Commentary

# A Common Origin for the Sagnac Effect and Einstein's Train-Embankment Thought Experiment

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**The Sagnac effect and the Einstein Train-Embankment thought experiment deal with the simultaneous emission of light and its non-simultaneous absorption. The original Einstein's thought experiment arrangement is slightly modified, also for pedagogical purposes, becoming a rectilinear version of Sagnac's interferometer. It is shown that at the base of the effect, lays the solution of the famous Achilles and the tortoise paradox, a problem of pursuit. An account is given of the failure of simultaneity at a distance, forming the basis of what Einstein called the relativity of simultaneity. The effect experienced by the moving observer which relies also on time dilation is also discussed.**

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## 1. Introduction

Sagnac's original experimental setup involved a source of light, a prism, several mirrors, and one interferometer arranged in a particular way. Sagnac measured the interference of two counter-propagating light waves, emitted simultaneously, and absorbed by an interferometer in motion in a closed path. The interference showed that the two waves did not reach the detector simultaneously.

Several descriptions and derivations of the effect, some quite complicated, have been presented in the literature [1][2][3][4][5][6] also recently.

A recent explanation on the origin of the Sagnac delay [1]: "... is the non-mid-point measurement of arrival times of counter-propagating waves leading to unequal path lengths traversed by the oppositely directed light rays in reaching the interferometer". More simply, the change of position of the absorber changes the path lengths of two counter-propagating waves, in a closed loop. The effect relies on the independence of the speed of light, as a wave, from the speed of the source, it also relies on the invariance of the out-and-back speed of light. Laser Gyros, which rely on optical fibers in a close loop, are the most widespread applications of the effect. The ”Sagnac correction” is a fundamental contribution to synchronizing GPS base stations [5].

Einstein conceived his thought experiment to give an account of the relativity of simultaneity, in support of Special Relativity.

This paper aims to keep the descriptions of the configurations, the derivations, and comparisons as simple as possible, without losing generality and dealing only when necessary with higher-order effects involved by the relativity of time.

In the second paragraph a simple explanation of the Sagnac effect is provided, showing that the simultaneous emission of counter-propagating signals, in a loop, corresponds to their non-simultaneous detection, for a non-stationary observer. The relevant derivation of the times involved is provided according to an inertial observer, at rest in the non-rotating frame.

The famous Einstein Train-Embankment thought experiment (E-TETE) is illustrated in the third paragraph, dealing as well, with the topic of simultaneity [7]. The E-TETE is changed from its original version, according to what is suggested in [1] and [6] to align it with Sagnac's working principle. The purpose of Einstein's thought experiment was to give an account of the relativity of simultaneity. This feature is shown in its exact form, accounting for the times involved.

The fourth paragraph shows that both configurations are governed by a first principle: “light chasing a moving target”, a version of Achille's paradox solution.

In the last paragraph, the relativity of time is applied to predict how moving observers would detect the same effect previously measured by stationary observers.

## 2. A simple derivation of Sagnac effect

A disk of radius $R$ rotates with angular speed $\omega$ as in Fig.1 . A device, denoted as ED, on the disk acts as both an emitter and detector of light. ED emits two light waves which propagate along the circumference of the disk, in opposite directions, and get detected. The detected effect is a phase difference of waves emitted from the same source (coherent, in phase).

