that exact world and studies *asymptotic order geometries*. A ladder may fail to be order-flat and fail to be periodic, yet still develop a rigid large-order profile. In dominant spectral families the first three ladder scales—$q_n - \lambda$, Δq_n, and K_n—encode not only the dominant spectral root but also the degree of its polynomial amplitude. Thus asymptotic order geometry reads external spectral data from internal curvature.

The cleanest source models are exponential polynomials.

Definition 8.1. An *exponential polynomial* is an entire function of the form

$$F(z) = \sum_{j=1}^{m} P_j(z) e^{\lambda_j z},$$

where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ are distinct and the P_j are polynomials, not all zero.

Lemma 8.2 (Ratio asymptotics for polynomial-type sequences). *Let $K \subset \mathbb{C}$ be compact. Suppose that S_n is a sequence of holomorphic functions on a neighborhood of K satisfying*

$$S_n(z) = \alpha n^d + \beta(z) n^{d-1} + O_K(n^{d-2})$$

uniformly on K, where $d \geq 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, and β is holomorphic near K. Then

$$\frac{S_{n+1}(z)}{S_n(z)} = 1 + \frac{d}{n} + O_K(n^{-2})$$

uniformly on K.

Proof. Uniformly on K one may write

$$S_n(z) = \alpha n^d \left(1 + \frac{\beta(z)}{\alpha n} + O_K(n^{-2})\right),$$

and similarly

$$S_{n+1}(z) = \alpha (n+1)^d \left(1 + \frac{\beta(z)}{\alpha(n+1)} + O_K(n^{-2})\right).$$

Therefore

$$\frac{S_{n+1}(z)}{S_n(z)} = \left(1 + \frac{1}{n}\right)^d \frac{1 + \frac{\beta(z)}{\alpha(n+1)} + O_K(n^{-2})}{1 + \frac{\beta(z)}{\alpha n} + O_K(n^{-2})} = 1 + \frac{d}{n} + O_K(n^{-2}),$$

uniformly on K. □

Theorem 8.3 (Asymptotic order-dimension law for a single spectral root). *Let*

$$f(z) = e^{\lambda z} P(z),$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and P is a nonzero polynomial of degree d. Then on every compact set $K \subset \mathbb{C}$,

$$q_n(z) = \lambda \left(1 + \frac{d}{n} + O_K(n^{-2})\right), \tag{8.1}$$

$$\Delta q_n(z) = -\frac{d\lambda}{n^2} + O_K(n^{-3}), \tag{8.2}$$

$$K_n(z) = \frac{2d\lambda}{n^3} + O_K(n^{-4}). \tag{8.3}$$

Consequently,

$$\lim_{n\to\infty} n\left(\frac{q_n(z)}{\lambda}-1\right) = \lim_{n\to\infty} -\frac{n^2}{\lambda}\Delta q_n(z) = \lim_{n\to\infty} \frac{n^3}{2\lambda}K_n(z) = d \tag{8.4}$$

locally uniformly on $\mathbb{C}$.

Proof. Using $D(e^{\lambda z}h) = e^{\lambda z}(D+\lambda)h$, one obtains

$$f^{(n)}(z) = e^{\lambda z}(D+\lambda)^n P(z) = e^{\lambda z}\sum_{k=0}^{d}\binom{n}{k}\lambda^{n-k}P^{(k)}(z).$$

Set

$$S_n(z) := \lambda^{-n}e^{-\lambda z}f^{(n)}(z) = \sum_{k=0}^{d}\binom{n}{k}\lambda^{-k}P^{(k)}(z).$$

Let a_d be the leading coefficient of P. Since $P^{(d)} = d!a_d$ is constant and

$$\binom{n}{d} = \frac{n^d}{d!} + O(n^{d-1}), \binom{n}{d-1} = \frac{n^{d-1}}{(d-1)!} + O(n^{d-2}),$$

we have, uniformly on K,

$$S_n(z) = a_d\lambda^{-d}n^d + \beta(z)n^{d-1} + O_K(n^{d-2})$$

for a holomorphic function β depending on P and λ. Lemma 8.2 therefore gives

$$\frac{S_{n+1}(z)}{S_n(z)} = 1 + \frac{d}{n} + O_K(n^{-2}).$$

Since

$$q_n(z) = \frac{f^{(n+1)}(z)}{f^{(n)}(z)} = \lambda\frac{S_{n+1}(z)}{S_n(z)},$$

this proves (8.1). Taking first and second forward differences of (8.1) yields (8.2) and (8.3). The limits in (8.4) follow immediately. □

