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Bell’s Theorem and Counter factual Definiteness CH

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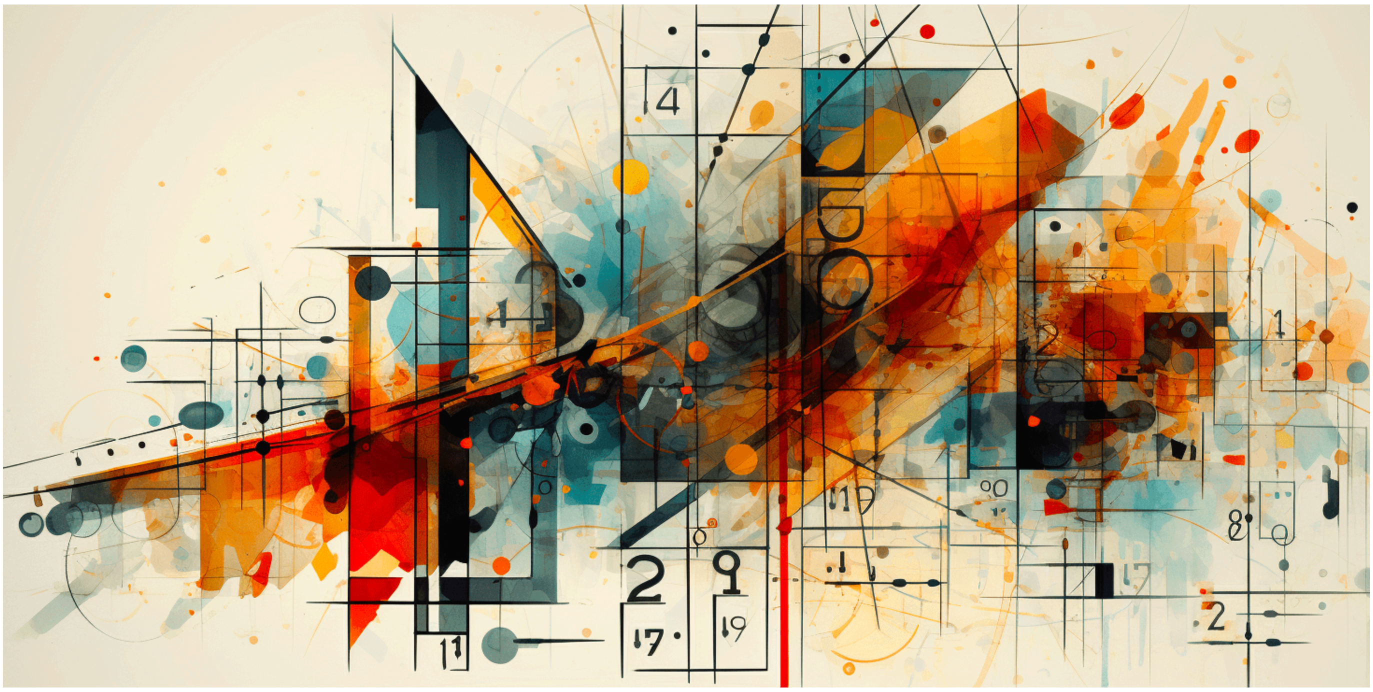

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Abstract

The decisive simplicity of Bell's no-go results has inspired a seemingly endless search for their undeclared assumptions. That of counter-factual definiteness has seen a recent crescendo of interest. This article aims to assess the claims of (i) Hess and Philipp and (ii) De Raedt, Michielsen, Zukowski and Brukner that Bell's theorem incorporates this assumption. It argues that these claims do not succeed, except with the aid of circular reasoning or implausible presuppositions. It argues further that they share a commonality: all of them need to be supplemented with additional, metaphysical content in order to secure the link between this attribution and various of Bell's quantitative premises (such as the factorisability condition), which lead to the canonical inequality.