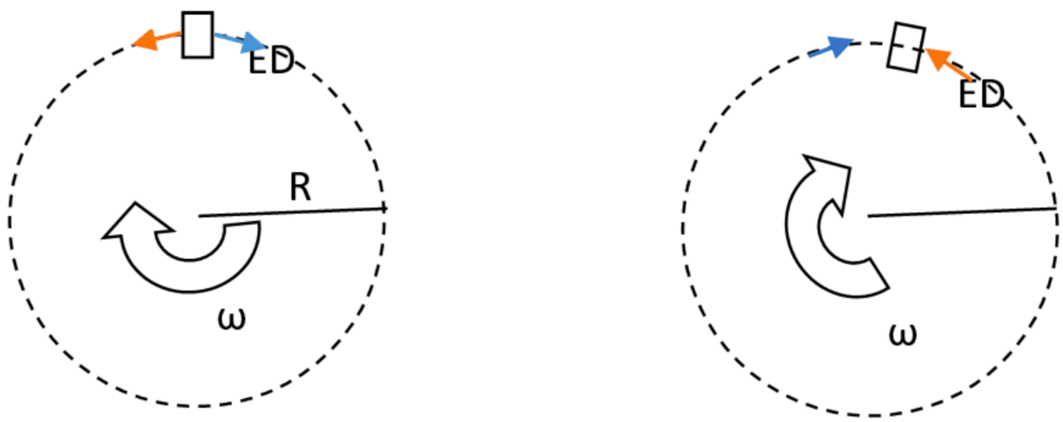


**Figure 1.** Sagnac effect: simultaneous emission and non-simultaneous detection

That phase difference originates from the unequal path lengths covered by the waves, traveling at the speed of light in the frame of the axis of rotation. One pulse arrives before the other, hence their phase difference is detected by the interferometer ED. From the picture, it is clear that the path length of the clockwise wave (blue) in reaching ED is longer than the other (red).

The times $T_+$ and $T_-$ taken by the waves to return to the detector ED are measured by a clock stationary with the axis of rotation, the relevant lengths covered by the waves before being absorbed by ED are $cT_+$ and $cT_-$, respectively. The kinematic relations for the lengths can be written as follows:

$$cT_+ = 2\pi R + \omega R T_+, cT_- = 2\pi R - \omega R T_- \tag{1}$$

By solving these equations for $T_+$ and $T_-$, we get:

$$T_+ = \frac{2\pi R}{c - \omega R}, T_- = \frac{2\pi R}{c + \omega R}$$

The difference in the arrival times of the two waves, $\Delta T_{\text{Sagnac}}$, is:

$$\Delta T_{\text{Sagnac}} = T_+ - T_- = \frac{2\pi R}{c - \omega R} - \frac{2\pi R}{c + \omega R} = \frac{4\omega\pi R^2}{c^2\left(1 - \frac{\omega^2 R^2}{c^2}\right)} \approx \frac{4\omega\pi R^2}{c^2} \tag{2}$$

This represents the time interval between the arrivals of the two waves at the detector ED.

Sagnac experimentally tested the following relation: $\Delta\phi = \frac{4\pi A\omega}{\lambda c}$ where $\Delta\phi$ is the phase difference of the waves , $A$ is the area enclosed by the loop, $\lambda$ is the wavelength of the light, and $c$ is the speed of light. Using the relation between phase difference and time difference: $\Delta t = \frac{\Delta\phi\lambda}{\pi c}$ and substituting $A = \pi R^2$ into Sagnac's formula:


doi.org/10.32388/WTULIK

$$\Delta t = \frac{4\pi A\omega}{\lambda c} \times \frac{\lambda}{\pi c} = \frac{4\omega\pi R^2}{c^2}$$

This matches the previously found first-order approximation for $\Delta T_{\text{Sagnac}}$.

The effect is valid for a generic loop and depends on its length $L$, regardless of its shape [1]. For a loop of length $L \ (= 2\pi R)$ and linear speed $v \ (= \omega R)$ :

$$T_+ = \frac{L/c}{1 - v/c}, T_- = \frac{L/c}{1 + v/c}, \Delta T_{\text{Sagnac}} = T_+ - T_- = \frac{2vL\gamma^2}{c^2} \tag{3}$$

with $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. That is the difference of the times of arrival of the waves at the detector ED as measured by a clock stationary with the axis of rotation.

## 3. Einstein's train-embankment thought experiment revisited

Einstein's TETE configuration involved a stationary platform, the embankment, and a wagon of a train moving on the platform beside an emitter and absorbers. He aimed to demonstrate that flashes of light reaching a stationary observer on the platform simultaneously cannot reach a moving observer at once.

In the original version of the thought experiment proposed by Einstein, emitters and detectors were separate entities. There were two sources of light placed at some positions on the embankment and two detectors one stationary M and one traveling M'.

In the configuration proposed, also considered in [1], and represented below (Fig. 2), the wagon is simplified as a moving emitter/detector, and mirrors are deployed to avoid the application of clocks and synchronization procedures. Such configuration aligns with the Sagnac interferometer, where no synchronization procedure between clocks is necessary.

