Graded quantum noise in quantum field theories

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August 29, 2024

https://doi.org/10.32388/VQHIH7

The aim of this article is to introduce into quantum field theory, $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum stochastic calculus with the aim of generalizing supersymmetric quantum stochastic calculus to situations where rather than just two kinds of particles, Bosons and Fermions, we can have particles of n^2 kinds graded by $\mathbb{Z}_n \times \mathbb{Z}_n$ rather than by $\mathbb{Z}_2$. Following the suggestions at the end of the book [1] by Timothy Eyre, we introduce the $\mathbb{Z}_n \times \mathbb{Z}_n$ graded tensor product by means of a bi-character constructed from a primitive n^{th} root of unity and then proceed to grade both system and noise operators in Boson Fock space. This enables us to construct consistent Lie algebra supercommutation relations graded by $\mathbb{Z}_n \times \mathbb{Z}_n$ for appropriately defined graded quantum stochastic processes. Using the $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum noise for driving the Hudson-Parthasarathy noisy Schrodinger equation combined with an appropriate non-demolition counting measurement process, we formulate the problem of $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum filtering as a generalization of the Belavkin filter. After that, we discuss how to construct Bosonic quantum noise out of these graded quantum noise processes by tensoring with appropriately graded system operators and then how to add such noise to Bosonic field theories including quantum gravity. We also discuss quantum noise in string theories and a little bit of string field theories wherein the action for the string field is chosen based on the BRST charge quantization process that defines the condition for a state to be physical in the absence of ghosts. The addition of higher degree terms in the string field action is then based on the condition that the action should satisfy the quantum master equation for invariance of the matrix elements under the choice of the gauge fixing functional. We suggest methods by which quantum noise fields can also be taken into account in string field theories. The entire aim of adding quantum noise to a quantum field theory is motivated by the fact that the resulting noisy Hamiltonian should describe a Hudson-Parthasarathy noisy Schrodinger equation with unitary evolution on the joint System-Bath Hilbert space so that after tracing out over the bath, one obtains a quantum dynamical semigroup of TPCP maps describing system evolution alone for open quantum systems. For noisy quantum field theories, finally, we explain how to compute propagator corrections caused by noise.

0.1 $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum stochastic differential equations and filtering

In what follows, we first define a $\mathbb{Z}_n \times \mathbb{Z}_n$ graded tensor product, then describe how to formulate $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum stochastic processes in Boson Fock space in such a way that these processes satisfy $\mathbb{Z}_n \times \mathbb{Z}_n$ graded Lie algebra super commutation relations and then using such noisy processes, we introduce a generalized form of quantum nojsy field theory by describing the resulting dynamics in terms of quantum stochastic differential equations in the sense of Hudson and Parthasarathy. We then explain how the Belavkin quantum filter can be generalized to such processes, i.e., we outline a method for constructing a $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum filter using the classic reference probability method of John Gough and Kostler. After that, we explain how to add $\mathbb{Z}_n \times \mathbb{Z}_n$ graded noise into quantum field theories described by Lagrangian and Hamiltonian densities which enable us to express the field equations as quantum stochastic partial differential equations as noisy Heisenberg matrix mechanics.

0.2 $\mathbb{Z}_n \times \mathbb{Z}_n$ graded tensor product

Let z be a primitive n^{th} root of unity and define

$$\omega(a,b) = z^{a_1 b_2 - a_2 b_1}, a = (a_1, a_2), b = (b_1, b_2) - - - (1)$$

Given the standard matrix $n^2 \times n^2$ $E(a,b)$ that is one at entry (a,b) and zero otherwise, define its $\mathbb{Z}_n \times Z_n$ grading as $(a_1 - b_1 mod n, a_2 - b_2 mod n)$ where $0 \le a, b \le n^2 - 1$ and $a = na_1 + a_2, b = nb_1 + b_2$ with $0 \le a_1, a_2, b_1, b_2 \le n - 1$. Now consider a $\mathbb{Z}_n \times \mathbb{Z}_n$ graded Hilbert space $\mathcal{H}$ expressed as a direct sum

$$\mathcal{H} = \bigoplus_{a_1, a_2 \in \mathbb{Z}_n} \mathcal{H}_{a_1, a_2} = \bigoplus_{a \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{H}_a - - - (2)$$

We write

$$\sigma(E(a,b)) = (a_1 - b_1, a_2 - b_2) mod n \in \mathbb{Z}_n \times \mathbb{Z}_n - - - (3)$$

Let $P_{a_1, a_2} = P_a, a = (a_1, a_2)$ denote the projection of $\mathcal{H}$ onto $\mathcal{H}_{a_1, a_2}$ for $a_1, a_2 \in \mathbb{Z}_n = \{0, 1, ..., n-1\}$. Thus, we have

$$P_a P_b = P_a \delta(a,b), a, b \in \mathbb{Z}_n \times \mathbb{Z}_n - - - (4)$$

and of course,

$$\sum_{a \in A} P_a = 1 - - - (5)$$

where we are using the notation

$$A = \mathbb{Z}_n \times \mathbb{Z}_n - - - (6)$$

Note that although A and $\mathbb{Z}_{n^2}$ are both Abelian groups of the same size n^2, they are not isomorphic because writing $a = na_1 + a_2, b = nb_1 + b_2$ with $a, b \in \mathbb{Z}_{n^2}$, we can write $a + b = n(a_1 + b_1) + a_2 + b_2$ and hence a+b mod $n^2 = n(a_1 + b_1 mod n) + a_2 + b_2$ but we cannot replace in this expression $a_2 + b_2$ by $a_2 + b_2 mod n$ because we are considering $mod n^2$. Another way to see this is to compute the irreducible (one-dimensional) characters of these groups and show that they are not the same. Now take $n^2 \times n^2$ matrices A, B and assuming

$$\sigma(A) = (a_1, a_2), \sigma(B) = (b_1, b_2) - - - (7)$$

as above, i.e., if $A = E(\alpha,\beta)$ where $\alpha = np_1 + p_2, \beta = nq_1 + q_2$, then $a_1 = p_1 - q_1, a_2 = p_2 - q_2$, both mod n. We have the bicharacter

$$\omega(\sigma(A),\sigma(B)) = z^{a_1 b_2 - a_2 b_1} - - - (8)$$

Assume that we have defined a linear map $A \to \Lambda_A$ from the linear space of $n^2 \times n^2$ matrices into the space of quantum stochastic processes. Consider now the supercommutator

$$[\Lambda_A, \Lambda_B]_\omega = \Lambda_A.\Lambda_B - \omega(\sigma(A),\sigma(B))\Lambda_B.\Lambda_A - - - (9)$$

We grade these quantum stochastic operator processes as

$$\sigma(\Lambda_A) = \sigma(A) \in A = \mathbb{Z}_n \times Z_n - - - (10)$$

In order to get Lie superalgebra commutation relations, we wish that quadratic terms in quantum stochastic processes disappear when we compute the differential of this supercommutator. In other words, we should obtain only quadratic terms in the quantum stochastic differentials since such terms, by quantum Ito's formula, can be expressed again as quantum stochastic differentials. To this end, we observe that

$$d[\Lambda_A, \Lambda_B]_\omega = \Lambda_A \otimes d\Lambda_B + \omega(\sigma(A),\sigma(B))\Lambda_B \otimes d\Lambda_A + 1 \otimes d\Lambda_A.d\Lambda_B$$

$$-\omega(\sigma(A),\sigma(B))(\Lambda_B \otimes d\Lambda_A + \omega(\sigma(B),\sigma(A))\Lambda_A \otimes d\Lambda_B + 1 \otimes d\Lambda_B.d\Lambda_A) - - - (11)$$

Now observing that

$$\omega(\sigma(A),\sigma(B))\omega(\sigma(B),\sigma(A)) = \omega((a_1,a_2),(b_1,b_2)).\omega((b_1,b_2),(a_1,a_2)) = z^{a_1 b_2 - a_2 b_1}.z^{b_1 a_2 - b_2 a_1} = 1 - - - (12)$$

and making cancellations, we get the formula required for the $\mathbb{Z}_n \times \mathbb{Z}_n$-supercommutator of a quantum stochastic process to define super-Lie algebra commutation relations:

$$d[\Lambda_A, \Lambda_B]_\omega = 1 \otimes d\Lambda_A.d\Lambda_B - \omega(\sigma(A),\sigma(B)).1 \otimes d\Lambda_B.d\Lambda_A$$

$$= 1 \otimes (d\Lambda_{A.\delta.B} - \omega(\sigma(A),\sigma(B))d\Lambda_{B.\delta.A}$$

$$= 1 \otimes d\Lambda_{A.\delta.B - \omega(\sigma(A),\sigma(B))B.\delta.A} = 1 \otimes d\Lambda_{[A,B]_{\delta,\omega}} - - - (13)$$

with the obvious notation based on the Hudson-Parthasarathy quantum stochastic calculus:

$$[A,B]_{\delta,\omega} = A.\delta.B - \omega(\sigma(A),\sigma(B))B.\delta.A - - - (14)$$

Now let X be an operator in $\mathcal{H}$. We define $X_{(a_1,a_2)}$ $(a_1, a_2 \in \mathbb{Z}_n)$ to be that component of X that maps $\mathcal{H}_{b_1,b_2}$ into $\mathcal{H}_{b_1-a_1,b_2-a_2}$ for all $b_1, b_2 \in \mathbb{Z}_n$. To see that such a decomposition is indeed possible is unique and that $\sum_{a_1,a_2 \in \mathbb{Z}_n} X_{(a_1,a_2)} = X$, we note that

$$X = \sum_{a,b \in \mathbb{Z}_n \times Z_n} P_a X P_b = \sum_c \sum_{a,b:b-a=c} P_a X P_b = \sum_{c \in \mathbb{Z}_n \times \mathbb{Z}_n} X_c - - - (15)$$

where

$$X_c = \sum_{b-a=c} P_a X P_b = \sum_b P_{b-c} X P_b, c \in \mathbb{Z}_n \times \mathbb{Z}_n --- (16)$$

where all summation indices range over $\mathbb{Z}_n \times \mathbb{Z}_n$. It is clear that

$$\sum_c X_c = \sum_{b,c \in \mathbb{Z}_n \times \mathbb{Z}_n} P_{b-c} X P_b = \sum_{a,b \in \mathbb{Z}_n \times \mathbb{Z}_n} P_a X P_b = X --- (17)$$

Note that when we write $b-c$, we actually mean $(b_1 - c_1 modn, b_2 - c_2 modn) \in \mathbb{Z}_n \times \mathbb{Z}_n$ where $b = (b_1, b_2), c = (c_1, c_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$. Also note that X_c maps $\mathcal{H}_a$ into $\mathcal{H}_{a-c}$ because

$$X_c P_a = \sum_b P_{b-c} X P_b P_a = P_{a-c} X P_a --- (18)$$

since $P_b P_a = \delta(b,a) P_a$ for all $a, b \in \mathbb{Z}_n \times \mathbb{Z}_n$.

The uniqueness of this decomposition is proved as follows. Suppose $X = \sum_c \tilde{X}_c$ where $\tilde{X}_c$ maps $\mathcal{H}_b$ into $\mathcal{H}_{b-c}$ for each $b, c \in \mathbb{Z}_n \times \mathbb{Z}_n$. Then, $\tilde{X}_c P_b = P_{b-c} \tilde{X}_c P_b$ for all b, c. and hence

$$X P_b = \sum_c \tilde{X}_c P_b = \sum_c P_{b-c} \tilde{X}_c P_b --- (19)$$

from which it follows that

$$P_{b-c} X P_b = P_{b-c} \tilde{X}_c P_b \forall b, c --- (20)$$

Also since by hypothesis, $\tilde{X}_c$ maps $\mathcal{H}_b$ into $\mathcal{H}_{b-c}$ for each b, it follows that

$$P_a \tilde{X}_c P_b = 0, a \neq b - c --- (21)$$

Thus, we get

$$P_{b-c} X P_b = P_{b-c} \tilde{X}_c P_b = \sum_a P_a \tilde{X}_c P_b = \tilde{X}_c P_b --- (22)$$

Summing this equation over all b, then gives us the desired uniqueness result:

$$X_c = \sum_b P_{b-c} X P_b = \sum_b \tilde{X}_c P_b = \tilde{X}_c --- (23)$$

Now define the operators

$$\theta_a = \sum_{b \in \mathbb{Z}_n \times \mathbb{Z}_n} \omega(a,b) P_b, a \in \mathbb{Z}_n \times \mathbb{Z}_n --- (24)$$

This is the same as saying that

$$\theta_{a_1,a_2} = \sum_{b_1,b_2\in\mathbb{Z}_n} z^{a_1b_2-a_2b_1} P_{b_1,b_2}. ---(25)$$

Now for two operators X, Y in $\mathcal{H}$, define their graded tensor product as

$$X \otimes_g Y = \sum_a X_a \otimes \theta_a Y ---(26)$$

where the sum is over all $a = (a_1, a_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$. Then, we get

$$(X \otimes_g Y).(U \otimes_g V) = \sum_{a,b} (X_a \otimes \theta_a Y).(U_b \otimes \theta_b V)$$

$$= \sum_{a,b} (X_a U_b \otimes \theta_a Y \theta_b V) = \sum_{a,b,c} (X_a U_b \otimes \theta_a Y_c \theta_b V) ---(27)$$

Now,

$$\theta_a Y_c \theta_b = \sum_{d,f} \omega(a,d)\omega(b,f) P_d Y_c P_f = \sum_{d,f} \omega(a,d)\omega(b,f)\delta(f-c,d) P_d Y_c P_f$$

$$= \sum_f \omega(a, f-c)\omega(b,f) P_{f-c} Y_c P_f ---(28)$$

Now observing that

$$P_{f-c} Y_c P_e = 0, e \neq f ---(29)$$

since Y_c maps $\mathcal{H}_e$ into $\mathcal{H}_{e-c}$ for each e, it follows that

$$P_{f-c} Y_c P_f = \sum_e P_{f-c} Y_c P_e = P_{f-c} Y_c ---(30)$$

and thus we get

$$\theta_a Y_c \theta_b = \sum_f \omega(a, f-c)\omega(b,f) P_{f-c} Y_c = \sum_f \omega(a,f)\omega(b, f+c) P_f Y_c ---(31)$$

Note that all summations are taken over the Abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$. Now observe that

$$\omega(a,f)\omega(b,f+c) = z^{a_1f_2-a_2f_1+b_1(f_2+c_2)-b_2(f_1+c_1)} = z^{(a_1+b_1)f_2-(a_2+b_2)f_1}.z^{b_1c_2-b_2c_1}$$

$$= \omega(a+b, f)\omega(b,c) ---(32)$$

This could also be seen from the bicharacter property of ω:

$$\omega(a,f)\omega(b,f+c) = \omega(a,f)\omega(b,f)\omega(b,c) = \omega(a+b,f)\omega(b,c) ---(33)$$

Thus, we get

$$\theta_a Y_c \theta_b = \omega(b,c)\sum_f \omega(a+b,f)P_f Y_c = \omega(b,c)\theta_{a+b}Y_c - - - (34)$$

Summing over c, we get

$$\theta_a Y \theta_b = \theta_{a+b}.\sum_c \omega(b,c)Y_c - - - (35)$$

Combining these results, we get

$$(X \otimes_g Y).(U \otimes_g V)$$

$$= \sum_{a,b,c}(X_a U_b \otimes \theta_a Y_c \theta_b V) =$$

$$\sum_{a,b,c}\omega(b,c)(X_a U_b \otimes \theta_{a+b} Y_c V) - - - (36)$$

In particular, when $Y = Y_c, U = U_b$ for fixed b, c we get

$$(X \otimes_g Y).(U \otimes_g V) = (X \otimes_g Y_c).(U_b \otimes_g V) = \omega(b,c)\sum_a X_a U_b \otimes \theta_{a+b} Y_c V$$

$$= \omega(b,c)\sum_d (XU)_d \otimes \theta_d YV = \omega(b,c) XU \otimes_g YV - - - (37)$$

which is the desired formula to be satisfied by the $Z_n \times \mathbb{Z}_n$ graded tensor product.

