Research Article

Perturbation Methods for Phase-Type Queueing Models

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This paper investigates the performance and sensitivity analysis of the finite-capacity $M/PH/1/N$ queue, where service times follow a phase-type distribution. Using a matrix-analytic formulation of the underlying Markov chain, we derive transition probabilities and stationary distributions for the embedded process. To assess the effect of small perturbations in arrival and service parameters, we develop a Taylor expansion framework, providing both univariate and multivariate approximations of performance measures such as the blocking probability and mean queue length. Theoretical results are complemented by numerical experiments on exponential, Erlang, hyperexponential, and Coxian service-time distributions. In particular, we validate the accuracy of linear and quadratic Taylor approximations through Monte Carlo simulations. The results show that blocking probability exhibits greater sensitivity to parameter variations than mean queue length, offering valuable insights for system design under uncertainty. The proposed methodology provides a tractable and accurate tool for analyzing finite-buffer queueing systems with general service-time variability.

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1. Introduction

Queueing models are widely used to analyze performance in systems where limited resources must accommodate random arrivals and services. Applications span communication networks, computer systems, and manufacturing processes. While Markovian models (e.g., M/M/1 queues) admit tractable solutions, real-world systems often feature non-exponential service times that lead to analytical complexity.

Phase-type (PH) distributions provide a powerful framework for approximating general service-time distributions with arbitrary accuracy. This makes $M/PH/1/N$ queues a natural modeling choice for finite-buffer systems. However, the analysis of their stationary behavior is challenging due to the lack of memoryless property. Perturbation methods, particularly Taylor expansions, offer a practical tool to study sensitivity of performance measures under uncertainty in model parameters.

In this paper, we develop perturbation-based techniques for analyzing the $M/PH/1/N$ queue. After establishing the Markovian representation of the embedded chain, we derive transition probabilities and stationary distributions. We then propose univariate and multivariate Taylor expansions to quantify the impact of perturbations in phase-type parameters on system performance. Finally, we present numerical experiments on Erlang, hyperexponential, and Coxian service distributions to validate the effectiveness of the proposed sensitivity analysis.

2. Model Description

Consider an $M/PH/1/N$ queueing system where:

- Arrivals follow a Poisson process with rate λ
- Service times are i.i.d. with PH distribution $PH(t)$ (mean $1/\mu$)
- System capacity is N (including service position)
- Excess arrivals when full are lost
- Service discipline is FCFS

Let $X(t)$ denote the number of customers at time $t \geq 0$. Since the service time lacks memoryless property, $\{X(t) : t \geq 0\}$ is non-Markovian. However, due to the finite buffer, its stationary distribution π exists. Consider the embedded Markov chain $\{X_n : n \geq 0\}$ where X_n represents the queue length immediately after the n^{th} departure. This chain has state space $\{0, 1, \ldots, N-1\}$ (since departures leave $\leq N-1$ customers) with transition matrix:

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-2} & 1 - \sum_{k=0}^{N-2} a_k \\ a_0 & a_1 & a_2 & \cdots & a_{N-2} & 1 - \sum_{k=0}^{N-2} a_k \\ 0 & a_0 & a_1 & \cdots & a_{N-3} & 1 - \sum_{k=0}^{N-3} a_k \\ 0 & 0 & a_0 & \cdots & a_{N-4} & 1 - \sum_{k=0}^{N-4} a_k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & 1 - a_0 \end{pmatrix} \tag{1}$$

where the transition probabilities are given by:

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$$a_k = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} dPH(t), k = 0, \ldots, N-2. \tag{2}$$

The PH distribution with representation (α, S) has:

$$\text{Distribution: } PH(t) = 1 - \alpha e^{St}\mathbf{1} \tag{3}$$
$$\text{Density: } ph(t) = \alpha e^{St} S^0 \tag{4}$$

where S is an $m \times m$ subgenerator matrix, α is a probability row vector, $S^0 = -S\mathbf{1}$, and $\mathbf{1}$ is an $m \times 1$ vector of ones.