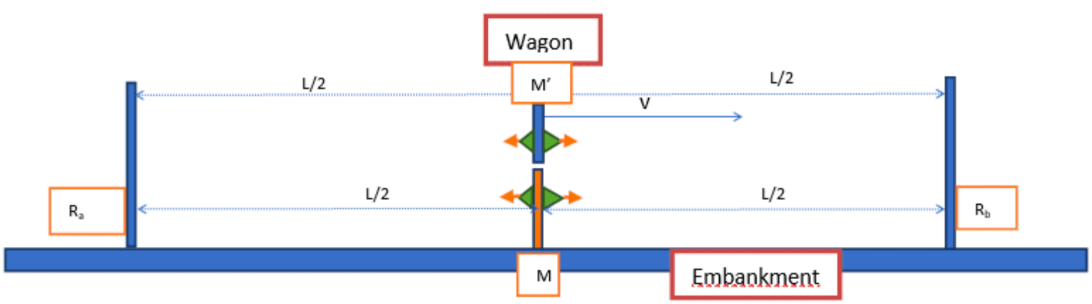


**Figure 2.** Simultaneous emission of waves from the wagon and embankment

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In the one-dimensional configuration reported in Fig. 2, light is emitted simultaneously in the same position M, from the embankment. The emitter/observer M stationary on the Embankment is placed at a distance L/2 from each mirror Ra and Rb .

The wagon is an "open wagon" in this case, just represented by the moving emitter/observer M' at speed v from left to right. The event of overlapping M and M' triggers the emission of two pairs of wave pulses propagating in opposite directions. It also sets the time reference to zero.

Due to the independence of the speed of light from the speed of the source, the four beams strike the mirrors $R_a$ and $R_b$ simultaneously. (Fig. 3).

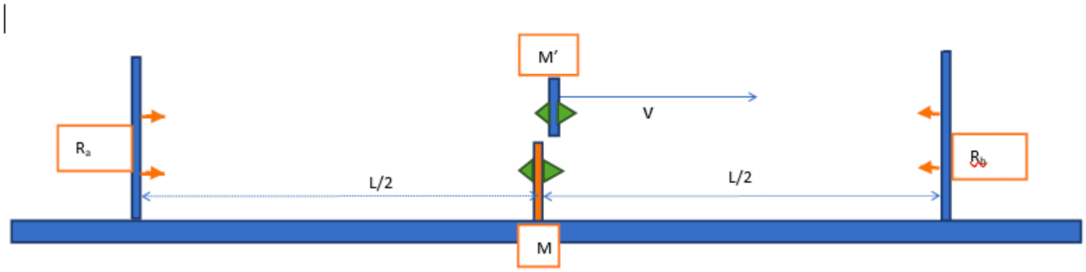


**Figure 3.** Simultaneous absorption of waves by the embankment

The light time to reach the mirrors, measured by a clock stationary on the embankment is TRa = TRb = ½ L/c, as measured from the embankment. The condition illustrated above is the same as having two pairs of sources, spatially separated, shooting at once.

The waves arrive simultaneously at the mirrors (as seen from the Embankment), what makes a difference is the time taken by the reflected waves to cover the distance with the two detectors.

The next event is the detection of the first wave, corresponding to the event of detection D'b, by M' (Fig. 4).

The following event is detection of both pulses by M, stationary on the embankment occurring after the time $T_D$ = L/c.


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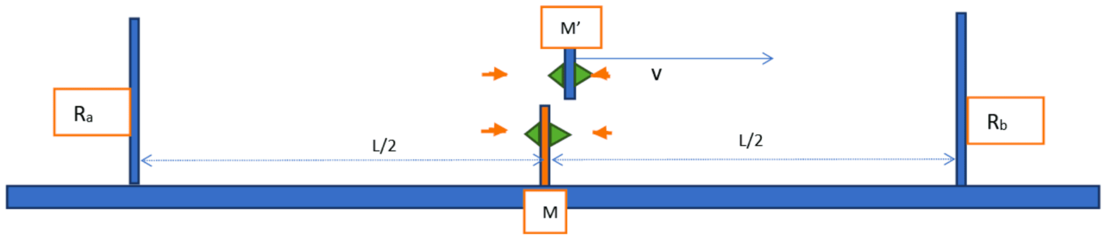


**Figure 4.** First absorption of a wave by the moving observer M'

Considering $T_{D'a}, T_{D'b}$ the time of the event of detection by $M'$, we can equate the lengths crossed by light and by the moving detector. That is similar to what was already proposed in the previous paragraph.