0.3 $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum stochastic processes with $\mathbb{Z}_n \times \mathbb{Z}_n$ graded Lie algebra commutation relations

For $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$, define

$$\phi(a,b) = a_1 b_2 - a_2 b_1 - - - (38)$$

Thus,

$$\omega(a,b) = z^{\phi(a,b)}, z = exp(2\pi i/n) - - - (39)$$

Let us also use the notation a for $na_1 + a_2$ and likewise b for $nb_1 + b_2$. This is justified because any element a of $\mathbb{Z}_{n^2} = \{0, 1, ..., n^2 - 1\}$ can be expressed in a unique way as $na_1 + a_2$ where $a_1, a_2 \in \mathbb{Z}_n = \{0, 1, ..., n-1\}$. Thus by a we can mean either the element (a_1, a_2) of $\mathbb{Z}_n \times \mathbb{Z}_n$ or else the element $na_1 + a_2$ of $\mathbb{Z}_{n^2}$.

Let now $\ngeq^a_b$ with $a, b = 0, 1, ..., n^2 - 1$ or equivalently $a, b \in \mathbb{Z}_n \times \mathbb{Z}_n$ denote the generalized quantum noise processes of the Hudson-Parthasarathy quantum stochastic calculus acting in the Boson Fock space $\Gamma_s(L^2(\mathbb{R}_+)\otimes\mathbb{C}^{n^2})$. They satisfy the quantum Ito formula

$$d\Lambda_b^a.d\Lambda_d^c=\epsilon_d^a d\Lambda_b^c---(40)$$

We now introduce the grading operators in this Boson Fock space:

$$G(t,a)=exp((2\pi i/n).\sum_b\phi(b,a)\Lambda_b^b(t))---(41)$$

where

$$\phi(b,a)=b_1a_2-b_2a_1$$

As usual, it is understood that a,b are elements of either $\mathbb{Z}_n\times\mathbb{Z}_n$ or equivalently, of $\mathbb{Z}_{n^2}$. Note that we can equivalently write since

$$\omega(b,a)=z^{b_1a_2-b_2a_1},z=exp(2\pi i/n)---(42)$$

$$G(t,a)=\Pi_b\omega(b,a)^{\Lambda_b^b(t)}=exp(\sum_n ln(\omega(b,a))\Lambda_b^b(t))---(43)$$

Remark: The grading operator $G(t,a-b)$ is the analogue of

$$G(t)^{\sigma(a,b)}=(-1)^{\sigma(a,b)\sum_c\sigma(c)\Lambda_c^c)(t)}$$

$$=(-1)^{\sigma(a,b)\Lambda_t(H)}=exp(i\pi.\sigma(a,b)\Lambda_t(H))=\Gamma(exp(i\pi H_t))^{\sigma(a,b)}=\Gamma(K_t)^{\sigma(a,b)}$$

in $\mathbb{Z}_2$ graded quantum stochastic calculus. Here,

$$\sigma(a)=0,a=1,...,r,\sigma(a)=1,a=r+1,r+2,...,N$$

and

$$\sigma(a,b)=\sigma(a)+\sigma(b),$$
$$H=diag[\sigma(a),a=1,2,...,N],H_t=H.\chi_{[0,t]},$$
$$K=exp(i\pi H)=diag[I_r,-I_{N-r}]$$

and

$$K_t=exp(i\pi H_t)=K.\chi_{[0,t]}+I.\chi_{(t,\infty)}$$

This grading operator has the important property

$$G(t)^{\sigma(a,b)}.d\Lambda_d^c(s)G(t)^{\sigma(a,b)}=(-1)^{\sigma(a,b)\sigma(c,d)}.d\Lambda_d^c(s),s<t$$

Now, coming back to the $\mathbb{Z}_n\times\mathbb{Z}_n$ situation, we find that for $s<t$,

$$<e(u)|G(t,a-b)^*d\Lambda_d^c(s).G(t,a-b)|e(v)>$$

$$=<G(t,a-b)e(u)|d\Lambda_d^c(s)|G(t,a-b)e(v)>=exp((2\pi/n)i\phi(c,a-b)-\phi(d,a-b))v_c(s)\bar{u}_d(s)ds---(44)$$

and therefore,

$$G(t,a-b)^* d\Lambda^c_d(s).G(t,a-b) = exp(2\pi i\phi(c-d,a-b))/n)d\Lambda^c_d(s) = z^{\phi(c-d,a-b)}d\Lambda^c_d(s) = \omega(c-d,a-b)d\Lambda^c_d(s)---($$

This suggests that we should define the grading of $d\Lambda^c_d(s)$ as

$$\sigma(d\Lambda^c_d(s)) = c-d = (c_1-d_1, c_2-d_2) --- (46)$$

The above formula can be expressed in elegant notation as

$$G(t,\sigma(A))^*.d\Lambda_s(B).G(t,\sigma(A)) = \omega(\sigma(B),\sigma(A))d\Lambda_s(B) ----(47)$$

Remark: The grading of E^a_b is $\sigma(E^a_b) = a-b$ where $a-b = (a_1-b_1, a_2-b_2)$ with $a = (a_1,a_2), b = (b_1,b_2)$.

The precise meaning of this equation is obtained by taking

$$G(t,\sigma(A)) = G(t,a-b), A = E^a_b --- (48)$$

and

$$d\Lambda_s(B) = \sum_{a,b} B^a_b.d\Lambda^a_b(s), B = \sum_{a,b} B^a_b E^a_b --- (49)$$

and

$$\omega(\sigma(B),\sigma(A))d\Lambda_s(B) = \sum_{c,d} B^c_d.\omega(\sigma(E^c_d),\sigma(A))d\Lambda_s(E^c_d) = \sum_{c,d} B^c_d\omega(c-d,a-b)d\Lambda^c_d(s)---(50)$$

Since $G(t,\sigma(A))$ is a unitary operator, we can equivalently write

$$d\Lambda_s(B).G(t,\sigma(A)) = \omega(\sigma(B),\sigma(A))G(t,\sigma(A))d\Lambda_s(B), s<t --- (51)$$

Of course, we are assuming here that $A = E^a_b$ for some $a,b \in \mathbb{Z}_n \times \mathbb{Z}_n$. Then, define the quantum stochastic process

$$\xi_t(B) = \int_0^t G(s,\sigma(B))d\Lambda_s(B), B = E^a_b --- (52)$$

or more precisely,

$$d\xi^a_b(t) = G(t,a-b)d\Lambda^a_b(t) --- (53)$$

Then consider for $s<t$,

$$d\xi_s(A).d\xi_t(B) = G(s,\sigma(A))d\Lambda_s(A).G(t,\sigma(B))d\Lambda_t(B) =$$

$$\omega(\sigma(A),\sigma(B))G(s,\sigma(A)).G(t,\sigma(B)).d\Lambda_s(A)d\Lambda_t(B) --- (54)$$

Note that for any A, B and any s, t $G(s, \sigma(A))$ commutes with $G(t, \sigma(B))$ because $G(t, \sigma(B))$ is a function of $\sum_f \phi(f\, a - b)\Lambda_f^f(t)$ where $\sigma(B) = a - b$ while $G(s, \sigma(A))$ is a function of $\sum_f \phi(f, c - d)\Lambda_f^f(s)$ and we have

$$[d\Lambda_f^f(t), d\Lambda_g^g(s)] = 0, s < t - - - (55)$$

while

$$[d\Lambda_f^f(s), d\Lambda_g^g(s)] = \epsilon_g^f.d\Lambda_f^g(s) - \epsilon_f^g.d\Lambda_g^f(s) = 0 - - - (56)$$

since ϵ_g^f is one for $f = g \geq 1$ and zero otherwise. Thus, using

$$\omega(\sigma(B), \sigma(A)).\omega(\sigma(A), \sigma(B)) = 1 - - - (57)$$

we get

$$d\xi_s(A).d\xi_t(B) - \omega(\sigma(A), \sigma(B)).d\xi_t(B).d\xi_s(A) = 0, s < t - - - (58)$$

which is the desired supercommutation relation that we are looking for. Further,

$$d\xi_s(A).d\xi_s(B) = G(s, \sigma(A)).d\Lambda_s(A).G(s, \sigma(B)).d\Lambda_s(B)$$

$$= G(s, \sigma(A))G(s, \sigma(B)).d\Lambda_s(A).d\Lambda_s(B)$$

$$= G(s, \sigma(B)).G(s, \sigma(A)).d\Lambda_s(A).d\Lambda_s(B)$$

$$= G(s, \sigma(A) + \sigma(B)).d\Lambda_s(A).d\Lambda_s(B) - - - (59)$$

since

$$exp((2\pi i/n)\sum_c \phi(c, \sigma(A))\Lambda_c^c(s)).exp((2\pi i/n).\sum_c \phi(c, \sigma(B)).\Lambda_c^c(s))$$

$$= exp((2\pi i/n)\sum_c \phi(c, \sigma(A) + \sigma(B))\Lambda_c^c(s)) - - - (60)$$

Therefore, we find

$$[d\xi_s(A), d\xi_s(B)]_\omega = d\xi_s(A).d\xi_s(B) - \omega(\sigma(A), \sigma(B)).d\xi_s(B).d\xi_s(A) =$$

$$= G(s, \sigma(A)+\sigma(B))[d\Lambda_s(A).d\Lambda_s(B) - \omega(\sigma(A), \sigma(B)).d\Lambda_s(B).d\Lambda_s(A)] - - - (61)$$

Now observing that by the quantum Ito formula,

$$d\Lambda_b^a(s).d\Lambda_d^c(s) = \epsilon_d^a.d\Lambda_b^c(s)$$

we can write by noting that $A = A_b^a E_a^b$ (The Einstein summation over the repeated indices a, b being implied),

$$d\Lambda_s(A).d\Lambda_s(B) = A_a^b B_c^d \epsilon_d^a d\Lambda_b^c(s) = (A\epsilon.B)_c^b d\Lambda_b^c(s) = d\Lambda_s(A\epsilon.B) - - - (62)$$

so we get

$$[d\xi_s(A), d\xi_s(B)]_\omega$$

$$= G(s, \sigma(A) + \sigma(B)) d\Lambda_s([A, B]_{\epsilon,\omega}) = d\xi_s([A, B]_{\epsilon,\omega}) - - - (63)$$

because

$$\sigma(A\epsilon B) = \sigma(A) + \sigma(B) - - - (64)$$

Note that

$$[A, B]_{\epsilon,\omega} = A.\epsilon.B - \omega(\sigma(A), \sigma(B))B.\epsilon.A - - - (65)$$

Remark: More precisely, we have the following:

$$[d\xi^a_b(s), d\xi^c_d(s)]_\omega = G(s, a-b).G(s, c-d).d\Lambda^a_b(s).d\Lambda^c_d(s) - \omega(a-b, c-d)G(s, c-d).G(s, a-b)d\Lambda^c_d(s).d\Lambda^a_b(s)$$

$$= G(s, a - b + c - d)[\epsilon^a_d d\Lambda^c_b(s) - \omega(a - b, c - d)G(s, a - b + c - d)\epsilon^c_b d\Lambda^a_d(s)]$$

$$= G(s, c - b)\epsilon^a_d d\Lambda^c_b(s) - \omega(a - b, c - d)G(s, a - d)\epsilon^c_b d\Lambda^a_d(s) - - - (66)$$

So that writing

$$A = A^b_a E^a_b, B = B^d_c E^c_d,$$

we get

$$[d\xi_s(A), d\xi_s(B)]_\omega = A^b_a \epsilon^a_d B^d_c G(s, c-b) d\Lambda^c_b - \omega(a-b, c-d) B^d_c \epsilon^c_b A^b_a G(s, a-d) d\Lambda^a_d(s)$$

$$= A^b_a \epsilon^a_d B^d_c d\xi^c_b - \omega(a - b, c - d) B^d_c \epsilon^c_b A^b_a d\xi^a_d(s)$$

$$= d\xi_s(A\epsilon.B - \omega(\sigma(A), \sigma(B))B\epsilon.A) - - - (67)$$

Combining these facts, we obtain the following quantum stochastic Lie algebra representation of $\mathbb{Z}_n \times \mathbb{Z}_n$ graded matrix Lie algebras:

$$[\xi_s(A), \xi_t(B)]_\omega = \xi_s(A)\xi_t(B) - \omega(\sigma(A), \sigma(B))\xi_t(B).\xi_s(A) =$$

$$\xi_{min(s,t)}(A\epsilon B - \omega(\sigma(A), \sigma(B))B\epsilon.A) - - - (68)$$