Corollary 8.4 (Dominant spectral component). *Let*

$$f(z) = \sum_{j=1}^{m} P_j(z)e^{\lambda_j z},$$

where the λ_j *are distinct, the* P_j *are nonzero polynomials, and one index* $*$ *satisfies*

$$|\lambda_*| > \max_{j\neq *}|\lambda_j|.$$

Let $d = \deg P_*$. *Then on every compact set* $K \subset \mathbb{C}$,

$$q_n(z) = \lambda_* \left(1 + \frac{d}{n} + O_K(n^{-2})\right), \Delta q_n(z) = -\frac{d\lambda_*}{n^2} + O_K(n^{-3}), K_n(z) = \frac{2d\lambda_*}{n^3} + O_K(n^{-4}),$$

uniformly on K. In particular,

$$q_n \to \lambda_*$$

locally uniformly on $\mathbb{C}$, and the dominant spectral root λ_* together with the degree d of its polynomial prefactor can be recovered from the asymptotic ladder data.

Proof. After relabeling, assume $* = 1$ and write $\lambda = \lambda_1$, $d = \deg P_1$. By the proof of Theorem 8.3,

$$(e^{\lambda z} P_1(z))^{(n)} = \lambda^n e^{\lambda z} \left(\alpha n^d + O_K(n^{d-1})\right)$$

for some $\alpha \neq 0$. For $j \geq 2$, Leibniz' rule gives

$$\left(P_j(z) e^{\lambda_j z}\right)^{(n)} = O_K\left(|\lambda_j|^n n^{\deg P_j}\right).$$

Hence, with

$$\rho := \max_{j \geq 2} \frac{|\lambda_j|}{|\lambda|} < 1,$$

one has

$$f^{(n)}(z) = \lambda^n e^{\lambda z} \left(\alpha n^d + O_K(n^{d-1}) + O_K(\rho^n n^M)\right),$$

where $M := \max_j \deg P_j$. Because the last term is exponentially smaller than n^{d-1}, the ratio asymptotics of Theorem 8.3 remain valid with λ and d replaced by λ_* and d. □

Theorem 8.5 (Abstract asymptotic rigidity of convergent ladders). *Let $V \subset U$ be a domain on which every q_n is holomorphic. If*

$$q_n \to q$$

locally uniformly on V for some holomorphic function q, then q is constant on V.

Proof. By the universal ladder law,

$$q_n' = q_n(q_{n+1} - q_n) = q_n \Delta q_n.$$

Because $q_n \to q$ locally uniformly, we also have $q_{n+1} \to q$ locally uniformly, hence $\Delta q_n = q_{n+1} - q_n \to 0$ locally uniformly. Therefore $q_n' \to 0$ locally uniformly on compact subsets of V.

But locally uniform convergence of holomorphic functions implies locally uniform convergence of derivatives, so $q_n' \to q'$ locally uniformly on compact subsets as well. Hence $q' \equiv 0$, and q is constant. □

Corollary 8.6 (Normal-family consequence). *Let $V \subset U$ be a domain on which every q_n is holomorphic. Assume that (q_n) is a normal family on V and that*

$$\Delta q_n \to 0$$

locally uniformly on V. Then every subsequential locally uniform limit of (q_n) is constant.

Proof. If $q_{n_j} \to q$ locally uniformly on compact subsets, then $q_{n_j+1} = q_{n_j} + \Delta q_{n_j} \to q$ locally uniformly as well. Apply Theorem 8.5 to the subsequence (q_{n_j}). □

Remark 8.7 (Asymptotic curvature as a spectral fingerprint).

In the rigid models of Theorem 6.2, one has $K_n \equiv 0$ identically. In the asymptotically rigid exponential-polynomial models of Theorem 8.3, the scaled curvature converges to a nonzero constant exactly when the dominant spectral packet carries a nontrivial polynomial prefactor. Thus K_n is both an exact defect invariant at finite level and an asymptotic fingerprint of hidden spectral multiplicity.

Remark 8.8 (Need for strict spectral dominance).

If several spectral roots share maximal modulus, the ladder need not converge; periodic packets are the clearest examples. The strict-dominance hypothesis in Corollary 8.4 isolates the regime in which asymptotic ladder data reduce to a single dominant root and a single integer d.