Introduction

The decisive simplicity of Bell's no-go results has inspired a seemingly endless search for their undeclared assumptions. Counter-factual definiteness, freedom of choice with respect to measurements performed, and various doctrines of realism all stand among the hypotheses Bell has been accused of taking for granted – accusations made perhaps all the more alluring by his compact style of presentation. The first of these, although dating back at least as far as the S-matrix formalism introduced by H. P. Stapp in 1971, has seen a recent crescendo of interest. G. Blaylock, in his 2009 "The EPR paradox, Bell's inequality, and the question of locality," contends that the derivation of one of Bell's canonical inequalities smuggles in counter-factual definiteness by assuming single definite sequences of outcomes corresponding to measurements not performed.[1] M. Zukowski and C. Brukner draw the same conclusion, denying that the inequality is derivable from locality assumptions alone, and arguing that Bell relies on a Kolmogorovian axiomatization of the probability space over which are calculated the characteristic expectation values of pairs of spin measurement outcomes.[2] In their 2021 "Note on Bell's Theorem Logical Consistency" J. Lambare and R. Franco dismiss the relevance of counter-factual definiteness for the theorem as "unnecessary and inconsistent" on the basis of a reconstruction of Bell's argument.[3] K. Hess, in publications with W. Philipp, H. De Raedt and K. Michielsen, contends that Bell's implicit treatment of time variables in the theorem involves counterfactual logic, narrows the scope of his argument and leads to a loss of generality, counter-posing an alternative model.[4] Rubin claims nakedly that the theorem "depends crucially on counterfactual reasoning."[5]

The essay analyses the contributions of two main groups of critics of Bell's theorem, being (i) Hess and Philipp; (ii) Hess, De Raedt, Michielsen, Zukowski and Brukner. It reconstructs the arguments submitted by each, and contends that these arguments fail.

These critical expositions (i) and (ii) of Bell's arguments concern two alleged weaknesses. Firstly, Hess and his collaborators argue that the failure of Bell's analysis to take "time-like correlated parameters"[6] into account limits the generality of the theorem, opening up the potential for alternative models parameterised temporally. This failure consists in the absence of the time-variable from the parameter space used to define Bell's probability measures, and is deemed responsible for the independence of outcomes of spin-measurement interactions on distant measurements in his argument, in the sense that the factorisability condition it employs follows only if one excludes these parameters from any space which features in Bell's premises. In the course of this argument, the absence of these time-variables is asserted to be a necessary condition for counter-factual definiteness. Secondly, Hess, De Raedt, Michielsen, Zukowski and Brukner claim that Bell's use of probabilistic reasoning in his derivation commits him to proving the applicability of Kolmogorovian probability axioms, and that these axioms in turn entail an assumption of counter-factual definiteness. They argue from here that Bell's analysis is incomplete in a fundamental sense. The authors' alternative model, enriched with additional

[1] Blaylock, G. (2009).
[2] Zukowski, M. and Brukner, C. (2015).
[3] Lambare, J. and Franco, R. (2021).
[4] Hess, K., De Raedt, H., and Michielsen, K. (2016).
[5] Rubin, M. (2001). p. 1
[6] Hess, K and Philipp, W. (2001a).

variables, posits a decomposition of the Bell parameter space into two subspaces with associated probability measures each dependent on the properties of distinct measurement apparatus. They claim that this dependency invalidates key aspects of Bell's reasoning about hidden variable theories, qualifies their amended model as nonlocal, and supplant any refutation of this model by purported violations of Bell's inequality in actual experiments.

This essay aims to assess these claims that Bell's theorem assumes counter-factual definiteness. It argues that these claims do not succeed, except with the aid of circular reasoning or implausible assumptions. It argues further that they share a commonality: all of them need to be supplemented with additional, metaphysical assumptions in order for this attribution to be made. These metaphysical assumptions are required to secure the link between this attribution and various of Bell's quantitative premises (such as the factorisability condition) which lead to the canonical inequality. Without them, this link rests on a weak textual basis.

Counter-factual definiteness and parameter space completeness

The argument of Hess and Philipp is spread across multiple publications, with the hard labour carried out in their 2001 "A possible loophole in the theorem of Bell," and "Einstein-separability, Time Related Hidden Parameters for Correlated Spins, and the Theorem of Bell," as well as in the 2016 "Counterfactual Definiteness and Bell's Inequality" by Hess, H. De Raedt and K. Michielsen. The first of their two attacks focuses on the alleged incompleteness of the parameter space used by Bell to formulate the theorem. Its overall logic proceeds as follows:

Argument A

1. Bell inequality tests violate local hidden variable theories only if counter-factually definite
2. Local hidden variable theories are counter-factually definite only if their spin measurement settings are time-independent[7]
3. Their spin measurement settings are time-independent only if hidden variable theories are incomplete
4. Hidden variable theories are complete if product probability measures obeying Einstein-separability are identifiable
5. Product probability measures obeying Einstein-separability are identifiable
6. Therefore, Bell inequality tests fail to violate local hidden variable theories