Now consider the following quantum stochastic differential equation:

$$dU(t) = (L^a_b \otimes_g d\xi^b_a(t))U(t) - - - (69)$$

where L^a_b has the grading

$$\sigma(L^a_b) = a - b - - - (70)$$

Note that ξ^b_a has the grading $b - a$ so $L^a_b \otimes d\xi^b_a$ has the grading zero as it should be since $U(t)$ to be unitary must have the grading zero because $U(t)$ and $U(t)^*$ are required to have the same grading and $I = U(t)^*U(t)$ has the grading zero. Of course, we could more generally require that $U(t)$ maps $\mathcal{H}_a$ into $\mathcal{H}_{a-c}$ so that $U(t)$ has the grading c and then $U(t)^*$ will map $\mathcal{H}_{a-c}$ into $\mathcal{H}_a$ which would imply that $U(t)^*$ has the grading $-c$ and then $I = U(t)^*U(t)$ will have the grading $-c + c = 0$ as required. With this understanding, we are free to toy with the idea that L^a_b has an arbitrary grade. However, it is absurd to think that a unitary operator does not have zero grade because such an operator is the exponential of i times a Hermitian operator and a Hermitian operator H must necessarily have zero grade because $H^* = H$ and so if H maps $\mathcal{H}_a$ to $\mathcal{H}_{a-c}$, then $H = H^*$ will map $\mathcal{H}_a$ to $\mathcal{H}_{a+c}$ which means that $c = 0$. So for consistency of analysis, we shall assume that $\sigma(L_b^a) = a - b$ and proceed. Note that we are using the graded tensor product between the system and noise operators. If A is a system operator and B a noise operator, then we have

$$(A \otimes_g B)^* = (\sum_a (A_a \otimes \theta_a B))^* = \sum_a A_a^* \otimes B^* \theta_a^* - - - (71)$$

where we recall that

$$\theta_a = \sum_b \omega(a,b) P_b, \omega = z^{a_1 b_2 - a_2 b_1}, z = exp(2\pi i/n) - - - (72)$$

so that

$$\theta_a^* = \sum_b \omega(-a,b) P_b = \theta_{-a} - - - (73)$$

Also note that

$$\sigma(A_a) = a, \sigma(A_a^*) = -a - - - (74)$$

because A_a maps $\mathcal{H}_b$ into $\mathcal{H}_{b-a}$ and hence A_a^* maps $\mathcal{H}_{b-a}$ into $\mathcal{H}_a$. Thus, we get

$$(A \otimes_g B)^* = \sum_a A_a^* \otimes B^* \theta_{-a} = \sum_a (A^*)_{-a} \otimes B^* \theta_{-a} = \sum_a (A^*)_a \otimes B^* \theta_a - - - (75)$$

Now observe that

$$A \otimes_g B = \sum_a (A_a \otimes I).(I \otimes \theta_a B) = \sum_a (I \otimes \theta_a B).(A_a \otimes I) - - - (76)$$

and hence,

$$(A \otimes_g B)^* = \sum_a (A_a \otimes I)^* (I \otimes \theta_a B)^* - - - (77)$$

Suppose for definiteness,

$$\sigma(A) = a, \sigma(B) = b - - - (78)$$

Then,

$$A \otimes_g B = A \otimes \theta_a B - - - (79)$$

$$(A \otimes_g B)^* = A^* \otimes (\theta_a B)^* = A^* \otimes B^* \theta_{-a} - - - (80)$$

Now recall the formula

$$\theta_a Y_c \theta_b = \omega(b,c) \theta_{a+b} Y_c - - - (81)$$

for any operator Y, which gives in particular,

$$\theta_a \theta_b = \theta_{a+b} - - - (82)$$

In particular,

$$\theta_a \theta_{-a} = \theta_0 = I - - - (83)$$

since $\omega(0,b) = 1 \forall b$. Note that I has the grading zero, i.e., $\sigma(I) = (0,0)$. Then, since $\sigma(A^*) = -a, \sigma(B^*) = -b$, it follows that

$$B^* \theta_{-a} = \theta_{-a} \theta_a B^* \theta_{-a} = \theta_{-a}.\omega(-a,-b) B^* = \omega(a,b) \theta_{-a} B^* - - - (84)$$

so that

$$(A \otimes_g B)^* = \omega(a,b) A^* \otimes \theta_{-a} B^* = \omega(a,b) A^* \otimes_g B^* - - - (85)$$

This is a fundamental formula. Another way to see this is:

$$(A \otimes_g B)^* = ((A \otimes_g I)(I \otimes_g B))^* = (I \otimes_g B)^* (A \otimes_g I)^* - - - (86)$$

But,

$$(A \otimes_g I)^* = (A \otimes_g \theta_a)^* = A^* \otimes \theta_{-a} = A^* \otimes_g I,$$
$$(I \otimes_g B)^* = (I \otimes B)^* = I \otimes B^* = I \otimes_g B^*$$

Combining these,

$$(A \otimes_g B)^* = (I \otimes_g B^*).(A^* \otimes_g I) = \omega(\sigma(B^*), \sigma(A^*)) A^* \otimes_g B^*$$
$$= \omega(-b,-a) A^* \otimes_g B^* = \omega(b,a) A^* \otimes_g B^* - - - (87)$$

0.4 $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum stochastic Belavkin filter

Now, let X be a system operator of definite grade and consider the quantum stochastic differential equation

$$dU(t) = (L_b^a \otimes d\xi_a^b(t)) U(t) - - - (88)$$

where it is assumed that the tensor product $\otimes$ is the graded tensor product $\otimes_g$, so that we avoid writing the subscript g at each tensor product. The system operators L_b^a are chosen so that $U(t)$ is unitary for all t. Using quantum Ito's formula in the form

$$d(U^* U) = dU^*.U + U^*.dU + dU^*.dU - - - (89)$$

we see that the condition for this to vanish so that U is a unitary evolution is that

$$(L_b^a)^* + L_a^b + (L_b^d)^* \epsilon_c^d L_a^c = 0 - - - (90)$$

with summation over the repeated indices c, d being implied. In shorthand notation, this equation is the same as

$$L^* + L + L^* \epsilon.L = 0 - - - (91)$$

where L is the "block operator" $((L_b^a))_{0 \le a,b \le n^2-1}$. We assume that

$$\sigma(L_b^a) = a - b, \sigma(d\xi_a^b) = b - a - - - (92)$$

so

$$\sigma(L_b^a \otimes d\xi_a^b) = 0 - - - (93)$$

Then, according to the above discussion,

$$(L_b^a \otimes d\xi_a^b)^* = \omega(\sigma(L_b^a), \sigma(d\xi_a^b))(L_b^a)^* \otimes d\xi_b^a) = \omega(a-b, b-a).()(L_b^a)^* \otimes d\xi_b^a) = (L_b^a)^* \otimes d\xi_b^a - - - (94)$$

since

$$(d\xi_b^a(t))^* = (G(t, a-b)d\Lambda_b^a(t))^* = G(t, b-a)d\Lambda_a^b(t) = d\xi_a^b(t) - - - (95)$$

Note that

$$G(t, a)^* = G(t, -a) = G(t, a)^{-1}, a \in \mathbb{Z}_n \times \mathbb{Z}_n - - - (96)$$

X being a system operator of definite grade, we have

$$j_t(X) = U(t)^* X U(t), t \ge 0 - - - (97)$$

We shall now derive a formula for $dj_t(X)$ which can be termed as a $\mathbb{Z}_n \times \mathbb{Z}_n$ graded version of the Evans-Hudson flow describing noisy Heisenberg evolution of a system observable in the context of the ungraded Hudson-Parthasarathy noisy Schrodinger equation. Such a graded Evans-Hudson flow can then be used to derive a $\mathbb{Z}_n \times \mathbb{Z}_n$ graded version of the Belavkin quantum filter. Note that in [Harish Parthasarathy, Qeios], we have already derived a $\mathbb{Z}_2$-graded, i.e. supersymmetric version of the Belavkin quantum filter.

Quantum Ito's formula gives

$$dj_t(X) = dU(t)^* X U(t) + U(t)^* X dU(t) + dU(t)^* X dU(t)$$

$$= j_t(((L_b^a)^* \otimes d\xi_b^a(t)^*)(X \otimes 1) + (X \otimes 1).(L_b^a) \otimes d\xi_a^b(t)) + (L_b^a)^* \otimes d\xi_b^a(t)).(X \otimes 1).(L_d^c \otimes d\xi_c^d(t)))$$

$$= \omega(b-a, \sigma(X)) j_t((L_b^a)^* X.G(t, b-a)) d\Lambda_a^b(t) +$$

$$j_t(X L_b^a G(t, b-a)) d\Lambda_a^b(t) + \omega(a-b, \sigma(X)) j_t((L_b^a)^* X L_d^c.G(t, a-b+d-c)) d\Lambda_b^a(t).d\Lambda_c^d(t) - - - (98)$$

Now consider the input measurement process

$$Y_i(t) = c(a)\xi_a^a(t) = c(a)\Lambda_a^a(t) - - - (99)$$

with $c(a)'s$ being some real constants and the sum over all $a \in \mathbb{Z}_{n^2}$ being implied. This process forms an Abelian family of operators in the Boson Fock space $\Gamma_s(L^2(\mathbb{R}_+) \otimes \mathbb{C}^{n^2})$. We form the output measurement process

$$Y_o(t) = U(t)^* Y_i(t) U(t) - - - (100)$$

Now observing that for $T \geq t$, $d\xi^a_b(T) = G(T, a-b) d\Lambda^a_b(T)$ commutes with $\Lambda^c_c(t)$ because both $G(T, a-b)$ and $d\Lambda^a_b(T)$ commute with $\Lambda^c_c(t)$, it follows that

$$Y_o(t) = U(T)^* Y_i(t) U(T), T \geq t - - - (101)$$

It follows that for any system observable X, since X commutes with $Y_i(t)$, that $j_T(X) = U(T)^* X U(T)$ will commute with $Y_o(t)$ for any $T \geq t$. Note that $Y_i(t)$ can be regarded as a weighted sum of the number counts up to time t of the particles of different types of grade (A $\mathbb{Z}_n \times \mathbb{Z}_n$ generalization of the $\mathbb{Z}_2$ Boson-Fermion counting process). It also follows that since $Y_i(.)$ forms and Abelian family and $U(T)$ is unitary that $Y_o(t) = U(T)^* Y_i(t) U(T), 0 \leq t \leq T$ forms an Abelian family for any $T \geq 0$. In other words, for all $t \geq 0$,

$$\eta_o(t) = \sigma(Y_o(s) : s \leq t) - - - (102)$$

is an Abelian Von-Neumann algebra. Since further $X.G(t, a-b)$ commutes with $Y_i(s), s \leq t$ for any system operator X, because $\Lambda^c_c(s)$ commutes with $\Lambda^d_d(t)$ for any s, t, c, d, it follows that $j_t(X.G(t, a-b)) = U(t)^* X G(t, a-b) U(t)$ commutes with $Y_o(s) = U(t)^* Y_i(s) U(t), s \leq t$ and hence

$$\pi_{t,a-b}(X) = \mathbb{E}(j_t(X.G(t, a-b)) | \eta_o(t)), X \in L(\langle) - - - (103)$$

is defined for all $t \geq 0, a, b \in \mathbb{Z}_{n^2}$. Since $\eta_o(t)$ is an Abelian algebra, it follows that $\pi_{t,a-b}(X), t \geq 0$ is Abelian family adapted to $\eta_o(.)$ and we can assume therefore in view of quantum Ito's formula for the quantum Poisson processes $\Lambda^c_c(t)$ that

$$d\pi_{t,a-b}(X) = F_{t,a-b}(X) dt + G_{t,a-b}(X) dY_o(t) - - - (104)$$

where $F_{t,a-b}(X), G_{t,a-b}(X) \in \eta_o(t)$.

We calculate

$$dj_t(X.G(t, a-b)) = d(U(t)^* X.G(t, a-b) U(t))$$

$$= U(t)^* X.dG(t, a-b).U(t) + dU(t)^* X.G(t, a-b).U(t) + U(t)^* X G(t, a-b) dU(t) + dU(t)^* X.dG(t, a-b).U(t) + U(t)^*$$

Now,

$$dG(t, a-b) = d.exp((-2\pi i/n) \sum_c \phi(a-b, c) d\Lambda^c_c(t))$$

$$= G(t, a-b). \sum_c (exp(-(2\pi i/n)\phi(a-b, c)) - 1).d\Lambda^c_c(t) - - - (106)$$

Remark: For any matrix H of size $n^2 \times n^2$, we have with $H_t = H.\chi_{[0,t]}$,

$$< e(u)|exp(\Lambda_t(H))|e(v) >=< e(u)|e(exp(H_t)v) >= exp(< u|exp(H_t)|v >) = exp(\int_0^t < u(s)|exp(H)|v(s) > -$$

and hence,

$$d < e(u)|exp(\Lambda_t(H))|e(v) >=< e(u)|exp(\Lambda_t(H)|e(v) > . < u(t)|exp(H){-}1|v(t) > dt$$

$$=< e(u)|exp(\Lambda_t(H))d\Lambda_t(exp(H) - 1)|e(v) > - - -(108)$$

Therefore,

$$dexp(\Lambda_t(H)) = exp(\Lambda_t(H)).d\Lambda_t(exp(H) - 1) - - - (109)$$

In particular, taking

$$H = \sum_c f(c)E_c^c = diag[f(c) : c \in \mathbb{Z}_{n^2}]$$

we get

$$exp(H) - 1 = \sum_c (exp(f(c)) - 1)E_c^c$$

and therefore,

$$dexp(\sum_c f(c)\Lambda_c^c(t)) = exp(\sum_c f(c)\Lambda_c^c(t)) \sum_c (exp(f(c)) - 1)d\Lambda_c^c(t) - - - (110)$$

in agreement with the classical formula for the differential of the exponential of a superposition of independent Poisson processes.

(Note that by $j_t(X.G(t, a{-}b))$, we actually mean $U(t)^*(X \otimes G(t, a{-}b))U(t)$).