9. Examples and source models

We record several model ladders illustrating the theory and, in particular, the asymptotic layer.

Example 9.1 (The two rigid archetypes). If $f(z) = Ce^{\lambda z}$, then

$$q_n(z) = \lambda, R_n(z) = 1, K_n \equiv 0$$

for every $n \geq 0$. If

$$f(z) = C(z - z_0)^A$$

with a branch chosen on a simply connected component of $U \setminus \{z_0\}$, then

$$q_n(z) = \frac{A-n}{z-z_0}, R_n(z) = \frac{A-n-1}{A-n}, K_n \equiv 0.$$

These are the only nonpolynomial ladders of finite difference order on the derivative-order axis.

 doi.org/10.32388/VQBNBD

Example 9.2 (A quadratic order ansatz is impossible). Suppose one tries to impose an order-quadratic form

$$q_n(z) = a(z)n^2 + b(z)n + c(z).$$

Then q_n' has order n^2 in the derivative-order variable, while $q_n \Delta q_n$ has order n^3 unless $a \equiv 0$. Thus the universal ladder law immediately rules out every genuinely quadratic order geometry. This is the finite-order collapse theorem in its most visible form: higher-order polynomial ladders are forbidden.

Example 9.3 (Exact near-flatness: a one-dimensional spectral defect). Let

$$f(z) = e^{\lambda z}(z-a), \lambda \neq 0.$$

Then

$$f^{(n)}(z) = e^{\lambda z} \lambda^{n-1} \left(\lambda(z-a) + n\right),$$

so

$$q_n(z) = \lambda \frac{\lambda(z-a)+n+1}{\lambda(z-a)+n} = \lambda + \frac{\lambda}{\lambda(z-a)+n}.$$

Hence the ladder is not order-flat, but it is asymptotically flat:

$$\Delta q_n(z) = -\frac{\lambda}{(\lambda(z-a)+n)(\lambda(z-a)+n+1)},$$

$$K_n(z) = \frac{2\lambda}{(\lambda(z-a)+n)(\lambda(z-a)+n+1)(\lambda(z-a)+n+2)}.$$

Thus

$$q_n = \lambda + \frac{\lambda}{n} + O(n^{-2}), \Delta q_n = -\frac{\lambda}{n^2} + O(n^{-3}), K_n = \frac{2\lambda}{n^3} + O(n^{-4}),$$

in agreement with Theorem 8.3 for the case $d = 1$.

Example 9.4 (Dominant spectral packet). Let

$$f(z) = e^{2z} + e^{-z}.$$

Then

$$f^{(n)}(z) = 2^n e^{2z} + (-1)^n e^{-z},$$

and therefore

doi.org/10.32388/VQBNBD

$$q_n(z) = \frac{2^{n+1}e^{2z} + (-1)^{n+1}e^{-z}}{2^n e^{2z} + (-1)^n e^{-z}}.$$

Since $|2| > |-1|$, the spectral root 2 dominates, and Corollary 8.4 yields

$$q_n(z) \to 2, \Delta q_n(z) \to 0, K_n(z) \to 0$$

locally uniformly on $\mathbb{C}$. Here the dominant polynomial degree is $d = 0$, so the ladder converges all the way to the exact exponential regime.

Example 9.5 (A nonflat packet with polynomial amplitude and secondary spectrum). Let

$$f(z) = e^z + ze^{2z}.$$

Then for every $n \geq 0$,

$$f^{(n)}(z) = e^z + 2^{n-1}e^{2z}(2z + n),$$

and hence

$$q_n(z) = \frac{e^z + 2^n e^{2z}(2z + n + 1)}{e^z + 2^{n-1}e^{2z}(2z + n)}.$$

This ladder is neither order-flat nor periodic. For instance,

$$q_n(0) = \frac{1 + (n+1)2^n}{1 + n2^{n-1}},$$

which is not affine in n, so exact collapse is impossible. Nevertheless the dominant spectral root is 2, and its dominant polynomial prefactor has degree 1. Therefore Corollary 8.4 gives, uniformly on compact sets,

$$q_n(z) = 2\left(1 + \frac{1}{n} + O(n^{-2})\right), \Delta q_n(z) = -\frac{2}{n^2} + O(n^{-3}), K_n(z) = \frac{4}{n^3} + O(n^{-4}).$$

Thus the asymptotic layer still detects the hidden one-dimensional polynomial amplitude attached to the dominant packet despite the presence of a secondary exponential component.