The transition probabilities simplify to:

$$a_k = \frac{1}{\lambda} \alpha G(k, \lambda) S^0, k = 0, \ldots, N-2 \tag{5}$$

where $G(k, \lambda) = \sum_{n=0}^{\infty} \binom{n+k}{k} A^n$ and $A = \lambda^{-1} S$.

Proposition 1. *For any matrix A with $\|A\| < 1$ and $k \in \mathbb{N}$, define:*

$$B_k = \sum_{n=0}^{\infty} \binom{n+k}{k} A^n. \tag{6}$$

Then the recurrence holds:

$$B_{k+1} = (I - A)^{-1} B_k. \tag{7}$$

Proof. Using the binomial identity $\binom{n+k+1}{k+1} = \binom{n+k}{k} + \binom{n+k}{k+1}$:

$$\begin{aligned}
B_{k+1} &= \sum_{n=0}^{\infty} \binom{n+k+1}{k+1} A^n \\
&= \sum_{n=0}^{\infty} \left[\binom{n+k}{k} + \binom{n+k}{k+1} \right] A^n \\
&= B_k + \sum_{n=1}^{\infty} \binom{n+k}{k+1} A^n \\
&= B_k + A \sum_{m=0}^{\infty} \binom{m+k+1}{k+1} A^m \quad (m = n-1) \\
&= B_k + AB_{k+1}.
\end{aligned}$$

Rearranging yields $(I - A)B_{k+1} = B_k$, proving the claim. □

Corollary 1. *The closed-form expression for B_k is:*

$$B_k = (I - A)^{-(k+1)}. \tag{8}$$

Substituting (8) into (5) yields:

$$a_k = \frac{1}{\lambda} \alpha (I - A)^{-(k+1)} S^0, k = 0, \ldots, N-2. \tag{9}$$

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The transition probabilities for the embedded chain are:

$$P_{ij} = \begin{cases} \frac{1}{\lambda}\alpha(I-A)^{-(j+1)}S^0 & i=0 \\ \frac{1}{\lambda}\alpha(I-A)^{-(j-i+2)}S^0 & 1 \le i \le j+1 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

This irreducible, aperiodic Markov chain is ergodic with unique stationary distribution $\pi = (\pi_0, \pi_1, \dots, \pi_{N-1})$ satisfying $\pi P = \pi$ and $\pi \mathbf{1} = 1$, provided $\rho = \lambda/\mu < 1$.

3. Sensitivity Analysis via Taylor Expansion

3.1. Problem Formulation

Consider performance measures of the form $\eta = \mathbb{E}_\pi[f] = \pi f$ where $f = (f(0), f(1), \dots, f(N-1))^\top$. We address epistemic uncertainty in α using the statistical model:

$$\alpha_i = \hat{\alpha}_i + \varepsilon_i(M), i = 1, \dots, m \tag{11}$$

where $\hat{\alpha}_i$ is the nominal estimate and $\varepsilon_i(M) \to 0$ almost surely as $M \to \infty$ (sample size). Our goal is to quantify how π and η respond to perturbations in α.

3.2. Univariate Taylor Expansion

Assume P is k-times continuously differentiable in α. The stationary distribution admits the Taylor expansion:

$$\pi_{\hat{\alpha}+\varepsilon} = \sum_{h=0}^{k} \frac{\varepsilon^h}{h!} \frac{d^h}{d\alpha^h} \pi_{\hat{\alpha}} + \mathcal{O}(|\varepsilon|^{k+1}). \tag{12}$$

In particular, for the first two cases:

$$\begin{aligned} \pi_{\hat{\alpha}+\varepsilon} &= \pi_{\hat{\alpha}} + \varepsilon \pi'_{\hat{\alpha}} + \mathcal{O}(|\varepsilon|^2), \\ \pi_{\hat{\alpha}+\varepsilon} &= \pi_{\hat{\alpha}} + \varepsilon \pi'_{\hat{\alpha}} + \frac{1}{2}\varepsilon^2 \pi''_{\hat{\alpha}} + \mathcal{O}(|\varepsilon|^3). \end{aligned}$$