Also

$$dU(t)^*XG(t, a - b)U(t) = U(t)^*d\xi_d^c(t)(L_d^c)^*XG(t, a - b)U(t)$$

$$= \omega(c{-}d, \sigma(X))U(t)^*(L_d^c)^*XG(t, a{-}b{+}c{-}d)U(t)d\Lambda_d^c(t) = \omega(c{-}d, \sigma(X))j_t((L_d^c)^*XG(t, a{-}b{+}c{-}d))d\Lambda_d^c(t){-}{-}{-}($$

Remark:

$$(L_d^c d\xi_c^d)^* = \omega(c{-}d, d{-}c)(d\xi_c^d)^*(L_d^c)^* = d\xi_d^c(L_d^c)^* = (L_d^c)^*G(t, c{-}d)d\Lambda_d^c{-}{-}{-}(112)$$

Likewise,

$$U(t)^*XG(t, a - b)dU(t) = j_t(XL_d^cG(t, a - b + d - c))d\Lambda_c^d(t) - - - (113),$$

$$dU(t)^*XG(t, a{-}b)dU(t) = \omega(c{-}d, \sigma(X){+}p{-}q)U(t)^*(L_d^c)^*XL_q^pG(t, a{-}b{+}c{-}d{+}q{-}p)\epsilon_p^c)U(t)d\Lambda_d^q(t)$$

$$= \omega(c{-}d, \sigma(X){+}c{-}q)j_t((1{-}\delta[c])(L_d^c)^*XL_q^c.G(t, a{-}b{+}q{-}d))d\Lambda_d^q(t){-}{-}{-}(114)$$

where $\delta[c]=0, d\geq 1, \delta[0]=1$. Further,

$$dU(t)^* X dG(t,a-b)U(t) =$$

$$\omega(c-d,\sigma(X))U(t)^*(L_d^c)^* X d\xi_d^c(t)G(t,a-b).(exp(-(2\pi i/n)\phi(a-b,p))-1).d\Lambda_p^p(t)U(t)$$

$$=\omega(c-d,\sigma(X))(exp(-(2\pi i/n)\phi(a-b,p))-1)j_t((L_d^c)^* XG(t,a-b+c-d))\epsilon_p^c d\Lambda_d^p(t)$$

$$=\omega(c-d,\sigma(X))(1-\delta[c])(exp(-(2\pi i/n)\phi(a-b,c))-1)j_t((L_d^c)^* XG(t,a-b+c-d))d\Lambda_d^c(t)---(115)$$

Likewise,

$$U(t)^* X dG(t,a-b).dU(t) =$$

$$U(t)^* XG(t,a-b).(exp(-(2\pi i/n)\phi(a-b,p))-1).d\Lambda_p^p(t)L_d^c d\xi_c^d(t)U(t)$$

$$=(exp(-(2\pi i/n)\phi(a-b,p))-1)j_t(XL_d^c.G(t,a-b+d-c))\epsilon_c^p d\Lambda_p^d(t)$$

$$=(1-\delta[c])(exp(-(2\pi i/n)\phi(a-b,c))-1)j_t(XL_d^c.G(t,a-b+d-c))d\Lambda_c^d(t)---(116)$$

In all these formulas, summation over repeated indices ranging from 0 to n^2-1 is implied or equivalently, summation over $\mathbb{Z}_n\times\mathbb{Z}_n$ with each index $a\in\mathbb{Z}_{n^2}$ being represented uniquely as $a=na_1+a_2$ or equivalently as $(a_1,a_2)\in\mathbb{Z}_n\times\mathbb{Z}_n$. Combining all these results, we can write,

$$dj_t(X.G(t,a-b)) = j_t(\theta(X|a-b,c,d,p,q).G(t,q))\omega(\sigma(X),p)d\Lambda_d^c(t)---(117)$$

where $X\to\theta(X|a-b,c,d,p,q)$ are linear maps in the space of system observables. We next calculate the output measurement noise differential making use of the above derived conditions for unitarity of $U(t)$ on L_b^a :

$$dY_o(t) = d(U(t)^* Y_i(t)U(t)) =$$

$$dY_i(t)+dU(t)^* dY_i(t)U(t)+U(t)^* dY_i(t)dU(t) =$$

$$c(a)d\Lambda_a^a(t)+j_t((L_b^a)^* d\xi_b^a(t).c(c)d\Lambda_c^c(t))+j_t(c(c)d\Lambda_c^c(t)L_b^a d\xi_a^b(t))$$

$$=c(a)d\Lambda_a^a(t)+j_t((L_b^a)^* G(t,a-b)d\Lambda_b^a(t).c(c)d\Lambda_c^c(t))+j_t(c(c)d\Lambda_c^c(t)L_b^a G(t,b-a)d\Lambda_a^b(t))$$

$$=c(a)d\Lambda_a^a(t)+j_t(c(a)(1-\delta[a])(L_b^a)^* G(t,a-b))d\Lambda_b^a(t)+j_t(c(a)(1-\delta[a])L_b^a G(t,b-a))d\Lambda_a^b(t)---(118)$$

This equation can be expressed as

$$dY_o(t) = C(a,b,c,d)j_t(M(c)G(t,d)d\Lambda_b^a(t)---(119)$$

where $C(a,b,c,d)$ are some complex constants and $M(c)$ are system operators. Using these formulas, it is easy to compute the filter coefficients $F_{t,a}(X) and G_{t,a}(X)$ based on the orthogonality principle (reference probability approach in the sense of John Gough and Kostler):

$$\mathbb{E}[(j_t(XG(t,a))-\pi_{t,a}(X))C(t)] = 0---(120)$$

where

$$dC(t) = C(t)f(t)dY_o(t), t\geq 0, C(0)=1---(121)$$

Taking the differential of this and using quantum Ito's formula with the arbitrariness of the complex function $f(t)$ gives us two equations

$$\mathbb{E}[(dj_t(X.G(t,a))) - d\pi_{t,a}(X)|\eta_o(t)] = 0 - - - (122),$$

$$\mathbb{E}[(j_t(X.G(t,a)) - \pi_{t,a}(X))dY_o(t)|\eta_o(t)] + E[(dj_t(X.G(t,a)) - d\pi_{t,a}(X))dY_o(t)|\eta_o(t)] = 0 - - - (123),$$

Using the formulas for the differentials of $j_t(X.G(t,a))$ and $Y_o(t)$ derived above and the homomorphism property of j_t, it is an elementary matter to obtain the filter coefficients in terms of $\pi_{t,a}$ evaluated at appropriate system operators, provided that we evaluate the expectation of $d\Lambda_b^a(t)$ in a coherent state of the bath $|\phi(u)>$ as $<\phi(u)|d\Lambda_b^a(t)|\phi(u)> = \bar{u}_a(t)u_b(t)dt$. We leave this derivation as an exercise for the interested reader. The calculations are similar to those in [1].

0.5 A remark on a more general way of imposing an $\mathbb{Z}_n \times \mathbb{Z}_n$ grading on $N^2 \times N^2$ matrices when $N \geq n$

For $a \in \{0, 1, ..., N-1\}\}$, let $\rho(a) \in \{0, 1, ..., n-1\}$, i.e., $\rho : \mathbb{Z}_N \to \mathbb{Z}_n$ is a map. Define the $\mathbb{Z}_n \times \mathbb{Z}_n$ grading of the $N^2 \times N^2$ matrix E_b^a where $a, b \in \{0, 1, ..., N^2 - 1\}$ by

$$\sigma(E_b^a) = (\rho(a_1) - \rho(b_1), \rho(a_2) - \rho(b_2)) = \rho(a) - \rho(b) - - - (124)$$

with

$$\rho(a) = (\rho(a_1), \rho(a_2)), \rho(b) = (\rho(b_1), \rho(b_2)) - - - (125)$$

where

$$a = Na_1 + a_2, b = Nb_1 + b_2 - - - (126)$$

with $a_1, a_2, b_1, b_2 \in \mathbb{Z}_N = \{0, 1, ..., N-1\}$. This is a consistent grading scheme in the sense that the grading of $E_b^a.E_c^b = E_c^a$ equals

$$\sigma(E_c^a) = (\rho(a_1) - \rho(c_1), \rho(a_2) - \rho(c_2))$$

$$= (\rho(a_1) - \rho(b_1), \rho(a_2) - \rho(b_2)) + (\rho(b_1) - \rho(c_1), \rho(b_2) - \rho(c_2)) = \sigma(E_b^a) + \sigma(E_c^b) - - - (127)$$

In accordance with this definition, following the general arguments of this paper, we can define a $\mathbb{Z}_n \times \mathbb{Z}_n$ graded tensor product on the space of $N^2 \times N^2$ matrices satisfying

$$(A_1 \otimes_g A_2).(B_1 \otimes_g B_2) = \omega(\sigma(A_2), \sigma(B_1)).A_1B_1 \otimes_g A_2B_2 - - - (128)$$

where the function $\omega : (\mathbb{Z}_n \times \mathbb{Z}_n) \times (\mathbb{Z}_n \times \mathbb{Z}_n) \to \mathbb{T}$ is as defined earlier with $\mathbb{T}$ being the unit circle in $\mathbb{C}$. The idea in defining this graded tensor product is that wherever $a, b \in \mathbb{Z}_n \times \mathbb{Z}_n$ appeared in the above arguments, we replace these by $\rho(a) = (\rho(a_1), \rho(a_2))$ and likewise $\rho(b) = (\rho(b_1), \rho(b_2))$.

We also introduce the grading equal to $\rho(a) - \rho(b)$ of the quantum stochastic process $\xi_b^a(t) = \int_0^t G(t, \rho(a) - \rho(b)) d\Lambda_b^a(s)$ where

$$G(t, \rho(a) - \rho(b)) = exp((-2\pi i/n). \sum_c \phi(\rho(a) - \rho(b), \rho(c)) \Lambda_c^c(t)) - - - (129)$$

whereas earlier,

$$exp(-2\pi i \phi(\rho(a) - \rho(b), \rho(c))/n) = \omega(\rho(a) - \rho(b), -\rho(c)) = z^{-\phi(\rho(a) - \rho(b), \rho(c))} - - - (130)$$

or equivalently,

$$\phi(\rho(a) - \rho(b), \rho(c)) = (\rho(a_1) - \rho(b_1))\rho(c_2) - (\rho(a_2) - \rho(b_2))\rho(c_1) - - - (132)$$

Note that if we define the processes

$$\sum_c \rho(c_k) \Lambda_c^c(t) = \tilde{\Lambda}_k(t), k = 1, 2, c = (c_1, c_2) = Nc_1 + c_2, - - -(133)$$

then we can write

$$G(t, \rho(a) - \rho(b)) = z^{-(\rho(a_1) - \rho(b_1))\tilde{\Lambda}_2(t) + (\rho(a_2) - \rho(b_2))\tilde{\Lambda}_1(t)}, z = exp(2\pi i/n) - - - (134)$$

Note that $\tilde{\Lambda}_k, k = 1, 2$ are mutually commuting Abelian processes.

We now consider the following identity that is crucial in defining the Lie $\mathbb{Z}_n \times \mathbb{Z}_n$ graded super-algebra supercommutation relations for quantum stochastic processes obtained as representations of $N \times N$ matrices with $N \geq n$: For $s < t$,

$$G(t, \rho(a) - \rho(b))|e(u)> = |e(H(a, b)u\chi_{[0,t]} + u.\chi_{(t,\infty)}) > - - -(135)$$

where

$$H(a, b) = exp((-2\pi i/n) \sum_c \phi(\rho(a) - \rho(b), \rho(c)) E_c^c) \in \mathbb{C}^{N \times N} - - - (136)$$

is a diagonal unitary matrix. Thus, we get

$$< e(v)|d\Lambda_d^c(s) G(t, \rho(a) - \rho(b))|e(u) >= \bar{v}_d(s)(H(a, b)u(s))_c ds < e(v)|G(t, \rho(a) - \rho(b))|e(u) >$$

$$= \bar{v}_d(s) exp((-2\pi i/n)\phi(\rho(a) - \rho(b), \rho(c)))u_c(s)ds. < e(v)|G(t, \rho(a) - \rho(b))|e(u) >,$$

$$= \omega(\rho(a) - \rho(b), -\rho(c))u_c(s)\bar{v}_d(s)ds. < e(v)|G(t, \rho(a) - \rho(b))|e(u) > - - -(137),$$

on the one hand, and on the other,

$$< e(v)|G(t, \rho(a) - \rho(b))d\Lambda_d^c(s)|e(u) >=$$

$$< e(H(a, b)^* v.\chi_{[0,t]} + v.\chi_{(t,\infty)})|d\Lambda_d^c(s)|e(u) >=$$

$$(H(a,b)v(s))_d.u_c(s)ds < e(v)|G(t,\rho(a)-\rho(b))|e(u) >$$

$$= \omega(\rho(a)-\rho(b), -\rho(d))u_c(s)\bar{v}_d(s)ds. < e(v)|G(t,\rho(a)-\rho(b))|e(u) > ---(138)$$

Comparing the two expressions and using the fact that exponential vectors are total in the Boson Fock space, i.e., their linear span is dense in the Boson Fock space, we derive the following generalization of the formula derived earlier,

$$\omega(\rho(a)-\rho(b),\rho(d))G(t,\rho(a)-\rho(b))d\Lambda^c_d(s) = \omega(\rho(a)-\rho(b),\rho(c)).d\Lambda^c_d(s).G(t,\rho(a)-\rho(b))---(139)$$

or equivalently,

$$d\Lambda^c_d(s)G(t,\rho(a)-\rho(b)) = \omega(\rho(a)-\rho(b),\rho(d)-\rho(c)).G(t,\rho(a)-\rho(b)).d\Lambda^c_d(s), s < t---(140)$$

From this formula and the commutativity of the operators $G(t,\rho(a)-\rho(b)), t \geq 0, a,b \in \mathbb{Z}_N \times \mathbb{Z}_N$ (because of the commutativity of the operators $\Lambda^c_c(t), t \geq 0, c \in \mathbb{Z}_N \times \mathbb{Z}_N = \mathbb{Z}_{N^2}$), it immediately follows that

$$\xi^c_d(s)\xi^a_b(t) - \omega(\rho(d)-\rho(c),\rho(a)-\rho(b))\xi^a_b(t)\xi^c_d(s) =$$

$$(\epsilon^c_b\xi^a_d - \epsilon^a_d.\omega(\rho(d)-\rho(c),\rho(a)-\rho(b))\xi^c_b)(min(t,s)) --- (141)$$

or equivalently, if we assign the grade

$$\sigma(\xi^a_b) = \rho(a)-\rho(b) --- (142)$$

to the quantum stochastic process ξ^a_b, then we can express the above equation as

$$\xi^c_d(s)\xi^a_b(t) - \omega(\sigma(\xi^c_d),\sigma(\xi^a_b))\xi^a_b(t)\xi^c_d(s)$$

$$= (\epsilon^c_b\xi^a_d - \epsilon^a_d.\omega(\sigma(\xi^c_d),\sigma(\xi^a_b))\xi^c_b)(min(t,s)) --- (142)$$

Note the way in which we have assigned the grade of ξ^a_b implies that

$$\sigma(\xi^a_b) = \sigma(E^a_b) --- (143)$$

This is natural to expect because of our definition

$$d\xi^a_b(t) = z^{-\sum_c \phi(\rho(a)-\rho(b),\rho(c))\Lambda^c_c(t)} d\Lambda_t(E^a_b) --- (143)$$

and the fact that $\Lambda^c_c = \Lambda_t(E^c_c)$ combined with the fact that $\sigma(E^c_c) = \rho(c)-\rho(c) = (0,0) = 0$.