Example 9.6 (Periodic spectral packets). Fix $p \geq 1$, let $\kappa \in \mathbb{C}$, and choose coefficients $c_\omega \in \mathbb{C}$ for the roots ω of $x^p - \kappa = 0$. Then

$$f(z) = \sum_{\omega^p = \kappa} c_\omega e^{\omega z}$$

satisfies $f^{(p)} = \kappa f$, so its ladder has period dividing p. The case $p = 2$ gives the hyperbolic and trigonometric examples

doi.org/10.32388/VQBNBD

$$f(z) = \cosh(sz), q_0 = s\tanh(sz), q_1 = s\coth(sz), q_2 = q_0,$$

and

$$f(z) = \cos(sz), q_0 = -s\tan(sz), q_1 = s\cot(sz), q_2 = q_0.$$

These ladders are periodic but not order-flat.

Example 9.7 (Second-order closure: Airy and Bessel fields). If f solves

$$f'' = A(z)f,$$

then Proposition 5.3 gives the Riccati closure

$$q_0' + q_0^2 = A(z), q_1 = \frac{A(z)}{q_0}, R_0 = \frac{A(z)}{q_0^2}.$$

For the Airy equation $f'' = zf$, this becomes

$$q_0' + q_0^2 = z, q_1 = \frac{z}{q_0}.$$

If f solves the Bessel equation

$$z^2 f'' + zf' + (z^2 - \nu^2)f = 0,$$

then on simply connected regions avoiding $z = 0$ and the zeros of f one has

$$q_0' + q_0^2 + \frac{1}{z}q_0 + 1 - \frac{\nu^2}{z^2} = 0, q_1 = -\frac{1}{z} - \frac{1 - \nu^2/z^2}{q_0}.$$

These are not rigid models, but they show how classical special functions inject structured coefficient fields into the ladder geometry.

10. Positioning, novelty, and outlook

What the paper establishes

The derivative ladder sits next to several established traditions, but the contribution of this paper is not a relabeling of those traditions. The core contribution is the emergence of a new coupled object and a theorem package around it.

First, the primary object is the full adjacent ladder $q_n = f^{(n+1)}/f^{(n)}$, not a single logarithmic derivative or pre-Schwarzian field. Second, the ladder carries an exact universal flow law

$$q_n' = q_n(q_{n+1} - q_n),$$

which upgrades successive derivative ratios from isolated quotients to a nonlinear differential-difference system on the order axis. Third, the order variable acquires an intrinsic calculus: order slope Δq_n, resonance R_n, and order curvature $K_n = (\log R_n)'$. Fourth, the theory is bidirectional: compatible ladders admit inverse realization once the bottom rung is integrated. Fifth, the curvature field is not merely formal. It drives exact rigidity, quantitative near-rigidity, finite-order collapse, and asymptotic spectral fingerprints. Sixth, periodic ladders are classified exactly by the constant-coefficient closure relation $f^{(p)} = \kappa f$, so derivative-order geometry connects directly to finite spectral packets of exponentials. Seventh, the asymptotic section shows that in dominant exponential-polynomial models the scaled ladder data recover both the dominant spectral root and the degree of its polynomial prefactor.

Taken together, these results produce a coherent theory with exact flow, exact inverse realization, exact singular fingerprints, exact finite-order collapse, exact periodic classification, quantitative near-flatness, and a nontrivial asymptotic fingerprint law.

Relation to neighboring literatures

Several neighboring theories touch individual rungs of the ladder.

The bottom rung $q_0 = f'/f$ belongs to the logarithmic-derivative tradition of Nevanlinna theory and complex differential equations [10]. The next rung $q_1 = f''/f'$ is the pre-Schwarzian derivative of f on zero-free regions of f', and is therefore adjacent to the Schwarzian and univalence literature [11][12]. Higher Schwarzian theories, including the operators of Tamanoi and the invariant higher Schwarzians of Kim and Sugawa, produce differential-polynomial invariants from finitely many derivatives of a single map [13][14]. Classical work on successive derivatives studies zeros, growth, normality, and final sets rather than a coupled ratio flow on the derivative index [1][2][3][4][5][6][7][8]. Grätsch's recent work on lifts of logarithmic derivatives shows that adjacent quotient fields also arise in quasi-normality questions, but in a different structural setting [15].