From 4. coupled with 5., it follows that hidden variable theories are complete. From the conjunction of this consequence and the third premise it follows that spin measurement settings are not time-independent. From the conjunction of this consequence and the second premise it follows that local hidden variable theories are not counter-factually definite. The overall conclusion then results by appeal to the first premise. The question of counter-factual definiteness (CFD) arises only indirectly, as a by-product of the question as to whether local hidden variable theories are provably subject to Bell's inequalities (in their various forms) without recourse to additional controversial assumptions. Its rejection is a intermediate step in deducing 6. by *modus tollens*: the existence of Einstein-separable measures rebuffs a series of necessary conditions relating to Bell inequality violation. So formulated, the Hess-Phillip argument seems to be valid. The main controversy arises in determining whether these premises are warranted.

Most of the analysis of Hess and Philipp focuses on a justification of Premise 5, which attempts to restore Einstein-separability within a hidden variable framework. In this respect, the clearest statement of the authors' characterisation of the connection between Einstein-separability and the measure constructed in their paper arises in the following two passages:

> If setting b at station S_2 is changed into setting c, the probability distribution governing the parameters Λ_1, a, t at station S_1 must remain unchanged…the average frequencies of the

[7] At times, the authors dismiss the difference between an outright refutation of hidden variable theories and the real content of Bell's research, which concerns the compatibility of these theories with various assumptions, e.g. locality; in the opening to the paper they gloss his objective as a demonstration "that a mathematical description of EPR-type experiments [2],[3] by a statistical (hidden) parameter theory [4] is not possible." These infidelities are put aside for the purposes of the present essay.

> parameters $f(\Lambda_1, a, t)$ in each of the intervals between -3 and $3n$ must not change when setting b is changed in the other station.[8]
>
> Therefore, if the setting b gets changed to the setting c, the random variables $f(\Lambda_1, a, t)$, $g(\Lambda_2, a, t)$ are also independent and there is no change in the distribution of $f(\Lambda_1, a, t)$ by changing from b to c. This should remove all suspicions of spooky action.[9]

An alternative framing of this view is provided by Myrvold. In his analysis of Hess' and his colleagues' case it is represented as follows:

> Since μ_{ab} is the same distribution for every a, b, changing the setting b will not affect the distribution, on μ_{ab}, of u, and changing the setting a will not affect the distribution of v.[10]

Hess' and colleagues' rationale is therefore the following: a model is findable fulfilling two conditions: firstly, the predictions of quantum mechanics as to distributions of spin measurement outcomes are reproduced; secondly, the model nonetheless posits hidden variables, in the sense that distributions of measurement outcomes show no dependence on space-like separated settings. Measurement sequences during which one of two causally isolated Stern-Gerlach apparatus is re-oriented based on whatever randomized factors one may choose leaves the results of the other's tests unaltered. This *prima facie* contradicts Bell's thesis as to the incompatibility of these two states of affairs: the inequalities predicted by hidden variable theory fail to square with the predictions of the quantum formalism, shorn of supplementation. Their claim incorporates what the literature following Bell has come to term *parameter independence.* Bell's crucial factorisability assumption, namely that $P(A, B|\lambda) = p(A|\lambda)p(B|\lambda)$ for space-like separated outcomes A, B, which he viewed as an implication of the heuristic principle of local causality, is crucial in the derivation of his inequality.[11] This assumption was prised apart by Jarrett and Shimony into two conditions, parameter independence and outcome independence, with the former capturing the indifference of the probability function for a given outcome on distant measurement settings, viz. $P_{a,b}(A|\lambda) = P_{a,c}(A|\lambda)$, for $b \neq c$, and the latter the independence of the actual *results* of each measurement, viz. $P_{a,b}(A, B|\lambda) = P_{a,b}(A|\lambda)P_{a,b}(B|\lambda)$.[12] For Hess and Philipp, then, it is the indifference of the probability distribution (corresponding to "average frequencies") to alterations in measurement settings which is the correlate of parameter independence.