It should be noted that in terms of $N^2 \times N^2$ matrices A, B, it follows by multiplying the above super-Lie-algebra relation by $A^d_c B^b_a$ that we can also express it in the following compact form:

$$\xi_s(A).\xi_t(B) - \omega(\sigma(A),\sigma(B)).\xi_t(B).\xi_s(A) = \xi_{min(t,s)}(A\epsilon B - \omega(A,B).B\epsilon A)---(144)$$

where it is understood that the matrices A, B have a definite grade.

0.6 On the design of TPCP maps using superstring Hamiltonians

The free Bosonic string Hamiltonian is

$$H_B = p^2/2 + \sum_{n\geq 1} \alpha(-n).\alpha(n) - - - (145)$$

while the free Fermionic string Hamiltonian is

$$H_F = \sum_{n\geq 0} S(-n).S(n) - - - (146)$$

where

$$S(-n).S(n) = S^a(-n).S^a(n), \alpha(-n).\alpha(n) = \alpha^\mu(-n).\alpha_\mu(n) - - - (147)$$

Now we allow the Bosonic component to interact with a string gauge field potential $B_{\mu\nu}(X)$ and the superstring supercurrent $J^a = X^\mu_{,b}\rho^b\rho^a\psi_\mu$ to interact with the gravitino field χ_a. Note that by virtue of the superstring field equations

$$\partial^b\partial_b X^\mu = 0, \rho^a\partial_a\psi_\mu = 0 - - - (148),$$

we have the result that the supercurrent is conserved:

$$\partial_a J^a = 0 - - - (149)$$

which can be derived by making use of the anticommutation relations

$$\rho^b\rho^a + \rho^a\rho^b = 2\eta^{ab}, \eta = ((\eta^{ab})) = diag[1, -1] - - - (150)$$

The total Lagrangian of the superstring taking into account interaction with gravitons $e^\alpha_a(\tau, \sigma)$, with gravitinos $\chi^\alpha_a(\tau, \sigma)$ and with the string gauge field $B_{\mu\nu}(X)$ is given by (where we include the more general situation when the string moves in a curved background space-time and also the situation when the world sheet has a curved metric $h_{ab}(\tau, \sigma)$:

$$L(X, \psi) = e.(1/2)h_{ab}g_{\mu\nu}(X)X^\mu_{,a}X^\nu_{,b} + (1/2)B_{\mu\nu}(X)\epsilon_{ab}X^\mu_{,a}X^\nu_{,b}.e$$

$$+\psi^{\mu T}\rho^0\rho^a\psi_{\mu,a}.e + g_{\mu\nu}(X)\chi^T_a X^\mu_{,b}\rho^0\rho^b\rho^a\psi^\nu.e - - - (160)$$

where

$$e = e(\tau, \sigma) = \sqrt{-h}, h = det((h_{ab})), \psi_\mu = g_{\mu\nu}\psi^\nu - - - (161)$$

It should be noted that in the case when space-time is flat, $g_{\mu\nu}(X) = \eta_{\mu\nu}$, then for the gravitino-interaction term, we get the supercurrent

$$J^a = X^\mu_{,b}\rho^0\rho^b\rho^a\psi_\nu - - - (162)$$

which is conserved for free strings:

$$\partial_a J^a = 0 - - - (163)$$

(assuming that by means of an appropriate world-sheet reparametrization and conformal transformation, we have got $h_{ab} = \eta_{ab}$ so that $e = 1$), in view of the free string equations

$$\eta^{ab} X^\mu_{,ab} = 0, \rho^a \partial_a \psi_\nu = 0 - - - (164)$$

because

$$\rho^a \rho^b + \rho^b \rho^a = 2\eta^{ab}.I_2 - - - (165)$$

In the curved space-time scenario, the supercurrent would be given by the gravitino interaction term as

$$J^a = g_{\mu\nu}(X) X^\mu_{,b} \rho^0 \rho^b \rho^a \psi^\nu - - - (166)$$

which would again be conserved because of the free field because now the string equations are

$$\eta^{ab} \partial_a (g_{\mu\nu} X^\nu_{,b}) = 0, \rho^a \partial_a \psi_\nu = 0 - - - (167)$$

Note that the spin connection of the world sheet can be taken as zero because it is two-dimensional. This means that we can replace a world sheet covariant derivative

$$\nabla_a \psi_\mu = (\partial_a + \omega_a)\psi_\mu, a = 0, 1 - - - (168)$$

by $\partial_a \psi_\mu$ where $\omega_a(\tau, \sigma)$ is a 2×2 matrix. This is because, the two-dimensional world sheet metric, by means of an appropriate reparametrization, can be brought to the form $\phi(\tau, \sigma) diag[1, -1]$ which means that the term $\psi^T \rho^0 \rho^a \omega_a \psi$ in the Fermionic Lagrangian $\psi^T \rho^0 \nabla_a \psi = \psi^T \rho^0 \rho^a (\partial_a + \omega_a)\psi$ can be replaced by $\psi^T \omega \psi$ where ω is a symmetric matrix and since ψ anti-commutes with itself, we have $\psi^T \omega \psi = 0$.

In fact, we note that the worldsheet spin connection

$$\omega_\alpha = (1/2) e^{a\beta} e^b_{\beta:\alpha} \rho_{ab}, \rho_{ab} = [\rho_a, \rho_b], a, b = 0, 1 \rho_0 = \sigma_2, \rho_1 = i\sigma_1 - - - (169)$$

is simply a scalar field times $\rho_{01} = 2\sigma_3$ and hence the spin connection contribution to the Fermionic field action given by

$$e^\alpha_a \psi^{\mu T} \rho^0 \rho^a \omega_\alpha \psi_\mu - - - (170)$$

vanishes because $\rho^0 \rho^0 \omega_\alpha$ and $\rho^0 \rho^1 \omega_\alpha$ are respectively proportional to σ_3 and I_2, both of which are symmetric matrices and $\psi^{\mu T} A \psi_\mu = 0$ for any symmetric matrix A owing to the Fermi statistics satisfied by the Fermionic wave function.

We seek to evaluate the Hamiltonian density corresponding to this Lagrangian density. The canonical momentum fields $P^X_\mu(\tau, \sigma), P^\psi_\mu(\tau, \sigma)$ that are conjugate to the canonical position fields $X^\mu(\tau, \sigma)$ and $\psi^\mu(\tau, \sigma)$ are given by

$$P^X_\mu = \partial L / \partial X^\mu_{,0}, P^\psi_\mu(\tau, \sigma) = \partial L / \partial \psi^\mu_{,0} - - - (171)$$

We leave it as an exercise in algebra to evaluate these and hence the superstring Hamiltonian.

0.7 String field theory

The quantum master equation for the string action S is given by

$$2ih\Delta S + (S, S) = 0 - - - (172)$$

where the Laplacian Δ is calculated in field space. This equation can be derived from the hypothesis of the gauge invariance of the quantum action. Specifically, let $S[\chi, \chi^*]$ denote the action. Here, the antifield $\chi^* = \delta\psi[\chi]/\delta\chi$ where ψ is a Fermionic functional of the fields χ. We require that the path integral which is used to compute the quantum effective action

$$\int exp(iS/h)D\chi - - - (173)$$

be gauge invariant, i.e., it should not change when the gauge fixing functional ψ is changed by a small amount $\delta\psi$. Noting that $S = S[\chi, d\psi[\chi]/d\chi], \chi^* = d\psi[\chi]/d\chi$, this means that

$$\int exp(iS/h)(\delta S/\delta\chi^*).(d\delta\psi[\chi]/d\chi)D\chi = 0 - - - (174)$$

Integrating this by parts in field space, this gives

$$\int d/d\chi(exp(iS/h)(\delta S/\delta\chi^*))\delta\psi[\chi]D\chi = 0 - - - (175)$$

which gives on expanding,

$$\int((i/h)(\delta S/\delta\chi).(\delta S/\delta\chi^*) + \delta^2 S/\delta\chi.\delta\chi^*))exp(iS/h)\delta\psi[\chi]D\chi = 0 - - - (176)$$

which yields the quantum master equation for the action S:

$$-ih\Delta S + (\delta S/\delta\chi).(\delta S/\delta\chi^*) = 0 - - - (177)$$

which actually means

$$-ih\int(\delta^2 S/\delta\chi(x)).\delta\chi^*(x))d^4x + \int(\delta S/\delta\chi(x)).(\delta S/\delta\chi^*(x))d^4x = 0 - - - (178)$$

The various solutions to this equation yield the admissible class of actions, namely, only those actions for which the amplitudes computed using path integrals in field space do not depend on the choice of the gauge fixing functional.

Remark: Note that in fact,

$$(d/d\chi)(exp(iS/h)\delta S/\delta\chi^*) = exp(iS/h)((i/h)\delta S/\delta\chi + (i/h)(\delta S/\delta\chi^*).(\delta^2\psi/\delta\chi.\delta\chi)).\delta S/\delta\chi^*$$

$$= exp(iS/h).(i/h)(\delta S/\delta\chi).(\delta S/\delta\chi^*) - - - (179)$$

since the contribution of the term

$$(\delta S/\delta\chi^*).(\delta^2\psi/\delta\chi.\delta\chi).(\delta S/\delta\chi^*) - - - (180)$$

vanishes. Note that $\chi^* = \delta\psi[\chi]/\delta\chi$ is Fermionic if χ is Bosonic and is Bosonic if χ is Fermionic, i.e., χ and χ^* have opposite statistics and $\delta^2\psi/\delta\chi.\delta\chi$ is antisymmetric if χ is Bosonic and is symmetric if χ is Fermionic. This equation should be read as

$$\int (\delta S/\delta\chi^*(y)).(\delta^2\psi/\delta\chi(x).\delta\chi(y)).(\delta S/\delta\chi^*(x)d^4xd^4y = 0 - - - (181)$$

To see this, we note that if χ is Bosonic, then χ^* is Fermionic and hence $\delta S/\delta\chi^*$ is Fermionic which means that $\delta S/\delta\chi * (x)$ anticommutes with $\delta S/\delta\chi^*(y)$ while $\delta^2\psi/\delta\chi(x).\delta\chi(y)$ is a symmetric kernel. On the other hand, if χ is Fermionic, then $\delta^2\psi/\delta\chi(x).\delta\chi(y)$ is an antisymmetric kernel while since now χ^* is Bosonic, $\delta S/\delta\chi^*$ is Bosonic and hence $\delta S/\delta\chi^*(x)$ commutes with $\delta S/\delta\chi * (y)$. So in both cases, the above integral vanishes, proving the validity of the quantum master equation, namely the equation that any action must satisfy in order that the quantum mechanical amplitudes computed using it should be independent of the choice of the gauge fixing functional.

String field theory is based on selecting an appropriate action for the string field namely for the wave functional of the Bosonic string and the Fermionic string. Such a choice of action starts with the BRST equation

$$Q\Psi(X,S) = 0 - - - (182)$$

where Q superstring BRST super charge operator (Green Schwarz and Witten, Superstring theory vol.1), X is the Bosonic string function $X(\sigma), \sigma \in [0, 2\pi)$ and S is the Fermionic string function $S(\sigma), \sigma \in [0, 2\pi)$. Specifically, we expand the Bosonic string in terms of the Bosonic creation and annihilation operators and construct the position operator q(n) corresponding to each creation-annihilation pair. Likewise, we expand the Fermionic string in terms of the Fermionic creation and annihilation operator and retain only the annihilation operators. Denoting the Bosonic position operator sequence by q and the Fermionic annihilation operators by S, we observe that q forms a complete set of commuting observables for the Bosonic string and S form a complete set of anticommuting variables for the Fermionic string and just as in superfield theories in supersymmetry, the string field wave function Ψ is a function of q, S, i.e. $\Psi(q, S)$ and this function can be expanded as

$$\Psi(q,S) = \Psi(q(0), q(1), q(2), q(3), , S_0, S_1, S_2, S_3, ...) = \sum_{0 \leq n_0 < n_1 < n_2 < ...} \Psi(q|n_0, n_1, n_2, ...)S_{n_0}S_{n_1}S_{n_2}... - - - (183)$$

This field satisfies in the unperturbed situation, the superstring BRST equation

$$Q\Psi(q,S) = 0 - - - (184)$$

This equation can be derived from the action principle

$$\delta < \Psi|Q|\Psi >= 0 - - - (185)$$

In order to describe interactions of the string field that lead to appropriate values of quantum mechanical scattering amplitudes of the superstring, add higher degree terms to the quadratic action $< \Psi|Q|\Psi >$ like $\sum_{n\geq 3,c_1,...,c_1=0,1} < W_n(c_1,...,c_n)|\Psi^{c_1}\otimes ...\Psi^{c_n} >$ where $\Psi^0 = \Psi, \Psi^1 = \Psi^*$. The coefficients W_n appearing here are subject to the constraint that the total string field action should satisfy the quantum master equation.