$$cT_{D'a} = L + vT_{D'a} \quad \text{and} \quad cT_{D'b} = L - vT_{D'b}. \tag{4}$$

Hence, by solving the equations the travel times are:

$$T_{D'b} = \frac{L/c}{1+v/c}, \text{and} \quad T_{D'a} = \frac{L/c}{1-v/c}.$$

For $v > 0$, it is $T_{Da} < T_D < T_{Db}$, it is indicating that the absorption of the four waves occurs at three distinct instants. The difference in the arrival times of the waves at $M'$ is given by:

$$T_{D'a} - T_{D'b} = \frac{\frac{2vL}{c^2}}{1-\left(\frac{v}{c}\right)^2}.$$

Finally, the difference of the arrival times of the waves is given by:

$$T_{D'a} - T_{D'b} = \Delta T_{\text{Sagnac}} = \frac{2vL}{c^2} \cdot \gamma^2.$$

where $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ is the Lorentz factor. The result is the same as Eq.(2)

The modified E-TETE becomes a rectilinear version of a closed loop in a classic Sagnac experimental setup, relying on the out-and-back constancy of the speed of light, peculiar to the original Sagnac configuration. That demonstrates that E-TETE and the Sagnac effect rely on the same principles.

By comparing the light-times $T_{D'a}, T_{D'b}$, to reach the moving observer $M'$, with $T_D$, the light-time to reach the stationary observer $M$, the following time-intervals are obtained:

$$\Delta T_{\text{Sim-a}} = T_{D'a} - T_D = \frac{vL}{c^2(1-\frac{v}{c})}, \text{and} \quad \Delta T_{\text{Sim-b}} = T_D - T_{D'b} = \frac{vL}{c^2(1+\frac{v}{c})}. \tag{5}$$


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This result represents two differences between times. Each of them is relevant to the same emission event, but different events of detection: one with a stationary observer on the embankment, and the second with a moving observer. All is measured by a stationary clock with the embankment.

Waves emitted simultaneously, by an emitter on the embankment and one on the wagon, are detected simultaneously by the detector on the embankment. They are not detected simultaneously by a moving detector. The extent of the failure of their simultaneity is expressed by Eq.(4).

The wave in the upper left position in Fig. 4, before its detection, chases the moving detector M' in a quite similar fashion as Achilles chases the Turtle, as illustrated in the next paragraph.

## 4. Zeno paradox of motion and the Sagnac effect - Achilles and the turtle configuration extended -

The time taken by Achilles or Flash to reach a moving target (turtle), as measured by a stationary observer, is a well-known problem in the literature. Here, it is quickly proposed again the problem and its solution with the only aim to extend its scope. Consider in Fig. 5 an emitter $E$ set at the origin and a moving absorber $P$ (turtle) traveling at speed $v$.

a. $P$ is at position $P_1$ when the light is emitted (or Achilles departs from $E$). The spatial interval $[E, P_1]$ has a length: $L = vt_0$, where $t_0$ is the time elapsed in the stationary frame. Light takes time $T_1$ to go across this interval: $T_1 = \frac{L}{c}$.

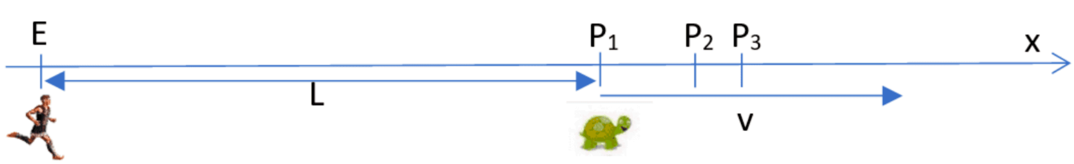


**Figure 5.** Achilles chasing the Turtle

b. During $T_1$, $P$ moves to a new position $P_2$. The displacement of $P$ is such that: $P_1 - P_2 = L_1 = vT_1$. Substituting $T_1 = \frac{L}{c}$, we find: $L_1 = v \cdot \frac{L}{c}$. To cross the distance $L_1$, light takes additional time $T_2$, given by: $T_2 = \frac{L_1}{c} = \frac{vL}{c^2}$.