Their justification for the crucial fifth premise, then, proceeds in two parts. One seeks to demonstrate that the quantum predictions are recovered within their own model, incorporating suitable time-dependencies they allege to be absent from Bell's work. The other involves proof that parameter independence obtains for this model. In respect of the former, Hess and Philipp show for an EPR-style case that the expectation value of a product of a pair of mixed parameters, depending both on time and each on one of the two measurement apparatus' settings, matches the quantum prediction in the following sense:

$$E\{A_{\boldsymbol{a},t}(\lambda^1, \cdot, f(\Lambda^1_{\boldsymbol{a},t}(\lambda^1, \cdot)))B_{\boldsymbol{b},t}(\lambda^2, \cdot, g(\Lambda^2_{\boldsymbol{b},t}(\lambda^2, \cdot)))\} = -\boldsymbol{a} \cdot \boldsymbol{b}$$ [13]

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[8] Hess, K and Philipp, W. (2001a). p. 19

[9] Hess, K and Philipp, W. (2001a). p. 21

[10] Myrvold, W. (2002). p. 9

[11] Bell, J. (2004). pp. 54, 243

[12] Shimony, A. (1990).

The second aspect of the justification of this premise, namely the demonstration of parameter independence, is significantly more involved, and is couched in the claim that, for a chosen parameter space, two functions $f(\Lambda^1_{\boldsymbol{a},t})$ and $g(\Lambda^2_{b,t})$, designed to mimic the spin-measurement outcomes in an EPR-style staging, are distributed uniformly over a probability space constructed by the authors.[14] If the presence of nonlocal effects between space-like separated systems hinges, in this case, on the dependence of certain parameters on measurement settings, a uniform distribution of these two parameters confutes any such correlation and, according to the authors, qualifies their model as appropriately local. Their demonstration attempts to show that the probability density governing $f(\Lambda^1_{a,t})$ and $g(\Lambda^2_{b,t})$, being $\frac{1}{N}\sum_{m=1}^{N}\rho_{\boldsymbol{ab}}(u,v,m)$, is a function only of an even integer number n which parameterises the size of the set Ω belonging to the relevant measure space $\Omega, F, \mu = \mu_{\boldsymbol{a},\boldsymbol{b}}$ and an additional variable θ which bounds the system's total mass by the equation $0 \leq \theta < {}^{1}\!/_{24}$. Consequently, this density displays no dependency on the spin measurement directions $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$, and measurements performed along $\boldsymbol{b}$ or $\boldsymbol{c}$ rather than $\boldsymbol{c}$ or $\boldsymbol{b}$ can have no effect on the probability of measuring various outcomes of $\boldsymbol{a}$.

Does this successfully underpin Premise 5? In his rebuttal of the Hess programme, W. Myrvold counter-poses a model he takes to be analogous to the foregoing staging of an EPR-style gedankenexperiment, aiming to undermine the connection between two claims: (i) that the probability measure for the two sets of space-like separated spin measurements is expressible as a product of two subsidiary measures each of which depends on the setting of one and only one apparatus, and (ii) parameter dependence fails for the model. Whereas the strategy of Hess and his colleagues involves finding a time-dependent probability space as a means to both preserve Einstein-separability and undermine the Bell factorisability condition, Myrvold aims to show that this space, despite the absence of occurrences of parameters labelling the spin measurements performed at a spatio-temporal remove from each probability measure, nonetheless embeds correlations between the relevant hidden variables. If successful, this demonstration would surely undermine their fourth premise. It is clear from the opening of "Einstein-Separability, Time Related Hidden Parameters for Correlated Spins, and the Theorem of Bell" that the authors view the product measure $\mu = \mu_{\boldsymbol{ab}}$ as a decisive lynchpin in salvaging Einstein-separability, going so far as to reify it as a "guarantee".[15]

Myrvold's argument considers two functions $\sigma^i_a(u)$ and $\tau^i_b(v)$ with u, v being hidden variables associated with the Stern-Gerlach apparatus. These functions amount to a probability density of the form $\rho^{ij}_{ab}(u,v) = \sigma^i_a(u)\,\tau^i_b(v)$ and, like the setting dependent subspace product measures (SDSPMs) of Hess and Philipp, reproduce the quantum mechanical prediction for the expectation value of products of spin measurements. These hidden variables are subsequently demonstrated to be correlated to the extent of satisfying parameter dependence.[16] In particular, the values of any one of the pair of hidden variables allows one to infer both of the two measurement orientations a and b.