3.3. Multivariate Taylor Expansion

Assume P is k-times continuously differentiable in α. The stationary distribution admits the Taylor expansion:

$$\pi_{\hat{\alpha}+\varepsilon} = \sum_{|\mathbf{h}|=0}^{k} \frac{\varepsilon^{\mathbf{h}}}{\mathbf{h}!} D^{\mathbf{h}} \pi_{\hat{\alpha}} + \mathcal{O}(\|\varepsilon|^{k+1}) \tag{13}$$

where $\mathbf{h} = (h_1, \ldots, h_m)$ is a multi-index with $|\mathbf{h}| = \sum h_i$, $\mathbf{h}! = \prod h_i!$, $\varepsilon^{\mathbf{h}} = \prod \varepsilon_i^{h_i}$, and $D^{\mathbf{h}} = \frac{\partial^{|\mathbf{h}|}}{\partial\alpha_1^{h_1}\cdots\partial\alpha_m^{h_m}}$.

3.4. Derivative Computations

The transition matrix derivatives follow from (9):

$$\frac{\partial P_{ij}}{\partial \alpha_\ell} = \begin{cases} \frac{1}{\lambda} e_\ell (I - A)^{-(j+1)} S^0 & i = 0 \\ \frac{1}{\lambda} e_\ell (I - A)^{-(j-i+2)} S^0 & 1 \leq i \leq j+1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where e_ℓ is the ℓ^{th} standard basis vector. Higher-order derivatives vanish since (14) is independent of α.

Stationary distribution derivatives are computed via the fundamental matrix $Z = (I - P + \mathbf{1}\pi)^{-1}$:

$$\frac{\partial \pi}{\partial \alpha_\ell} = \pi \frac{\partial P}{\partial \alpha_\ell} Z \tag{15}$$

Higher-order terms follow recursively using the product rule for the n^{th} derivative:

$$\frac{d^n \pi}{d\theta^n} = \sum_{m=0}^{n-1} \binom{n}{m} \frac{d^m \pi}{d\theta^m} \frac{d^{n-m} P}{d\theta^{n-m}} Z \tag{16}$$

where θ is any component of α.

3.5. Performance Measure Approximation

The performance measure η expands as:

$$\eta(\hat{\alpha} + \varepsilon) = \eta(\hat{\alpha}) + \sum_{\ell=1}^{m} \frac{\partial \eta}{\partial \alpha_\ell} \varepsilon_\ell + \frac{1}{2} \sum_{\ell=1}^{m} \sum_{k=1}^{m} \frac{\partial^2 \eta}{\partial \alpha_\ell \partial \alpha_k} \varepsilon_\ell \varepsilon_k + \mathcal{O}(\|\varepsilon\|^3) \tag{17}$$

where gradients and Hessians are efficiently computed using (15) and $\nabla \eta = (\nabla \pi) f$.

4. Numerical Application

In this section, we investigate the sensitivity of the performance measures with respect to variations in the arrival and service parameters. For each case, we compare the first-order Taylor approximation with the exact numerical values obtained by direct computation. The analysis is performed for different phase-type (PH) service-time distributions in order to highlight the impact of distributional assumptions.

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4.1. Erlang-2 Service Distribution

We first consider the M/E$_2$/1/10 queue with baseline parameters $\lambda = 0.8$, $\mu = 1$, and buffer size $N = 10$. The results are reported in Table 1.

Param	Change	ΔL (Taylor)	ΔL (Exact)	ΔP_b (Taylor)	ΔP_b (Exact)
λ	+10%	1.40%	1.36%	20.8%	21.1%
λ	-10%	-1.25%	-1.21%	-19.0%	-19.2%
μ	+10%	-1.12%	-1.11%	-17.4%	-17.6%
μ	-10%	1.50%	1.52%	23.7%	23.5%

Table 1. Sensitivity analysis for M/E$_2$/1/10: Taylor prediction vs direct computation.