The present paper is closest to these literatures, but it is not subsumed by them. The contribution here is the derivative-order geometry of the whole adjacent ladder together with the theorem package described above.

doi.org/10.32388/VQBNBD

Scope of the originality claim

The originality claim is precise. It is not that f'/f or f''/f' are new objects, nor that ratios of successive derivatives have never appeared before. It is that the literature appears not to isolate the full adjacent-ratio ladder as an exact differential-difference flow carrying, at once, inverse realization, singular transport laws, finite-order collapse, periodic spectral classification, quantitative near-flatness, and asymptotic order-dimension laws recovering hidden spectral multiplicity. That is the level at which the paper claims novelty.

Outlook

The theory now has two complementary faces. One face is exact and rigid: universal flow, inverse realization, transport, collapse, periodicity. The other face is asymptotic: near-flatness estimates, dominant spectral limits, and curvature fingerprints. The next natural questions are therefore not merely internal bookkeeping questions. They are external analytic problems:

1. develop a sharp stability theory for ladders with small curvature on large domains;
2. understand how Nevanlinna growth of q_n, R_n, and K_n encodes classical data about successive derivatives;
3. determine which broader classes of meromorphic functions exhibit asymptotic order-dimension laws;
4. extend the ladder formalism to directional derivatives in several complex variables.

In this sense, derivative-order geometry is not just an internal reformulation of familiar quotient identities. It is a structural theory of successive derivatives with exact laws, rigidity mechanisms, spectral closure, and a first genuine asymptotic fingerprint calculus.

Notes

2020 Mathematics Subject Classification: 30D35, 30D15, 30C45, 34A05, 39A30.

References

1. [a, b]*Pólya G (1943). "On the Zeros of the Derivatives of a Function and Its Analytic Character." Bull Amer Mat h Soc.* ***49****:178–191.*

2. a, b Rådström H (1949). "On the Zeros of Successive Derivatives of Entire Functions." Proc Natl Acad Sci U S A. **35**(7):399–404.
3. a, b Boas RP Jr, Reddy AR (1973). "Zeros of the Successive Derivatives of Entire Functions." J Math Anal Appl. **42**:466–473.
4. a, b Hellerstein S, Williamson J (1977). "Successive Derivatives of Entire Functions." Proc Amer Math Soc. **66**(1):105–108.
5. a, b Gethner RM (1986). "Accumulation Rate of Zeros of the Successive Derivatives of Meromorphic Functions." J London Math Soc (2). **33**(2):294–302.
6. a, b Bergweiler W, Eremenko A (2006). "Proof of a Conjecture of Pólya on the Zeros of Successive Derivatives of Real Entire Functions." Acta Math. **197**(2):145–166.
7. a, b Meyrath T, Müller J (2013). "On the Behaviour of the Successive Derivatives of Meromorphic Functions on the Final Set." J Anal Math. **120**(1):131–149.
8. a, b MacLane GR (1952). "Sequences of Derivatives and Normal Families." J Analyse Math. **2**:72–87.
9. ^ Crouse JE, Putt CW (1972). "Roots of Polynomials by Ratio of Successive Derivatives." NASA Technical Note D-6793. National Aeronautics and Space Administration.
10. a, b Cherry W, Ye Z (2001). "Logarithmic Derivatives." In: Nevanlinna's Theory of Value Distribution. Berlin: Springer. pp. 89–106.
11. a, b Osgood B (1998). "Old and New on the Schwarzian Derivative." In: Quasiconformal Mappings and Analysis. New York: Springer. pp. 275–308.
12. a, b Stowe D (1998). "Injectivity and the Pre-Schwarzian Derivative." Michigan Math J. **45**(3):537–546.
13. a, b Tamanoi H (1996). "Higher Schwarzian Operators and Combinatorics of the Schwarzian Derivative." Math Ann. **305**(1):127–151.
14. a, b Kim S-A, Sugawa T (2011). "Invariant Schwarzian Derivatives of Higher Order." Complex Anal Oper Theory. **5**(3):659–670.
15. a, b Grätsch M (2025). "Lifts of Logarithmic Derivatives." Math Scand. **131**(3):article a-158582.

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.

 doi.org/10.32388/VQBNBD