Remark:It should be mentioned that during the process of quantization of the Bosonic string using path integrals, we have to introduce reparametrization Fermionic ghost fields $c(\sigma), b(\sigma)$ appearing in a ghost action in order to take into account the determinant contribution coming from the path integral over the world sheet metric. From the ghost action, it follows that these ghost fields satisfy the CAR in which the $c's$ all mutually anticommute and so do the $b's$, but the c does not anticommute with the $b's$, in fact, their anticommutator is a delta function. The Fourier series expansion of c field thus yields mutually anticommuting operators $c(n), n \geq 0$. Likewise while quantization of the Fermionic string, we are forced to introduce Bosonic ghost fields $\beta(\sigma), \gamma(\sigma)$ appearing in a ghost action in order to take into account the determinant combing from Berezin path integration over the worldsheet gravitino. From the ghost field action, again it follows that these ghost fields satisfy CCR in which the $\beta's$ mutually anticommute and so do the $\gamma's$, but the commutator of the β field with the γ field is a delta function. Thus, if $\beta(n)$ are the Fourier series coefficients of the β field, these operators mutually commute. It follows then that the total Lagrangian and hence Hamiltonian of the superstring contains in addition Fermionic and Bosonic ghost terms. In short, the string field wave function $\Psi(q,S)$ should be such that q includes apart from the Bosonic string position operators, also the Bosonic ghosts $\beta(n), n \geq 0$ coming from Fermionic string quantization and S includes apart from the Fermionic string annihilation operators, also the Fermionic ghosts coming from Bosonic string quantization. Indeed, the nilpotent BRST supercharge operator Q is a quadratic function of the Bosonic and Fermionic string amplitudes and a cubic function of the Fermionic and Bosonic ghosts.

Remark: A physical state $|\psi >$ is characterized by the fact that it satisfies $Q|\psi >= 0$ where Q is the BRST operator. This equation generalizes the requirement that the action be invariant under gauge transformations of the fields. The generalization is required because we are required to accommodate ghost fields into the action based on the Faddeev-Popov method in order to replace determinants associated with the choice of a gauge fixing functional so that the path integral is invariant under the choice of the gauge fixing functional. Now the BRST operator Q is nilpotent, specifically, $Q^2 = 0$. Physical states $|\psi>$ are characterized by BRST invariance, i.e., $Q|\psi>=0$. Moreover, a state $|\psi>$ should be equivalent to any state of the form $|\psi>+Q|\chi>$ where $|\chi>$ is any state. Indeed $Q(|\psi>+Q|\chi>) = Q|\psi>$ because $Q^2 = 0$. In other words, physical states are characterized by the fact that they are elements of the Q-cohomology. Q-cohomology consists of the equivalence class of all states annihilated by Q modulo the range of Q. An observable O is said to be BRST-invariant if $[Q, O] = 0$. If $|\psi_k>, k = 1, 2$ are physical states, then the matrix elements of any observable O w.r.t these states are BRST invariant because $<\psi_1|O|\psi_2>$ changes to $<\psi_1|[Q, O]|\psi_2>=0$ since $<\psi_1|Q = 0, Q|\psi_2>=0$. Moreover, if O is any BRST invariant operator, i.e., $QO = OQ$, then its matrix elements w.r.t physical states $|\psi_k>, k = 1, 2$ depends only on the cohomology classes to which $|\psi_1>, |\psi_2>$ belong. This is because

$$< \psi_1+Q\chi_1|O|\psi_2+Q\chi_2 >=< \psi_1|O|\psi_2 > + < \chi_1|QO|\psi_2 > + < \psi_1|OQ|\chi_2 > + < \chi_1|QOQ|\chi_2 >$$

$$=< \psi_1|O|\psi_2 > + < \chi_1|OQ|\psi_2 > + < \psi_1|QO|\chi_2 > + < \chi_1 OQ^2|\chi_2 >=< \psi_1|O|\psi_2 > ---(186)$$

Since matrix elements of observables w.r.t physical states are all that we measure, it follows that BRST invariant observables and the Q-cohomology of states are all that matter in a valid physical theory wherein experiments can be performed, observables measured and transition probabilities computed.

0.8 Quantum noise in string theory

Let $X^\mu(\tau, \sigma)$ denote the Bosonic string and $\psi^\mu(\tau, \sigma)$ the Fermionic string. The Bosonic string is assumed to interact with a quantum noisy string gauge potential $B_{\mu\nu}(\tau, X)$ in accordance with the interaction Lagrangian

$$(1/2)\int \epsilon(ab)B_{\mu\nu}(\tau, X(\tau, \sigma))X^\mu_{,a}(\tau, \sigma)X^\nu_{,b}(\tau, \sigma)d\sigma - - - (187)$$

while the Fermionic string is assumed to interact with a quantum noisy gauge potential $A_a(\tau, \sigma)$ in accordance with the interaction Lagrangian

$$\int \psi^{\mu T}(\tau, \sigma)\rho^0\rho^a\psi_\mu(\tau, \sigma)A_a(\tau, \sigma)d\sigma - - - (188)$$

We write

$$B_{\mu\nu}(\tau, X) = \sum_{n\geq 1}[W_n(\tau)B_{n\mu\nu}(X) + W_n(\tau)^*B_{n\mu\nu}(X)^*] - - - (189)$$

where the $B_{n\mu\nu}$ are given basis functions on the string state space $\mathbb{R}^d$. $W_n(\tau)$ equals the time derivative of the annihilation process appearing in the Hudson-Parthasarathy (HP) quantum stochastic calculus and $W_n(\tau)^*$ is its adjoint, namely, the time derivative of the creation process appearing in the HP quantum stochastic calculus. They satisfy the HP quantum Ito formula:

$$W_n(\tau)W_m(\tau)^* = (1/d\tau)\delta[n-m] - - - (190)$$

More generally, we can write for time discretization step sizes $d\tau$,

$$W_n(\tau)W_m(s)^* = \delta_{\tau,s}(d\tau)^{-1}\delta[n-m] = \delta(\tau-s)\delta[n-m] - - - (191)$$

Writing the annihilation process as

$$A_n(\tau) = \int_0^t W_n(s)ds - - - (192),$$

we see that we can write the above equation as

$$dA_n(\tau).dA_m(s)^* = \delta_{\tau,s}d\tau.\delta[n-m] = \delta(\tau-s)\delta[n-m]d\tau.ds - - - (193)$$

The interaction Lagrangian of the Bosonic string with the noisy string gauge field can now be expressed as

$$-V_1(\tau) = (1/2)\int \epsilon(ab)B_{\mu\nu}(\tau, X(\tau,\sigma))X^\mu_{,a}(\tau,\sigma)X^\nu_{,b}(\tau,\sigma)d\sigma$$

$$= (1/2)\sum_n [W_n(\tau)\int \epsilon(ab)B_{n\mu\nu}(X(\tau,\sigma))X^\mu_{,a}(\tau,\sigma)X^\nu_{,b}(\tau,\sigma)d\sigma + H.c] - - - (194)$$

Likewise, writing the noisy gauge potential A_a as

$$A_a(\tau,\sigma 0 = \sum_n [W_n(\tau)A_{na}(\sigma) + H.c] - - - (195)$$

where A_{na} is again a set of basis functions of the string length parameter, we can write for the interaction Lagrangian of the Fermionic string with this noisy gauge potential as

$$-V_2(\tau) = \int \psi^{\mu T}(\tau,\sigma)\rho^0\rho^a\psi_\mu(\tau,\sigma)A_a(\tau,\sigma)d\sigma$$

$$= \sum_n W_n(\tau)\int \psi^{\mu T}(\tau,\sigma)\rho^0\rho^a\psi_\mu(\tau,\sigma)A_{na}(\sigma)d\sigma + H.c - - - (196)$$

Our aim is to compute the corrections to the superstring Hamiltonian and propagator caused by these quantum noisy effects. First, however, we observe that corresponding to the canonical Bosonic position field $X^\mu(\tau,\sigma)$, the canonical momentum field is

$$P^X_\mu(\tau,\sigma 0 = \partial L/\partial X^\mu_{,0} = X_{\mu,0}(\tau,\sigma) + B_{\mu\nu}(\tau, X(\tau,\sigma))X^\nu_{,1}(\tau,\sigma) - - - (197)$$

and corresponding to the canonical Fermionic position field $\psi^\mu(\tau,\sigma)$, the canonical Fermionic momentum field is

$$P^\psi_\mu(\tau,\sigma) = \partial L/\partial \psi^\mu_{,0} = i\psi_\mu(\tau,\sigma) - - - (198)$$

The superstring Hamiltonian density obtained by the Legendre transformation is then

$$\mathcal{H} =$$

$$P_\mu^X X_{,0}^\mu + P_\mu^\psi \psi_{,0}^\mu - L =$$

$$(1/2)[X_{,0}^\mu X_{\mu,0} + X_{,1}^\mu X_{\mu,1}] - \psi^{\mu T}\sigma_3\psi_{\mu,1} - i\psi^{\mu T}\rho^0\rho^a\psi_\mu.A_a - - - (199)$$

In order to calculate the correction to the superstring propagator, we must first solve for the Bosonic and Fermionic string equations and express these solutions in terms of the free Bosonic and Fermionic string creation and annihilation operators and the quantum noise processes W_n, W_n^*. The Bosonic string equations are

$$X_{\mu,00} - X_{\mu,11} - \epsilon(ab)B_{\mu\nu}(\tau, X)_{,a}X_{,b}^\nu = 0 - - - (200)$$

while the Fermionic string equations are

$$[\partial_0 + A_0 + \sigma_3(\partial_1 + A_1)]\psi^\mu = 0 - - - (201)$$

We shall assume for simplicity that $B_{n\mu\nu}(X) = H_{n\mu\nu\rho}X^\rho$ where $H_{n\mu\nu\rho}$ are constants. This assumption amounts to assuming a constant string gauge field over the length scale of the string. Note that $\rho^0 = \sigma_2, \rho^1 = i\sigma_1$ so $\rho^0\rho^1 = \sigma_3$ and $(\rho^0)^2 = 1, (\rho^1)^2 = -1, \rho^0\rho^1 + \rho^1\rho^0 = 0$ which can be expressed in compact notation as

$$\rho^a\rho^b + \rho^b\rho^a = 2\eta^{ab} - - - (202)$$

The Bosonic string equations can be expressed as

$$X_{\mu,00}(\tau,\sigma) - X_{\mu,11}(\tau,\sigma) + [H_{n\mu\nu\rho}[W_n'(\tau)X^\rho(\tau,\sigma)X_{,1}^\nu(\tau,\sigma) +$$

$$W_n(\tau)(X_{,0}^\rho X_{,1}^\nu - X_{,1}^\rho X_{,0}^\nu)(\tau,\sigma))] + H.c.] = 0 - - - (203)$$

with the summation convention over the repeated indices n, ν, ρ being understood. Up to linear orders in the noise amplitudes W_n, W_n^*, the solution to this differential equation is

$$X_\mu(\tau,\sigma) = \sum_n a_\mu(n)exp(in(\tau-\sigma)) + \sum_n b_\mu(n,\tau)exp(-in\sigma) - - - (204),$$

where

$$\partial_\tau^2 b_\mu(n,\tau) + n^2 b_\mu(n,\tau) = f_\mu(n,\tau) - - - (205),$$

with

$$f_\mu(n,\tau) = (1/2\pi)\int_0^{2\pi} exp(in\sigma).f_\mu(\tau,\sigma) - - - (206),$$

where

$$f_\mu(\tau,\sigma) =$$

$$-\sum_m [H_{m\mu\nu\rho}[W_m'(\tau)X_0^\rho(\tau,\sigma)X_{0,1}^\nu(\tau,\sigma) +$$

$$W_m(\tau)(X_{0,0}^\rho X_{0,1}^\nu - X_{0,1}^\rho X_{0,0}^\nu)(\tau,\sigma))] + H.c.] - - - (207)$$

where

$$X_0(\tau,\sigma) = \sum_n a_\mu(n)exp(in(\tau-\sigma)) - - - (208)$$

Since we are assuming only right moving unperturbed string field, the term $X_{0,0}^\rho X_{0,1}^\nu - X_{0,1}^\rho X_{0,0}^\nu)(\tau,\sigma)$ vanishes and we get

$$f_\mu(\tau,\sigma) = -\sum_m [H_{m\mu\nu\rho}.W'_m(\tau).X_0^\rho(\tau,\sigma)X_{0,1}^\nu(\tau,\sigma)$$

$$= i\sum_{m,k,n} H_{m\mu\nu\rho}W'_m(\tau)a^\rho(k)a^\nu(n-k)exp(in(\tau-\sigma)) + H.c. - - - (209)$$

This gives

$$f_\mu(n,\tau) = i\sum_{m,k} H_{m\mu\nu\rho}W'_m(\tau)a^\rho(k)a^\nu(n-k)exp(in\tau)+H.c.(n <---> -n)---(210)$$

We note that the equation

$$\partial_\tau^2 b_\mu(n,\tau) + n^2 b_\mu(n,\tau) = f_\mu(n,\tau) - - - (211)$$

has the solution

$$b_\mu(n,\tau) = \int_0^\tau \frac{sin(n(\tau-s))}{n(\tau-s)} f_\mu(n,s)ds - - - (212)$$

if $n \neq 0$, while if $n = 0$, it has the solution

$$b_\mu(0,\tau) = \int_0^\tau (\tau-s)f_\mu(0,s)ds - - - (213)$$

This analysis of noise in quantum string theories can be cast into a more general framework as follows: Let $\phi_k(t,x), k = 1,2,...,N$ be a set of Bosonic fields and $\psi_k(t,x), k = 1,2,...,M$ a set of Fermionic fields on the space-time manifold $\mathbb{R}\times\mathbb{R}^n$, i.e., $t \in \mathbb{R}, x \in \mathbb{R}^n$. Assume that the total Lagrangian density of these fields is

$$L(\phi_k, \psi_m, \psi_m^*, \phi_{k,\mu}, \psi_{m,\mu}) - - - (214)$$

where $\mu = 0,1,...,n$ with $x^0 = t, x = (x^1,...,x^n)$ denoting respectively the time and spatial coordinates. Assume that this Lagrangian density has the special form

$$L = L_0(\phi(t,x),\phi_{,\mu}(t,x)) + \sum_{k,m}\psi_k(t,x)^*[L_{1km}^\mu(\phi(t,x),\phi_{,\nu}(t,x))\partial_\mu + L_{2km}(\phi(t,x),\phi_{,\nu}(t,x))]\psi_m(t,x)---(215)$$