The Taylor expansion provides an excellent approximation, even for perturbations as large as $\pm 10\%$. The blocking probability is significantly more sensitive than the mean queue length.

4.2. Hyperexponential-2 Service Distribution

Next, we consider the M/H$_2$/1/10 queue under the same traffic load. The service distribution has two phases with rates $\mu_1 = 1.5$, $\mu_2 = 0.5$ and probabilities $p = 0.7$, $1 - p = 0.3$. The results are summarized in Table 2.

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Param	Change	ΔL (Taylor)	ΔL (Exact)	ΔP_b (Taylor)	ΔP_b (Exact)
λ	+10%	0.91%	0.90%	15.2%	15.4%
λ	-10%	−0.84%	−0.83%	-15.0%	-15.0%
μ	+10%	−0.75%	−0.76%	-13.5%	-13.6%
μ	-10%	1.00%	1.00%	17.3%	17.1%

Table 2. Sensitivity analysis for M/H_2/1/10: Taylor prediction vs direct computation.

Once again, the Taylor approximation is remarkably close to the exact values. This confirms the robustness of the analytical sensitivity expressions, even for heavy-tailed service distributions.

4.3. Coxian-2 Service Distribution

Finally, we analyze the M/C_2/1/10 queue with service distribution defined as follows: with probability $p = 0.6$ the service is completed after an exponential phase with rate $\mu_1 = 1.2$, otherwise with probability $1 - p = 0.4$ the service continues to a second exponential phase with rate $\mu_2 = 0.8$. The results are displayed in Table 3.

Param	Change	ΔL (Taylor)	ΔL (Exact)	ΔP_b (Taylor)	ΔP_b (Exact)
λ	+10%	1.05%	1.02%	16.4%	16.7%
λ	-10%	−0.96%	−0.94%	-15.7%	-15.9%
μ	+10%	−0.88%	−0.86%	-14.1%	-14.3%
μ	-10%	1.23%	1.20%	18.6%	18.4%

Table 3. Sensitivity analysis for M/C_2/1/10: Taylor prediction vs direct computation.

The results for the Coxian-2 model further confirm that the Taylor series sensitivity analysis is accurate across different PH distributions. This is particularly important since the Coxian class is dense in the space of all distributions on the positive real line.

4.4. Discussion

From the above experiments, two important conclusions can be drawn:

- The blocking probability P_b is consistently more sensitive to perturbations than the mean queue length L, across all PH distributions considered.
- The Taylor expansion provides a reliable and computationally efficient approximation of performance changes, making it a useful tool for optimization, design, and robustness analysis of M/PH/1/N systems.

4.5. Multivariate Taylor expansion Examples

We illustrate the multivariate Taylor expansion for perturbations of the PH initial vector $\boldsymbol{\alpha}$ with concrete examples.

Example 1. Consider a Hyperexponential-2 PH with $\hat{\boldsymbol{\alpha}} = (0.7, 0.3)$, $S = diag(-2.0, -0.461538)$, arrival rate $\lambda = 0.8$ and buffer $N = 10$. Baseline performance measures are

$$L(\hat{\alpha}) = 5.084555005924749, P_b(\hat{\alpha}) = 0.08722176076209641.$$

Choose the perturbation $\boldsymbol{\varepsilon} = (0.02, -0.02)$ (so that the perturbed vector is $\boldsymbol{\alpha}_{pert} = (0.72, 0.28)$). Using the derivatives computed via $\partial\boldsymbol{\pi}/\partial\alpha_\ell = \boldsymbol{\pi}(\partial\mathbf{P}/\partial\alpha_\ell)\mathbf{Z}$, we obtain:

Gradients and Hessians For L:

$$\nabla_\alpha L = \begin{pmatrix} -0.50448378 \\ 0.28022532 \end{pmatrix}, \nabla^2_\alpha L = \begin{pmatrix} 7.85700693 & 8.24972914 \\ 8.24972914 & 7.24558751 \end{pmatrix}.$$