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c. During the interval $T_2$, the absorber $P$ moves further from $P_2$ to $P_3$. The displacement is given by: $P_2 - P_3 = L_2 = vT_2$. Substituting $T_2 = \frac{vL}{c^2}$: $L_2 = v \cdot \frac{vL}{c^2} = \frac{v^2 L}{c^2}$. Light takes additional time $T_3$ to cross this new interval: $T_3 = \frac{L_2}{c} = \frac{v^2 L}{c^3}$.

Adding all intervals: $(T_1 + T_2 + T_3 + \cdots + T_n)$ the total time taken by light (Achilles) to reach the moving target (turtle) becomes:

$$Z_{\text{time}+} = \frac{L}{c} + \frac{vL}{c^2} + \frac{v^2 L}{c^3} + \cdots + \frac{v^n L}{c^{n+1}} = \frac{L}{c}\left(1 + \frac{v}{c} + \frac{v^2}{c^2} + \cdots + \frac{v^n}{c^n}\right) = \frac{L}{c} * Sn$$

For n arbitrarily large, the sum Sn converges to: $S = \frac{1}{1-v/c}$ with $\frac{v}{c} < 1$. Thus, the time becomes:

$$Z_{\text{time}+} = \frac{L}{c} \cdot \frac{1}{1 - \frac{v}{c}}.$$

That is the time taken by Achilles to reach the turtle, initially separated by the distance L.

The same result could be obtained in a much easier way by just using Eq.(4). The longer derivation was necessary to familiarize better with the governing mechanism at the base of the effect.

It is now represented in Fig. 6 an extension of the original configuration. Another Achilles coming from an opposite direction to meet the Turtle (always departing from a distance $L$)

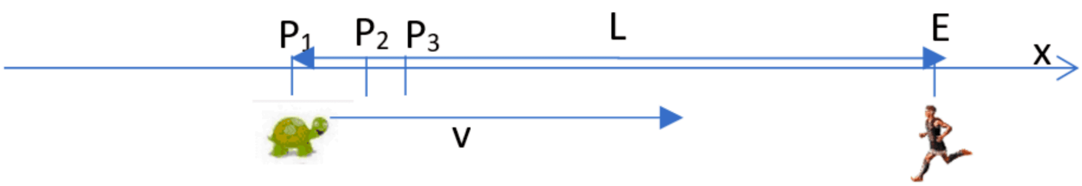


**Figure 6.** Achilles meeting the Turtle

The time taken to reach the Turtle is obtained by inverting one velocity:

$$Z_{\text{time}-} = \frac{L}{c} \cdot \frac{1}{1 + \frac{v}{c}}.$$

The simultaneous application of both configurations represented in Fig.7 is a more general case, where Achilles1 chases the Turtle, and Achilles2 runs toward the Turtle.


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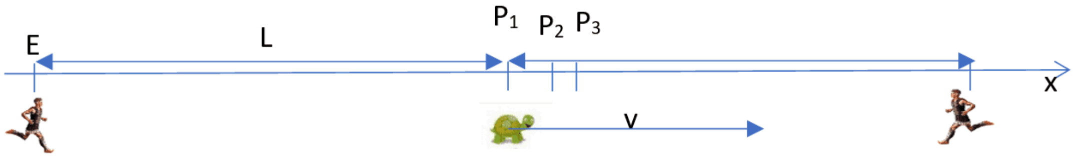


**Figure 7.** Two Achilles, one chasing and one meeting the Turtle

The difference in their arrival times is given by:

$$Z_{\text{time}+} - Z_{\text{time}-} = \frac{\frac{L}{c}}{1-\frac{v}{c}} - \frac{\frac{L}{c}}{1+\frac{v}{c}}. = \frac{\frac{2vL}{c^2}}{1-\left(\frac{v}{c}\right)^2} = \frac{2vL\cdot\gamma^2}{c^2}$$

The expression obtained is the same as Eq. (3). Although the final result in terms of time difference is the same, the departures of Achilles and the turtle must be ”coordinated”. It would be necessary to place, at the position of departure, Einstein synchronized clocks. It must be assumed that the frame, where Achilles and the turtle departed from, is inertial, to let the synchronization procedure work properly.

There is another way to obtain the same results and avoid synchronization procedures to coordinate the departure of Achilles and the turtle. Achilles and the Turtle can be arranged in a closed loop of length $L$ as represented in Fig. 8.

