

Peer Review

Review of: "Solution of Schrödinger equation for two different potentials using extended Nikiforov-Uvarov method and polynomial solutions of biconfluent Heun equation"

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We analyse the analytical eigenvalues obtained by means of the so-called extended Nikiforov-Uvarov method for two radial potentials. We show that in one case the analytical expression is extremely inaccurate. In the other case, the analytical expression exhibits a wrong behaviour with respect to the potential parameters. It seems that the approach proposed by the authors is inadequate for the treatment of this kind of quantum-mechanical models.

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1. Introduction

In a paper published in this journal, Karayer et al.^[1] applied the so-called extended Nikiforov-Uvarov (ENU) method to the Schrödinger equation with two radial potentials: $V_1(r) = -\alpha r^{-1/2}$ and $V_2(r) = kr^2 + \beta r - \alpha r^{-1}$. In the former example, they obtained an analytical expression for the bound-state eigenvalues but did not verify its accuracy. On the other hand, in the latter example, they derived *three* analytical expressions for the spectrum, which does not make sense.

The purpose of this Comment is the analysis of the results given in the paper just mentioned. In section 2, we consider the results of Karayer et al. for $V_1(r)$; section 3 is devoted to the analysis of the results obtained by those authors for $V_2(r)$. Finally, in section 4, we summarize the main results and draw conclusions.

2. First example

The first example treated by Karayer et al.^[1] is given by the eigenvalue equation

$$\frac{d^2 u(r)}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} + \frac{\alpha}{\sqrt{r}} \right] u(r) = 0, \quad (1)$$

where the energy is given by $E = k^2$. Clearly, there are bound states $E < 0$ for all $\alpha > 0$.

The authors defined $k = i\kappa$, $\kappa > 0$, and obtained an analytical expression for the discrete spectrum as

$$\kappa^3 = \frac{\alpha^2}{4(n+2l+2)}, \quad (2)$$

that leads to

$$E_{nl} = -\frac{\alpha^{4/3}}{[4(n+2l+2)]^{2/3}}. \quad (3)$$

Since the authors did not compare their results with those of Li and Dai^[2] whom they cited, we will do it here. For example, $E_{00}^{ENU} = -0.25$ and $E_{00}^{LD} = -0.4380$, for $\alpha = 1$, show that the error of equation (3) for the ground state is 43%. Note that the approximate analytical expression derived by Ishkhanyan^[3] is remarkably more accurate^[4]. Therefore, we conclude that the analytical expression (3) is far from providing a reasonable spectrum for the potential $-\alpha r^{-1/2}$.

3. Second example

For the second example, given by the potential

$$U(r) = kr^2 + \beta r - \frac{\alpha}{r}, \quad (4)$$

the authors derived the radial eigenvalue equation^[1]

$$\frac{d^2}{dr^2} R(r) + \frac{2}{r} \frac{d}{dr} R(r) + \left[2\epsilon - \frac{l(l+1)}{r^2} - kr^2 - \beta r + \frac{\alpha}{r} \right] R(r) = 0. \quad (5)$$

This radial equation does not look correct. If we assume that it was derived from the Schrödinger equation $H\psi = E\psi$, where

$$H = -\frac{\hbar^2}{2m} \nabla^2 + U(r), \quad (6)$$

then we conclude that $\epsilon = \frac{m}{\hbar^2} E$ and instead of k , α and β we should have $k' = \frac{2m}{\hbar^2} k$, $\beta' = \frac{2m}{\hbar^2} \beta$ and $\alpha' = \frac{2m}{\hbar^2} \alpha$, respectively.

It is well known that the eigenvalues E_{nl} of the Schrödinger equation with a radial potential satisfy $E_{nl} > E_{n'l'}$ if $n > n'$ and $E_{nl} > E_{n'l'}$ if $l > l'$. In the present case, since $\lim_{r \rightarrow \infty} U(r) = \infty$, we expect

$$\lim_{n \rightarrow \infty} E_{nl} = \infty. \quad (7)$$

Besides, the Hellmann-Feynman theorem^{[5], [6]} tells us that

$$\frac{\partial E}{\partial \beta} = \langle r \rangle > 0, \lim_{\beta \rightarrow 0} \frac{\partial E}{\partial \beta} = \langle r \rangle|_{\beta=0} > 0, \quad (8)$$

$$\frac{\partial E}{\partial \alpha} = -\left\langle \frac{1}{r} \right\rangle < 0, \quad (9)$$

$$\frac{\partial E}{\partial k} = \langle r^2 \rangle > 0. \quad (10)$$

In its most general form, the Schrödinger equation with the potential (4) is not exactly solvable, but one can obtain exact polynomial solutions for particular values of the model parameters. For this reason, such kinds of problems are commonly called quasi-exactly solvable or conditionally solvable (see^{[7], [8]} and references therein). In particular, such exact solutions to the Schrödinger equation with the potential (4) have been widely misunderstood by many researchers, as shown elsewhere^{[9], [10]}.

The Schrödinger equation with the potential (4) is exactly solvable in two cases: $\beta = \alpha = 0$ (harmonic oscillator) and $k = \beta = 0$ (Coulomb interaction).

Karayer et al.^[11] applied the ENU method to equation (5) and obtained

$$\epsilon^{ENU} = \begin{cases} \epsilon^a = -\frac{\beta^2}{8k} + \sqrt{k} \left(n + l + \frac{3}{2} \right), \\ \epsilon^b = -\frac{\beta^2}{8k} - \sqrt{k} \left(n - l + \frac{1}{2} \right), \\ \epsilon^c = -\frac{\beta^2}{8k} - \sqrt{k} \left(n + l + \frac{3}{2} \right), \end{cases} \quad (11)$$

where we have introduced the superscripts a , b , and c in order to refer to each solution separately. It is clear that ϵ^b and ϵ^c do not exhibit the correct behaviour with respect to n shown in equation (7) and should be discarded. Moreover, the three solutions exhibit a wrong behaviour with respect to the model parameters. Just compare

$$\frac{\partial \epsilon^{ENU}}{\partial \beta} = -\frac{\beta}{4k} < 0, \beta > 0, \lim_{\beta \rightarrow 0} \frac{\partial \epsilon^{ENU}}{\partial \beta} = 0, \quad (12)$$

$$\frac{\partial \epsilon^{ENU}}{\partial \alpha} = 0, \quad (13)$$

with equations (8) and (9). It is clear that the analytical expressions (11) derived by Karayer et al.^[11] are wrong.

4. Conclusions

In this Comment we have shown, beyond any doubt, that the eigenvalues obtained by Karayer et al.^[1] for the Schrödinger equation with two simple radial potentials are wrong. The only possible conclusion is that the ENU method is unsuitable for solving eigenvalue equations.

References

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Declarations

Potential competing interests: No potential competing interests to declare.