Review of: "Nonlinearity and Illfoundedness in the Hierarchy of Large Cardinal Consistency Strength"

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It is a common observation that the consistency strengths of the set-theoretic principles actually studied are linearly ordered. It was not obvious before investigation that this would be the case, all the more so since many of these principles superficially have nothing to do with one another. Therefore this phenomenon calls for explanation. This article is a philosophical enquiry, buttressed by mathematical results, into just this topic. While most scholars who deal with this hold that natural assertions are linearly ordered, even well-ordered, the author sets himself the task of challenging that belief.

Despite that goal, Hamkins implicitly shows the hypothesis of linearity a good deal of respect by calling his self-appointed task “hard”. Presumably it’s hard because the linearity hypothesis has strong evidence in its favor, even if Hamkins does not review such arguments here. What this admission of hardness does do is inspire the critic, at least this critic, to be gentle. It is after all easier to find a flaw in someone else’s position than to develop and defend one’s own. No doubt that’s why I find the more convincing parts of this work to be his critiques of the natural linearity hypothesis, rather than his claims for non-linearity.

Where I find Joel’s arguments for non-linearity and ill-foundedness wanting are in the fact that they all depend on computability phenomena, which seem to me beside the point. He sets the stage by showing that the consistency degrees are not just ill-founded but actually dense, and highly non-linear. These are all based on Gödelian-style self-referential statements in arithmetic, which Joel makes clear he knows are not what’s at stake in the natural set-theoretic consistency questions. But even when he does turn to set theory, issues of computability seem to be at the core of the examples. For instance, the statements “there are *n* inaccessible cardinals,” where *n* is a natural number written in unary (i.e. a fixed number of iterations of the successor operation applied to 0), are well-ordered by consistency strength; where he finds non-linearity is in statements like “the number of inaccessibles is the number of steps it takes this particular Turing machine to halt.” Indeed, Hamkins himself goes on to say that such arguments are not really about large cardinals, and then proves theorems of incomparable consistency strengths of assertions of membership and non-membership in computably enumerable sets. His next examples seem to me no different in this regard. He observes that the consistency statements we consider, like Con(ZF), are not so much the consistency of a theory as the consistency of a particular enumeration of a theory. One can well ask what would happen with a different enumeration of, say, ZF. For instance, take what he calls the cautious enumeration, which lists the axioms of ZF in some standard order until such time as a contradiction among them is found, and then stops. If you believe ZF is actually consistent, then you will believe that the cautious enumeration actually generates ZF. Yet he shows it has strictly weaker consistency strength. This I find to be one of the stronger parts of the paper, not because it convinces me of non-linear or non-well-founded phenomena, but because it introduces several interesting kinds of enumeration and does a good job analyzing their consistency strength.

The paper is best toward the end, as I pointed out, when he turns from expositing non-linear phenomena to questioning the arguments for linearity. The best criticism here is the discussion of naturality. Sure, the wildly-ordered consistency strengths explored earlier all have to do with computability, which do not rightly pertain to set theory, and arguably are not even natural mathematical statements. But what is natural? To make the argument solid, one would have to first explain and then justify what “natural” means. The most solid point of the paper I would say is his proposal for what to do with the concept of naturality. He makes the analogy with computability theory. The Turing degrees have a wild and complicated structure, but the natural Turing degrees are the ordinal-indexed jumps of 0. Hamkins brings out current topics of investigation, namely Martin’s conjecture and degree-invariant solutions to Post’s problem, as specific ways of explaining the ordering of the Turing degrees without reference to naturality. He focuses on the specific desired properties, the restriction of actions to Borel actions and degree-invariance, as concretizations of the intuition of naturality. He calls on set theorists to similarly deepen their intuition of naturality by developing precise properties they would like to see fulfilled. Then we could hope to prove, or disprove, that the consistency degrees of interest are linear or well-founded.