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[13] Hess, K and Philipp, W. (2001a). p. 19
[14] Hess, K and Philipp, W. (2001a). p. 21
[15] Hess, K and Philipp, W. (2001a). p. 3
[16] Myrvold, W. (2002). p. 11

In order to grasp this conclusion, consider the model's relationship to the parameter independence (PI) criterion $P_{a,b}(A|\lambda) = P_{a,c}(A|\lambda)$ which in this context takes the form $P_{a,b}(A|u,v) = P_{a,c}(A|u,v)$ for some outcome of interest A. Myrvold holds that this assumption fails for the Hess model, qualifying their model as surreptitiously nonlocal and "violating…the conditions assumed by Bell."[17] This requires inspecting the density function $\rho_{ab}^{ij}(u,v) = \sigma_a^i(u)\, \tau_b^i(v) k_j(u,v)$, where $k(u,v)$ denotes a function which takes on the value 1 for only one value of u and only one value of v which span the space $\Omega = [0,4) \times [0,4)$. As a result, for any choice of j, only four elements of Ω are non-zero and exactly one u together with exactly one v belonging to $u \times v$ generate a probability of 1 rather than 0. Further, the model constructed by Myrvold is such that, for each choice of i given any such choice of j, the product $\sigma_a^i(u)\, \tau_b^i(v)$ has probability 1 for only one combination of a, b (taking on one of two values). Therefore, most importantly, out of the possible choices of a, i, j, u, v, the distributions of $\sigma_a^i(u)$, $\tau_b^i(v)$ and $k_j(u,v)$ which result allow only one value of b consistent with a non-zero probability $\rho_{ab}^{ij}(u,v)$ which these functions determine.

This result can be deployed in order to prove violation of PI for the model by contradiction. Firstly, assume parameter independence holds. Secondly, take any set of choices of a, i, j, u, v. Thirdly, take some value, 0 or 1, for b such that $P(A) = 1$. By hypothesis, substituting for either of these values for b another value c, via simple maps $0 \to 1, 1 \to 0$, leaves the probability of the outcome A unchanged. This implies that, for fixed a, i, j, u, v, more than one value of b consists with a non-zero probability of $\rho_{ab}^{ij}(u,v)$. This, however, contradicts the upper bound established on the number of non-zero probabilities in the model. As a result, reasons Myrvold, the PI must be jettisoned.

How (dis)analogous are these two models? Does Myrvold's attack land successfully on the Hess SDSPM framework, or is he at all guilty of contriving straw men? Given that Myrvold's argument relies upon the construction of a new parameter space Ω in order to illustrate violation of PI, it might be questioned how relevant this space is to the Hess-Philipp argument, and whether the features of their model responsible for this outcome exist also in the latter. In particular, it is worth observing that the rejection of PI would follow even in the event of far weaker correlations between b and A (or between a and B): finding just one example of some values of b, c for choices of u, v, i, j such that $b \to c$ alters $P(A)$ would suffice to violate parameter independence; here it happens that for any combination of such choices well-defined by this model $b \to c$ alters $P(A)$, meaning $P_{a,b}(A|\lambda) \neq P_{a,c}(A|\lambda)$. Myrvold notes the extremity of this violation is an idiosyncrasy of his model *cf.* Hess-Philipp, but that the latter does not on that account circumvent spectres of nonlocality.[18] So what links the two?

Myrvold identifies the structure of the probability density $\rho_{ab}(u,v) = \sigma_a(u)\tau_b(v)\nu(u,v)$ which appears in two separate papers[19] as the entry point of PI violation, but is not specific about why

[17] Myrvold, W. (2002). p. 13
[18] Myrvold, W. (2002). p. 11
[19] Hess, K and Philipp, W. (2001a), (2001b)

this leads to the deficiencies other than to say as much is clear "by inspection."[20] He does deny, however, that the uniformity of the distribution is a sufficient condition for PI, a relation to which Hess and Philipp are clearly committed[21]:

> This shows that the joint density of $f(\Lambda^1_{a,t})$, $g(\Lambda^2_{b,t})$ is uniform … and therefore $f(\Lambda^1_{a,t})$ and $g(\Lambda^2_{b,t})$ considered as random variables are stochastically independent and themselves have uniform distribution over the appropriate intervals. Therefore, if the setting b gets changed to the setting c, the random variables $f(\Lambda^1_{a,t})$, $g(\Lambda^2_{b,t})$ are also independent and there is no change in the distribution of $f(\Lambda^1_{a,t})$by changing from b to c. This should remove all suspicions of spooky action.[22]