We now add quantum Bosonic noise and quantum Fermionic noise to this Lagrangian density by replacing $\phi(t,x)$ with

$$\phi(t,x) + \sum_k (W_k(t)\chi_k(x) + W_k(t)^*\chi_k(x)^*) - - - (216)$$

an $\psi(t,x)$ with

$$\psi(t,x)+\sum_k V_k(t)\eta_k(x)---(217)$$

so that $\psi(t,x)^*$ gets replaced with

$$\psi(t,x)^*+\sum_k V_k(t)^*\eta_k(x)^*---(218)$$

where $W_k(t)=A'_k(t)$ is white Bosonic annihilation noise, i.e., the time derivative of the annihilation process appearing in the Hudson-Parthasarathy quantum stochastic calculus while $V_k(t)=J'_k(t)$ is Fermionic annihilation noise, i.e., the time derivative of the Fermionic annihilation process again appearing in the Hudson-Parthasarathy quantum stochastic calculus. Note that

$$W_k(t)^*W_m(t)=A'_k(t)^*A'_m(t)=d\Lambda_k^m(t)/dt---(219)$$

is the time derivative of the Bosonic conservation processes in the Hudson-Parthasarathy theory and likewise

$$V_k(t)^*V_m(t)=J'_k(t)^*J'_m(t)=d\tilde{\Lambda}_k^m(t)/dt---(220)$$

is again a conservation process associated with the Fermions. Expanding the total Lagrangian density after making these noise substitutions up to second order in the noise amplitudes W_k, W_k^*, V_n, V_m^* gives us

$$L=L_0(\phi(t,x),\phi_{,\mu}(x))+\sum_k[L_{0k}(x,\phi(t,x),\phi_{,\mu}(t,x))W_k(t)+H.c]$$

$$+\sum_{km}L_{0km}(x,\phi(t,x),\phi_{,\mu}(t,x))d\Lambda_k^m(t)/dt$$

$$+\psi(t,x)^*[L_1^\mu(\phi(t,x),\phi_{,\nu}(t,x))\partial_\mu+L_2(\phi(t,x),\phi_{,\nu}(t,x))]\psi(t,x)$$

$$+\sum_k V_k(t)^*[\eta_k(x)^*[L_1^\mu(\phi(t,x),\phi_{,\nu}(t,x))\partial_\mu+L_2(\phi(t,x),\phi_{,\nu}(t,x))]\psi(t,x)$$

$$+\sum_k\psi(t,x)^*[L_1^\mu(\phi(t,x),\phi_{,\nu}(t,x))\partial_\mu+L_2(\phi(t,x),\phi_{,\nu}(t,x))]\eta_k(x)V_k(t)$$

$$+\sum_{km}\eta_k(x)^*[L_1^\mu(\phi(t,x),\phi_{,\nu}(t,x))\partial_\mu+L_2(\phi(t,x),\phi_{,\nu}(t,x))]\eta_m(x).d\tilde{\Lambda}_k^m(t)/dt$$

$$+\sum\psi(t,x)^*[W_k(t)(L_{1k}^\mu(x,\phi(t,x),\phi_{,\nu}(t,x))\partial_\mu+L_{2k}(x,\phi(t,x),\phi_{,\nu}(t,x))$$

$$+(d\Lambda_k^m(t)/dt)(L_{1km}^\mu(x,\phi(t,x),\phi_{,\nu}(t,x))\partial_\mu+L_{2km}(x,\phi(t,x),\phi_{,\nu}(t,x)))+H.c.]\psi(t,x)---(221)$$

where

$$L_{0k}(x,\phi(t,x),\phi_{,\mu}(t,x))=\sum_s[(\partial L_0(\phi(t,x),\phi_{,\mu})(t,x))/\partial\phi_s)\chi_{ks}(x)+(\partial L_0(\phi(t,x),\phi_{,\mu}(t,x))/\partial\phi_{s,\mu})\chi_{ks,\mu}(x)---$$

As an example of this, we can apply this formalism of adding quantum noise to the action for the gravitational field in general relativity interacting with the electromagnetic field of photons and the Dirac field of electrons and positrons. Let $\phi(t,x)$ denote the spatial components of the covariant metric tensor of space-time. In a synchronous frame, the Lagrangian density of the gravitational field has the form

$$(1/2)\phi_{,0}^T A_1(\phi)\phi_{,0} - (1/2)(\nabla\otimes\phi)A_2(\phi)(\nabla\otimes\phi) - - - (223)$$

where $A_1(\phi)$ and $A_2(\phi)$ are respectively complicated 6×6 and 18×18 matrix-valued functions of $\phi(t,x)$ (ie, these matrix-valued functions do not contain any space-time derivatives of ϕ).

Adding noise to the gravitational field $\phi(t,x)$ involves replacing $\phi(t,x)$ with $\phi(t,x)+\sum_n W_n(t).\chi_n(x)$.

0.9 Quantum noise associated with particles having $\mathbb{Z}_n\times\mathbb{Z}_n$ graded spins as a generalization of the situation for $\mathbb{Z}_2$ involving only Bosons and Fermions

We first recall some of the things discussed earlier.

We define the graded tensor product between the system and noise observables in such a way that

$$L\otimes W = \omega(\sigma(L),\sigma(W))W\otimes L$$

where if $\sigma(L)=(a,b)$ and $\sigma(W)=(c,d)$ with $a,b,c,d\in\mathbb{Z}_n$, then

$$\omega(\sigma(L),\sigma(W)) = z^{ad-bc}$$

with z being a primitive n^{th} root of unity, for example $z=exp(2\pi i/n)$. Assuming $W_b^a(t)dt = G(t,a-b)d\Lambda_b^a(t)$ where the noise grading operator $G(t,a-b)$ is defined by

$$G(t,a-b) = z^{-\sum_c \phi(a-b,c)\Lambda_c^c(t)}$$

with $\phi(a,b)=a_1b_2-a_2b_1$ where $a\in\mathbb{Z}_{n^2}$ is identified with $(a_1,a_2)\in\mathbb{Z}_n\times\mathbb{Z}_n$ via the formula

$$a = na_1+a_2$$

and likewise for b,c,d etc. Noting that

$$\Lambda_c^c(t) = \lambda(E_c^c\chi_{[0,t]})$$

and hence

$$exp(i\alpha.\Lambda_c^c(t))|e(u)>=|e(exp(i\alpha)E_c^c u\chi_{[0,t]}+u\chi_{(t,\infty)})>$$

and hence,

$$G(t, a-b)|e(u)>=|e(exp(iK(a-b))u.\chi_{[0,t]}+u.\chi_{(t,\infty)})>$$

where

$$K(a-b)=exp((2\pi i/n).\sum_c \phi(a-b,c)E_c^c)=\Pi_c\omega(a-b,c)^{E_c^c}=diag[\omega(a-b,c):c\in\mathbb{Z}_n\times\mathbb{Z}_n]$$

It is now easy to prove that for $s<t$, (by forming the matrix element w.r.t. $<e(v)|$ and $|e(u)>$) that

$$G(t,a-b)d\Lambda_d^c(s)=\omega(a-b,c-d)d\Lambda_d^c(s).G(t,a-b)$$

[Note that $a-b=(a_1-b_1,a_2-b_2), c-d=(c_1-d_1,c_2-d_2)$ and

$$\omega(a,b)=z^{a_1b_2-a_2b_1}$$

so

$$\omega(a-b,c-d)=z^{(a_1-b_1)(c_2-d_2)-(a_2-b_2)(c_1-d_1)}]$$

and hence the processes

$$\xi_b^a(t)=\int_0^t G(s,a-b)d\Lambda_b^a(s)$$

satisfy a $\mathbb{Z}_n\times\mathbb{Z}_n$ Lie algebra supercommutation relation:

$$\xi_b^a(t).\xi_d^c(s)-\omega(a-b,c-d)\xi_d^c(s).\xi_b^a(t)=\epsilon_d^a.\xi_b^c(min(t,s))-\omega(a-b,c-d).\epsilon_b^c.\xi_d^a(min(t,s))$$

Owning to these quantum stochastic processes satisfying a $\mathbb{Z}_n\times\mathbb{Z}_n$ Lie algebra, it is tempting to use such noise processes as a model for noise in quantum field theory. Specifically, let us assume that we have a set of $\mathbb{Z}_n\times\mathbb{Z}_n$ graded quantum fields $\phi_b^a(t,x), a,b\in\mathbb{Z}_n\times\mathbb{Z}_n$. Let us assume that the grade of the field ϕ_b^a is $a-b\in\mathbb{Z}_n\times\mathbb{Z}_n$. Then, the grade of $\phi_b^a(t,x)\otimes d\xi_a^b(t)$ is zero (this differential is therefore a Bosonic noise differential), and hence, given a Bosonic field $\chi(t,x)$, we can think of adding the quantum noise field $\phi_b^a(t,x)\otimes d\xi_a^b(t)/dt$ to this Bosonic field and thereby alter the Hamiltonian density $\mathcal{H}(\chi(t,x),\nabla\chi(t,x),P(t,x))$ to $\mathcal{H}(\chi(t,x)+\phi_b^a(t,x)W_a^b(t),\nabla\chi(t,x)+\nabla\phi_b^a(t,x)W_a^b(t),P(t,x))$ where $W_b^a(t)=d\xi_b^a(t)/dt$. For example, it is instructive to start with the charged Klein-Gordon Lagrangian density

$$L(\phi,\phi^*,\phi_{,\mu},\phi^*_{,\mu}))=(D^\mu\phi)^*(D_\mu\phi)-m^2\phi^*\phi, D_\mu=\partial_\mu+igV_\mu---(225)$$

calculate the momentum densities

$$P=\partial L/\partial\phi_{,0}, P^*=\partial L/\partial\phi^*_{,0}---(226)$$

and then the Hamiltonian density

$$\mathcal{H}(\phi,\phi^*,\nabla\phi,\nabla\phi^*,P,P^*)=P.\phi+P^*\phi^*-L---(227)$$

and then add quantum noise to the position fields ϕ, ϕ^* and then write down the Hudson-Parthasarathy quantum stochastic differential equation corresponding to the Hamiltonian $\int \mathcal{H} d^3x$. It should be noted that in this example, only linear and quadratic terms in the noise will appear in the Hamiltonian and quadratic terms can again be expressed as linear terms using the quantum Ito formula. In the general case, when we expand the Hamiltonian density in powers of the noise $d\xi_b^a(t)/dt$, we will obtain terms such as $d\xi_{b_1}^{a_1}(t)...d\xi_{b_n}^{a_n}(t)/dt^n$ which can again, using quantum Ito's formula, be expressed as linear terms in $d\xi_b^a(t)$. The denominator terms $1/dt^n$ can be replaced by $1/\Delta^n$ where Δ is a finite time discretization step size. This is just like introducing ultraviolet and infrared cutoffs in quantum field theory while evaluating Feynman diagrams.

Remark: Let X be observable in the system Hilbert space (i.e. the quantum field Bosonic Fock space). We wish to express the product $\phi_b^a d\xi_a^b.X$ in a form in which the noise differential appears to the right of all the system observables. Note that this product is to be interpreted as $(\phi_b^a \otimes d\xi_a^b).(X \otimes I)$ and we know from the basic theory of the graded tensor product that if X has a definite grading $\sigma(X) = (c, d)$, then this product equals

$$(\phi_b^a \otimes d\xi_a^b).(X \otimes I) = \omega(d\xi_a^b, \sigma(X))(\phi_b^a X \otimes d\xi_a^b) = \omega(d\xi_a^b, \sigma(X))(\phi_a^b X)d\xi_a^b$$

$$= \omega(b-a, c-d)\phi_a^b X.d\xi_a^b = z^{(b_1-a_1)(c_2-d_2)-(b_2-a_2)(c_1-a_1)}\phi_a^b.X.d\xi_a^b \quad --- (228)$$

0.10 Introducing quantum noise into quantum gravity theory in a consistent way

Let $\mathcal{H}(Q, \nabla Q, P)$ be the Hamiltonian density of a quantum field $Q(t, x)$. The Hamiltonian field equations are

$$\partial_t Q(t,x) = -\partial_P mathcal H(Q(t,x), \nabla Q(t,x), P(t,x)),$$

$$\partial_t P(t,x) = -\nabla_Q \mathcal{H}(t, Q(t,x), \nabla Q(t,x), P(t,x)) + div \nabla_{\nabla Q} \mathcal{H}(Q(t,x), \nabla Q(t,x), P(t,x)) --- (229)$$

When we seek to introduce a quantum noisy field into this theory, we cannot alter the momentum field because, by analogy with Newtonian mechanics, the first equation is a generalization of $\partial_t Q = P/m$ which merely states the relationship between the velocity field and the momentum field. However, in the more general case, when there is a vector potential $A(t, Q)$, the velocity equation in Newtonian mechanics gets modified to

$$\partial_t Q = (P + eA(t, Q))/m --- (230)$$

and we can alter this equation by adding a noise term to P owing to the fact that such an addition is equivalent to adding a noisy term to the vector potential. However, we shall assume that such a vector potential field function of the quantum field $Q(t, x)$ is not present, although, in the case of gravity in general relativity, it is present. Indeed, if we describe the metric tensor by six independent position fields $\phi(t, x)$, then the gravitational Lagrangian density has the form

$$L(\phi, \partial_\mu \phi) = (1/2)(\partial_t \phi)^T A_0(\phi)(\partial_t \phi) + \partial_t \phi^T A_1(\phi)(\nabla \otimes \phi)$$

$$-(1/2)(\nabla \otimes \phi)^T A_2(\phi).(\nabla \otimes \phi) --- (231)$$

Thus, the canonical momentum field is

$$P = \partial L / \partial \partial_t \phi = A_0(\phi) \partial_t \phi + A_1(\phi))(\nabla \otimes \phi)$$

and hence application of the Legendre transformation gives the Hamiltonian density as

$$\mathcal{H} = (P, \partial_t \phi) - L = (1/2)(\partial_t \phi)^T A_0(\phi)(\partial_t \phi) + (1/2)(\nabla \otimes \phi)^T A_2(\phi).(\nabla \otimes \phi)$$

$$= (1/2)(P - A_1(\phi)(\nabla \otimes \phi))^T (P - A_1(\phi)(\nabla \otimes \phi)) + (1/2)(\nabla \otimes \phi)^T A_2(\phi).(\nabla \otimes \phi) --- (231)$$