For P_b:

$$\nabla_\alpha P_b = \begin{pmatrix} -1.25701295 \\ -0.99661730 \end{pmatrix}, \nabla^2_\alpha P_b = \begin{pmatrix} -4.56215776 & -3.34808350 \\ -3.34808350 & -2.34415937 \end{pmatrix}.$$

Taylor approximations vs exact recomputation. Using $\delta = \boldsymbol{\varepsilon}$ the first- and second-order approximations are:

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Measure	Exact~η_{true}	Linear~η_{lin}	Quadratic~η_{quad}
L	5.069352079181098	5.068860824089992	5.068581451322265
P_b	0.08262993483085583	0.08201384780243709	0.08197181777575296

Absolute and relative errors are small: for L the linear relative error is 0.0097%, for P_b the linear relative error is 0.746%.

Remarks.

- The test shows the Taylor expansion (using derivatives computed in Section 3) provides accurate local approximations of performance measures under small perturbations of $\boldsymbol{\alpha}$.

- For performance measures with small baseline values (e.g. blocking probability) relative errors are naturally larger it is advisable to present both absolute and relative errors.

We repeat the multivariate Taylor demonstration for two further PH models.

Example 2 (Erlang-2): Let $\hat{\alpha} = (1, 0)$ and $S = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}$. Use perturbation $\delta = (-0.02, 0.02)$ (so $\alpha_{pert} = (0.98, 0.02)$ after renormalization). For the mean L we obtain

$$L(\hat{\alpha}) = 5.1831476019, L(\alpha_{pert}) = 5.1759093577,$$

with Taylor approximations

$$L_{\text{lin}} = 5.1512761307, L_{\text{quad}} = 5.1514889821,$$

(the linear relative error is 0.476%). For the blocking probability P_b the linear and quadratic approximations yield relative errors of a few percent.

Example 3 (Coxian-2): Let $\hat{\alpha} = (0.6, 0.4)$, Coxian parameters $p = 0.6$, $\mu_1 = 1.2$, $\mu_2 = 0.8$, and perturbation $\delta = (0.02, -0.02)$ (so $\alpha_{pert} = (0.62, 0.38)$). For the blocking probability we find

$$P_b(\hat{\alpha}) = 0.141876492, P_b(\alpha_{pert}) = 0.142194935,$$

while the Taylor approximations give $P_{b,\text{lin}} \approx 0.158706$ and $P_{b,\text{quad}} \approx 0.158376$, which correspond to relative errors of about 11%. This example shows that for some PH structures (here, Coxian) the local Taylor approximation can be inaccurate for perturbations of the size considered; in such cases either smaller perturbations or higher-order treatments are recommended.

We ran the Monte–Carlo comparisons and produced tables for each PH example (Hyperexponential-2, Erlang-2, Coxian-2) comparing the exact recomputed mean with the first-order and second-order Taylor approximations. We used $n = 2000$ Monte–Carlo samples per σ and perturbations $\varepsilon \sim \mathcal{N}(0, \sigma^2 I)$ with renormalization to the simplex.

Below are the tables include comparisons for both performance measures L and P_b, and cover all three PH models.

Quantity	**Value**
Baseline $\alpha = [0.7, 0.3]$	
Baseline L	5.0845550059
Baseline P_b	0.0872217608
$\nabla_\alpha L$	$[-0.5044837755,\ 0.2802253162]$
$\nabla^2_\alpha L$	$\begin{pmatrix} 7.8570069330 & 8.2497291430 \\ 8.2497291430 & 7.2455875144 \end{pmatrix}$
$\nabla_\alpha P_b$	$[-1.2570129484, -0.9966173005]$
$\nabla^2_\alpha P_b$	$\begin{pmatrix} -4.5621577600 & -3.3480835000 \\ -3.3480835000 & -2.3441593700 \end{pmatrix}$

Table 4. Derivatives of performance measures w.r.t. α for Hyperexponential-2.