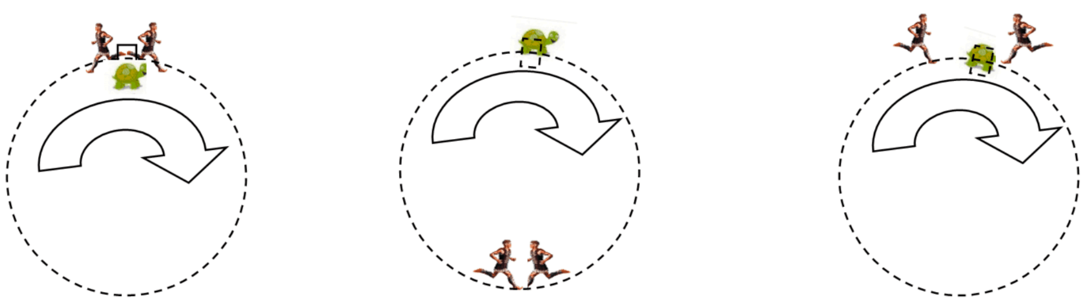

**Figure 8.** Zeno paradox of motion revisited and Sagnac

The two Achilles (light) and the Turtle (absorber) start their motion simultaneously from the same position then one Achilles manages to reach the Turtle first, after the time: $Z_{\text{time}-} = \frac{L/c}{1+v/c}$. and the other after the time $Z_{\text{time}+} = \frac{L/c}{1-v/c}$.


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A full analogy with the Sagnac effect is at hand, except for an important aspect. The sagnac effect occurs with a source already in motion (at constant speed but non-inertial in general). Only if it is considered that the speed of Achilles is not affected by the speed of the source where he departed from, then the analogy is complete. Only in the particular case where everybody departs at once from a rest position in the inertial frame, the analogy is sound. Otherwise, if Achilles took the inertia of the source, like a bullet would do, the time difference would be 0, hence the analogy would be lost.

To let the ”Achilles and the Turtle” be the governing mechanism of the Sagnac effect and the E-TETE, Achilles must be given a peculiar property. Achilles is requested to behave like a wave, a FLASH with his characteristic speed, regardless of the status of motion of the object from where he departed.

The enhanced Zeno paradox of motion with the wavelike property of Achilles, suites very well the E-TETE configuration. The Sagnac effect can be considered an extended version of the “pursuit or chasing problem” of light waves and a moving detector at constant speed.

## 5. The arrival times measured by the moving detector

The interval of time measured by a stationary clock on the embankment, as given by Eq. (2) and Eq. (4), is generally valid irrespective of the detector’s speed. However, these formulas are not suitable for a clock co-moving with the detector. In such cases, the “twin effect” or time dilation must be considered. The results already found must be multiplied by $\gamma^{-1}$. From Eq. (3):

$$\Delta T'_{\text{Sagnac}} = \frac{2vL\gamma}{c^2} \tag{6}$$

where $\Delta T'_{\text{Sagnac}}$ is the Sagnac time difference corrected for the co-moving clock. That would be the difference in the arrival times measured by the Turtle.

The difference in the arrival times for the light-time $L/c$, relevant to the condition when the source and observer are at rest, is expressed in Eq. (5). It must also be multiplied by $\gamma^{-1}$. The difference in arrival times between a light wave detected at rest ($L/c$) and the wave detected in motion becomes:

$$\Delta T'_{\text{Sima}} = \frac{vL}{c^2}\sqrt{\frac{1+v/c}{1-v/c}}, \text{and} \quad \Delta T'_{\text{Simb}} = \frac{vL}{c^2}\sqrt{\frac{1-v/c}{1+v/c}} \tag{7}$$

which has the same form as a relativistic Doppler shift.

This represents the extent of the failure of simultaneity as measured by a clock co-moving with the detector.

## 6. Conclusions

A very simple derivation of the times involved in the Sagnac effect and E-TETE has been provided, along with relations between the quantities at stake in their most general form. The modified E-TETE, in its essence, is just a rectilinear version of the Sagnac configuration. The mechanism at the base of both is a suitable extension for waves of the famous pursuit problem of "Achille's and the turtle".

The failure of simultaneity, in detecting simultaneously departing waves, has been described and mathematically formulated with simple descriptions. The times measured by observers at high speeds have also been calculated, by relying on the experimentally verified "twin effect".

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### Declarations


**Funding:** No specific funding was received for this work.

**Potential competing interests:** No potential competing interests to declare.