If now we add weak quantum noise $W(t, x)$ to $\phi(t, x)$, i.e., replace $\phi(t, x)$ by $\phi(t, x) + W(t, x)$, then it is clear that the Hamiltonian density, up to second-degree terms in the noise, would be given by an expression of the form

$$\mathcal{H} = (1/2)(P - A_1(\phi)(\nabla \otimes \phi))^T (P - A_1(\phi)(\nabla \otimes \phi)) + (1/2)(\nabla \otimes \phi)^T A_2(\phi).(\nabla \otimes \phi)$$

$$+ C_1(\phi)(P \otimes P \otimes W) + C_2(\phi)(P \otimes P \otimes W) + C_3(\phi)(P \otimes P \otimes W \otimes W) + C_4(\phi)(P)(P \otimes W \otimes W) +$$

$$+C_5(\phi)((\nabla\otimes W)\otimes(\nabla\otimes W))+C_6(\phi)((\nabla\otimes\phi)\otimes(\nabla\otimes W)\otimes W))$$
$$+C_7(\phi)((\nabla\otimes\phi)\otimes(\nabla\otimes\phi)\otimes W\otimes W)$$
$$+C_8(\phi)((\nabla\otimes\phi)\otimes(\nabla\otimes\phi)\otimes W)---(232)$$

In short, after expanding the position, momentum, and noise fields in terms of basis functions of the spatial coordinates x, we can express the total gravitational Hamiltonian (obtained by integrating the Hamiltonian density over the entire spatial volume), as

$$H(t,q(t),p(t))=$$
$$H_0(q(t),p(t))+V_{1kjm}(q(t))p_k(t)p_(t)W_m(t)+V_{2kj}(q(t))p_k(t)W_j(t)$$
$$+V_{3kjmr}(q(t))p_k(t)p_j(t)W_m(t)W_r(t)+V_{4kmr}(q(t))p_k(t)W_m(t)W_r(t)$$
$$+V_{5m}(q(t))W_m(t)+V_{6mr}(q(t))W_m(t)W_r(t)---(233)$$

This value of the noisy Hamiltonian must be substituted into the Hudson-Parthasarathy noisy Schrodinger equation after replacing $W_m(t)$ by the annihilation process differential $A'_m(t)$ and its adjoint and the products $W_m(t)W_r(t)$ by the differentials of the conservation process $\Lambda_j^{k'}(t)$ and finally adding to the noisy Schrodinger dynamics, quantum Ito correction terms in order to guarantee unitarity of the evolution.

0.11 The general form of the noisy propagator

In view of the above remarks, we consider the Lagrangian

$$L=L_0(\phi(t,x),\phi_{,\mu}(t,x))+L_{1k}(\phi(t,x),\phi_{,\mu}(t,x))W_k(t)---(234)$$

with summation over k where $W_k(t)'s$ are quantum white noise processes, these processes include Bosonic and Fermionic creation, annihilation, and conservation processes. With $\phi_k(t,x)$ being the position fields, some of which may be Bosonic and others Fermionic, the corresponding momentum fields are

$$P_k(t,x)=\partial L/\partial\phi_{k,0}=P_{k0}(t,x)+P_{km}(t,x)W_m(t)$$

with summation over the repeated indices m being understood. Here,

$$P_{k0}=\partial L_0/\partial\phi_{k,0},P_{km}=\partial L_{1m}/\partial\phi_{k,0}---(235)$$

The field equations are the Euler-Lagrange equations:

$$\partial_t P_{k0}+\partial_t(P_{km}W_m(t))=F_{k0}+F_{km}W_m(t)---(236)$$

where

$$F_{k0}=\partial L_0/\partial\phi_k-\partial_r\partial L_0/\partial\phi_{k,r},F_{km}=\partial L_{1m}/\partial\phi_k-\partial_r\partial L_{1m}/\partial\phi_{k,r}---(237)$$

where summation over the repeated index $r = 1, 2, 3$ is implied. Note that we are using the standard convention used in general relativity, namely, that Greek indices like μ, ν, ρ, σ stand for space-time indices, i.e., they assume values $0, 1, 2, 3$ while Roman indices like r, s, k, m assume values $1, 2, 3$ or perhaps more than three as for example when such an index appears in $W_m(t)$. The equal time canonical commutation relations (CCR) assuming all fields are Bosonic, are

$$[\phi(t,x), P(t,x')^T] = i\delta^3(x-x'), [\phi(t,x), \phi(t,x')^T] = 0, [P(t,x), P(t,x')^T] = 0---(238)$$

or equivalently,

$$[\phi_k(t,x), P_{j0}(t,x') + P_{jm}(t,x')W_m(t)] = i\delta_{kj}\delta^3(x-x') - - - (239)$$

To proceed further, we make an important assumption that is usually satisfied in most physical situations, namely that L_0 is quadratic in $\phi_{k,0}$ and L_1 is independent of $\phi_{k,0}$. Then, inverting the above nonlinear equation that relates P_k to $\phi_{k,0}, \phi_k$ and W_m, we get up to linear orders in the $W_m(t)$, an equation of the form

$$\phi_{k,0}(t,x) = M_{kj}(\phi(t,x), \phi_{,r}(t,x))P_j(t,x) + N_k(\phi(t,x), \phi_{,r}(t,x))W_k(t)---(240)$$

hence, the CCR can be expressed as

$$[\phi_k(t,x), \phi_{m,0}(t,x')] = iM_{mk}(\phi(t,x)), \phi_{,r}(t,x))\delta^3(x-x') - - - (241)$$

The field propagator matrix elements are defined by

$$\Delta_{km}(t,x|t',x') = < T(\phi_k(t,x)\phi_m(t',x')) > - - -(242)$$

or equivalently, in matrix notation,

$$\Delta(t,x|t',x') = < T(\phi(t,x)\phi(t',x')^T) >= \theta(t-t') < \phi(t,x)\phi(t',x')^T > +\theta(t'-t) < \phi(t',x')\phi(t,x)^T > ---(242)$$

where for simplicity, we are assuming that fields are real Bosonic fields, i.e., Bosonic fields that do not carry charge. Examples are the scalar chargeless Klein-Gordon field, the Yang-Mills non-Abelian gauge fields, and the gravitational field.

where the expectations are carried out in a vacuum state of the field and perhaps a coherent state of the noise. Up to linear orders in the noise, the field equations can be expressed as

$$\partial_t^2\phi(t,x) + A_1(\phi(t,x), \phi_{,r}(t,x))\partial_t\phi(t,x) + A_{2km}(\phi(t,x), \phi_{t,r}(x))\phi_{,km}(t,x)$$

$$+F(\phi(t,x), \phi_{,r}(t,x)) + \sum_k G_k(\phi(t,x), \phi_{,r}(t,x))W_k(t) = 0 - - - (243)$$

The aim is, of course, to derive a partial differential equation satisfied by the propagator. Elementary computations using this equation and the above CCR show that

$$\partial_t\Delta(t,x|t',x') = < T(\partial_t\phi(t,x)\phi(t',x')^T) >,$$

$$\partial_t^2\Delta(t,x|t',x') = i < M(\phi(t,x),\phi_{,r}(t,x))^T > \delta(t-t')\delta^3(x-x') + < T(\partial_t^2\phi(t,x)\phi(t',x')^T) >$$
$$= i < M(\phi(t,x),\phi_{,r}(t,x))^T > \delta(t-t')\delta^3(x-x') - < T(A_1(\phi(t,x),\phi_{,r}(t,x))\partial_t\phi(t,x).\phi(t',x')^T) >$$
$$- < T(A_{2km}(\phi(t,x),\phi_{,r}(t,x))\phi_{,km}(t,x)\phi(t',x')^T) > - < T(F(\phi(t,x),\phi_{,r}(t,x))\phi(t',x')^T) >$$
$$-\sum_k < T(G_k(\phi(t,x),\phi_{,r}(t,x))\phi(t',x')^T) >< W_k(t) > ---(244)$$

Note that $W_k(t)$ will have a nonzero mean in a coherent state in general if the coherent state is not the vacuum. Also note that $W_k(t)$ is a superposition of the time differentials of the creation, annihilation, and conservation processes in the Hudson-Parthasarathy quantum stochastic calculus theory. Note that in the further special case when A_1 is a constant matrix, the above equation simplifies to

$$\partial_t^2\Delta(t,x|t',x') =$$
$$i < M(\phi(t,x),\phi_{,r}(t,x))^T > \delta(t-t')\delta^3(x-x') - A_1\partial_t\Delta(t,x|t',x')$$
$$- < T(A_{2km}(\phi(t,x),\phi_{,r}(t,x))\phi_{,km}(t,x)\phi(t',x')^T) >$$
$$- < T(F(\phi(t,x),\phi_{,r}(t,x))\phi(t',x')^T) > - \sum_k < T(G_k(\phi(t,x),\phi_{,r}(t,x))\phi(t',x')^T) >< W_k(t) > ---(245)$$

If further, G_k is linear in ϕ and $\phi_{,r}$, so that it can be written as

$$G_k(\phi(t,x),\phi_{,r}(t,x)) = G_{k1}\phi(t,x) + G_{k2r}\phi_{,r}(t,x) --- (246)$$

(summation over $r = 1,2,3$ again being implied) where G_{k1} and G_{k2r} are constant matrices, and also F is linear in ϕ and $\phi_{,r}$ so that it can be written as

$$F(\phi(t,x),\phi_{,r}(t,x)) = F_1\phi(t,x) + F_{2r}\phi_{,r}(t,x) --- (247)$$

where $F_1, F_{2r}, r = 1,2,3$ are constant matrices, and further, A_{2km} are constant matrices, then the above propagator differential equation further simplifies to

$$\partial_t^2\Delta(t,x|t',x') =$$
$$= i < M(\phi(t,x),\phi_{,r}(t,x))^T > \delta(t-t')\delta^3(x-x') - A_1\partial_t\Delta(t,x|t',x')$$
$$-F_1\Delta(t,x|t',x') - F_{2r}\partial_r^x\Delta(t,x|t',x') - A_{2km}\partial_k^x\partial_m^x\Delta(t,x|t',x')$$
$$-(G_{k1}\Delta(t,x|t',x') + G_{k2r}\partial_r^x\Delta(t,x|t',x')) < W_k(t) > ---(248)$$

again summation over t,r being understood in the last term. In case that $< W_k(t) >= \mu(k,u)$ are independent of time but functions of the coherent state parameter u, and M is evaluated in a constant vacuum expected state ϕ_0 of ϕ, the above propagator equation in the four-momentum domain becomes

$$[(p^0)^2 - A_{2km}p^kp^m - iA_1p^0 + F_1 + iF_{2r}p^r + G_{k1}\mu(k,u) + iG_{2kr}p^r\mu(k,u))\Delta(p) = iM^T(\phi_0) ---(249)$$

Actually, in the above equation of motion of the field ϕ, the noise term coefficient G_k would more generally, be a function of not only ϕ and $\phi_{,r}$, but also of the double spatial partial derivatives $\phi_{,rs}$. In that case, assuming linear dependence on these partial derivatives, we would write

$$G_k(\phi(t,x),\phi_{,r}(t,x)) = G_{k1}\phi(t,x) + G_{k2r}\phi_{,r}(t,x) + G_{3krs}\phi_{,rs}(t,x) - - - (250)$$

where G_{3krs} are also constant matrices. In that case, the above propagator equation further generalizes to

$$[(p^0)^2 - A_{2rs}p^r p^s - iA_1 p^0 + F_1 + iF_{2r}p^r + G_{k1}\mu(k,u) + iG_{2kr}p^r\mu(k,u) - G_{3krs}p^r p^s\mu(k,u))\Delta(p)$$

$$= iM^T(\phi_0) - - - (251)$$

Thus, the propagator contains dissipative terms that are linear in p^μ and have imaginary coefficient matrices. It also contains mass terms, with masses given by the eigenvalues of the mass block matrix $((A_{2rs} + G_{3krs}\mu(k,u)))_{rs}$ explicitly showing how noise in a coherent state can contribute to particle mass.

0.12 Conclusions

In this article, we have firstly constructed a $\mathbb{Z}_n \times \mathbb{Z}_n$ graded tensor product in Hilbert space. Secondly, we have constructed $\mathbb{Z}_n \times \mathbb{Z}_n$-graded quantum stochastic processes in the sense of Hudson and Parthasarathy along the lines outlined for $\mathbb{Z}_2$-grading in the work of Timothy Eyre. These processes are shown to satisfy $\mathbb{Z}_n \times \mathbb{Z}_n$-graded super Lie algebra commutation relations. Thirdly using the graded tensor product between system Hilbert space and noise Boson Fock space, we have defined $\mathbb{Z}_n \times \mathbb{Z}_n$ graded quantum stochastic differential equations along the lines of Hudson and Parthasarathy. Fourthly, we have outlined an approach for generalizing Belavkin's quantum filter for such $\mathbb{Z}_n \times \mathbb{Z}_n$-graded quantum stochastic differential equations based on counting process non-demolition measurements for $\mathbb{Z}_n \times \mathbb{Z}_n$ noise. Fifthly, we have explained how graded quantum stochastic noise is to be incorporated into quantum field theory and quantum gravity via Lagrangian and Hamiltonian approaches. Finally, we have derived some approximate formulas for the correction to the propagator in quantum field theory caused by the presence of quantum stochastic noise. Formulas for corrections to particle masses caused by noise have been derived from such corrected propagators.

0.13 References

[1] Timothy Eyre, "Quantum Stochastic Calculus and Representations of Lie Super-algebras, Springer Lecture notes in mathematics.

[2] John Gough and C.Kostler, "Quantum Filtering in Coherent States, Arxiv.

[3] John Gough et.al., "The Fermionic Quantum Filter", Arxiv.

[4] K.R.Parthasarathy, "An introduction to quantum stochastic calculus", Birkhauser, 1992.

[5] Steven Weinberg, "The quantum theory of fields, vols.1,2,3, Cambridge University Press, 1995.

[6] M.Green, J.Schwarz and E.Witten, "Superstring Theory", vols.1,2, Cambridge University Press, 2012.

[7] Harish Parthasarathy, "Supersymmetry and Superstring Theory with Engineering Applications", Taylor and Francis, 2023.

[8] Harish Parthasarathy, "Fundamentals of classical and supersymmetric quantum stochastic filtering theory", Qeios, May 17, 2024.