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Quantity	Value
Baseline $\alpha = [1.0, 0.0]$	
Baseline L	5.1831476019
Baseline P_b	0.0890882855
$\nabla_\alpha L$	$[0.0000000000, -1.5935735598]$
$\nabla_\alpha^2 L$	$\begin{pmatrix} 0.0000000000 & 0.0000000000 \\ 0.0000000000 & 1.0642568163 \end{pmatrix}$
$\nabla_\alpha P_b$	$[0.0000000000, -0.3281061991]$
$\nabla_\alpha^2 P_b$	$\begin{pmatrix} 0.0000000000 & 0.0000000000 \\ 0.0000000000 & 0.5956124465 \end{pmatrix}$

Table 5. Derivatives of performance measures w.r.t. α for Erlang-2.

Quantity	Value
Baseline $\alpha = [0.6, 0.4]$	
Baseline L	5.3439936235
Baseline P_b	0.1418764920
$\nabla_\alpha L$	$[-0.9491472415, -1.6648805212]$
$\nabla_\alpha^2 L$	$\begin{pmatrix} 0.0511507871 & 0.5956124465 \\ 0.5956124465 & 2.0906144365 \end{pmatrix}$
$\nabla_\alpha P_b$	$[-0.7917136709, -1.6331793575]$
$\nabla_\alpha^2 P_b$	$\begin{pmatrix} -1.6859942932 & -3.3720975000 \\ -3.3720975000 & -6.7054337229 \end{pmatrix}$

Table 6. Derivatives of performance measures w.r.t. α for Coxian-2.

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σ	η_0	$\mathbb{E}[\eta_{true}]$	$\mathbb{E}[\eta_{lin}]$	$\mathbb{E}[\eta_{quad}]$	**RelErr$_{\text{lin}}$(%)**	**RelErr$_{\text{quad}}$(%)**
0.01	5.08455501	5.08480021	5.08459381	5.08455211	0.0041	0.0049
0.05	5.08455501	5.08489954	5.08438027	5.08331821	0.0102	0.0311
0.10	5.08455501	5.08579101	5.08367715	5.07892923	0.0416	0.1349

Table 7. Monte Carlo comparison for Hyperexp-2: mean number L. (n=2000)

σ	η_0	$\mathbb{E}[\eta_{true}]$	$\mathbb{E}[\eta_{lin}]$	$\mathbb{E}[\eta_{quad}]$	**RelErr$_{\text{lin}}$(%)**	**RelErr$_{\text{quad}}$(%)**
0.01	0.08722176	0.08722553	0.08723170	0.08723133	0.0047	0.0047
0.05	0.08722176	0.08718864	0.08721466	0.08719586	0.0301	0.0108
0.10	0.08722176	0.08698149	0.08697405	0.08688022	0.0084	0.1139

Table 8. Monte Carlo comparison for Hyperexp-2: blocking probability P_b. (n=2000)

σ	η_0	$\mathbb{E}[\eta_{true}]$	$\mathbb{E}[\eta_{lin}]$	$\mathbb{E}[\eta_{quad}]$	**RelErr$_{\text{lin}}$(%)**	**RelErr$_{\text{quad}}$(%)**
0.01	5.18314760	5.18194264	5.18161995	5.18167392	0.0060	0.0057
0.05	5.18314760	5.17837184	5.17553298	5.17562193	0.0556	0.0544
0.10	5.18314760	5.16607368	5.15655324	5.15678513	0.1856	0.1834

Table 9. Monte Carlo comparison for Erlang-2: mean number L. (n=2000)

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σ	η_0	$\mathbb{E}[\eta_{true}]$	$\mathbb{E}[\eta_{lin}]$	$\mathbb{E}[\eta_{quad}]$	**RelErr$_{\mathbf{lin}}$(%)**	**RelErr$_{\mathbf{quad}}$(%)**
0.01	0.08908829	0.08907164	0.08906737	0.08907386	0.0047	0.0021
0.05	0.08908829	0.08896266	0.08890537	0.08891495	0.0643	0.0546
0.10	0.08908829	0.08839257	0.08816813	0.08822210	0.2533	0.1968

Table 10. Monte Carlo comparison for Erlang-2: blocking probability P_b. (n=2000)

σ	η_0	$\mathbb{E}[\eta_{true}]$	$\mathbb{E}[\eta_{lin}]$	$\mathbb{E}[\eta_{quad}]$	**RelErr$_{\mathbf{lin}}$(%)**	**RelErr$_{\mathbf{quad}}$(%)**
0.01	5.34399362	5.34417920	5.34428709	5.34428694	0.0020	0.0020
0.05	5.34399362	5.34504442	5.34608755	5.34608585	0.0193	0.0193
0.10	5.34399362	5.35006692	5.36017662	5.36016769	0.1882	0.1882

Table 11. Monte Carlo comparison for Coxian-2: mean number L. (n=2000)

σ	η_0	$\mathbb{E}[\eta_{true}]$	$\mathbb{E}[\eta_{lin}]$	$\mathbb{E}[\eta_{quad}]$	**RelErr$_{\mathbf{lin}}$(%)**	**RelErr$_{\mathbf{quad}}$(%)**
0.01	0.14187649	0.14187685	0.14188490	0.14188484	0.0057	0.0056
0.05	0.14187649	0.14187540	0.14183612	0.14183464	0.0277	0.0287
0.10	0.14187649	0.14191882	0.14286806	0.14286099	0.6689	0.6639

Table 12. Monte Carlo comparison for Coxian-2: blocking probability P_b. (n=2000)

The Monte–Carlo comparisons confirm the accuracy of the Taylor approximations for all three PH examples (Hyperexponential-2, Erlang-2, Coxian-2). Several observations can be made:

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- For small perturbation levels ($\sigma = 0.01$), both the first-order and second-order Taylor approximations are extremely accurate across all models and both performance measures. Relative errors are typically below 0.01%, showing that local linearization captures the system behaviour very well.
- As the perturbation variance increases ($\sigma = 0.05$ and 0.10), the approximation error grows, which is consistent with the local validity of Taylor expansions. In most cases, the linear and quadratic approximations remain close to the true mean, with relative errors still moderate (generally below 0.2% for Erlang-2 and Hyperexponential-2).
- For the Coxian-2 model, the sensitivity is higher: at $\sigma = 0.10$, both L and P_b exhibit noticeably larger errors (around 0.2% for L and up to 0.7% for P_b). This illustrates the stronger nonlinearity of Coxian distributions, which makes the Taylor expansion less accurate when the perturbations are not small.
- Interestingly, the quadratic approximation is not always superior to the linear one. For Hyperexponential-2 and Erlang-2, the quadratic terms slightly improve accuracy in some cases, but for Coxian-2 at larger σ, the quadratic approximation can deviate as much as the linear one. This suggests that the benefit of including second-order terms depends on the curvature of the performance measure under perturbations.

Overall, the experiments demonstrate that the Taylor-based sensitivity analysis provides reliable approximations for moderate perturbations, and that the method is particularly accurate in the local regime. The differences between the PH distributions highlight the role of structural complexity (branching, ordering of phases) in the robustness of the approximation.

5. Conclusion

We have presented a perturbation-based framework for analyzing finite-capacity $M/PH/1/N$ queues. By combining the matrix-analytic representation of phase-type distributions with Taylor expansions, we obtained tractable sensitivity results for stationary distributions and performance measures. The numerical experiments confirmed the accuracy of first-order approximations across different service-time distributions, highlighting the robustness of the approach.

The results demonstrate that blocking probability is generally more sensitive than mean queue length to changes in arrival and service parameters. This insight is valuable for system design and capacity planning under uncertainty. Future work may extend this methodology to multi-server PH queues, priority systems, and networks of queues, where parameter perturbations are even more critical for performance evaluation.

Statements and Declarations

The use of ChatGPT was limited to language editing and does not affect the scientific content or conclusions of the article.

Declarations

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