Naturally, then, the relation due for analysis becomes the link between the proposition that the joint density of $f(\Lambda^1_{a,t})$, $g(\Lambda^2_{b,t})$, defined as $\rho_{ab}(u,v) = \sigma_a(u)\tau_b(v)\nu(u,v)$, is uniform over some interval, and PI: $P_{a,b}(A|\lambda) = P_{a,c}(A|\lambda)$. Does the "Einstein-Separability" paper prevail in securing this link or is Myrvold right to eschew it? Note here that $u, v \in \lambda$, these being the hidden variables that determine the spin measurement outcomes alongside the apparatus orientation. First of all, observe that the uniformity of the joint density pertains to the probability of products of σ_a and τ_b, not to individual outcomes. Inferring characteristics of the probability distribution of $P_{a,b}(A|\lambda)$ *vis à vis* $P_{a,c}(A|\lambda)$ requires the integration over the relevant interval of some function $A(a,u)$ with $\rho_{ab}(u,v)$ with respect to hidden variable u. In turn, the invariance of this outcome under the transformation $b \rightarrow c$ requires the independence of u and b, c. But unless u and v are independent this property is not assured by the uniformity of the product $\sigma_a(u)\tau_b(v)\nu(u,v)$. Otherwise, counter-veiling variations in the constituent functions $\sigma_a(u)\tau_b(v)$, which themselves embed significant correlations, could be responsible for the uniformity of $\rho_{ab}(u,v)$. Passing from this uniformity to the invariance of the probability of an individual spin outcome with respect to shifts in distant measurement orientations involves a non-sequitur.

In order to envisage how this impinges on the argument of Hess and Philipp, recall, the fifth premise: Product probability measures obeying Einstein-separability are identifiable. If their measure is no such example, the persuasiveness of this premise attenuates materially (although, of course, outright refutation of this premise would require a proof, not just that this particular model does not suffice, but that those models are impossible). Above, it was shown that the authors' understanding of Einstein-separability is tantamount to the assumption of parameter independence operative in Bell factorisability. The contention of the superjacent paragraphs is that their ascription of parameter independence to Bell-like models stands unjustified. The Hess-Philipp thesis that Bell's theorem fails to reveal inconsistencies between quantum-mechanical and local hidden variable-theoretic predictions is not proven by their argument.

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[20] Myrvold, W. (2002). p. 12

[21] Myvold's is, like that of Hess, a uniform distribution.

[22] Hess, K and Philipp, W. (2001a). p. 21

Counterfactual definiteness and probability

The second distinctive line of argument aiming to ascribe assumptions of counter-factual definiteness to Bell's theorem spans a greater range of commentators than Hess and Philipp alone. It focuses on the probability space invoked by Bell to tease out the implications of hidden variable theory for spin measurement combinations. Preliminarily, it is useful to recapitulate Hess' definition of a counterfactually definite theory, being one which is

> Described by a function (or functions) that map(s) tests onto numbers. The variables of the function(s) argument(s) must be chosen in a one to one correspondence to physical entities that describe the test(s) and must be independent variables in the sense that they can be arbitrarily chosen from their respective domains.[23]

Phrased as such, this defines a necessary rather than sufficient condition for a theory to be counterfactually definite: CFD requires that any function which predicts measurement outcomes ("tests") must take as inputs ("variables of the function(s) argument(s)") only independent variables. The authors also refer to a more commonsensical and less mathematical definition of CFD in terms of the existence of determinate results for unperformed tests, following that of Peres and Leggett.[24] Hess proceeds to argue that, without deleterious modification, Bell's no-go results – although he is unspecific about which – fail to meet this condition. If the outcomes predicted by Bell, taking the form of statistical measures of combinations of spin measurements made at space-like separation, enable his theory to satisfy CFD, the spin measurements along with whatever other variables are involved must be independent variables. Hess, De Raedt and Michielsen deny this for a combination of two variables: spin measurements along an arbitrarily chosen axis $\boldsymbol{j}$ and the time variable t: "setting and time variables of EPRB experiments cannot be defined on one probability space," since "that space would necessarily contain impossible events (such as different settings for the same measurement times,"[25]) with non-zero probability. Therefore, the authors conclude, the space of independent variables must be artificially restricted so that such conflicts do not arise. The authors' reasoning can be represented as follows:

Argument B

1. Bell's theorem requires CFD
2. CFD holds for the parameter space of a theorem only if all variables in the space are independent
3. All variables are independent for the space of Bell's theorem only if this space is incomplete
4. Therefore Bell's theorem involves an incomplete parameter space

Two observations are due here, one relating to logical structure and the second relating to the import of the third premise and conclusion. First of all, it is noteworthy that Hess' argument contains no *justification* for the claim that Bell's theorem *relies* on CFD. It presupposes rather than proves that this is the case. This presupposition is then deployed in inferring the limitations which must be applied the parameter space utilised by Bell to avoid what they take to be objectionable results. An independent justification is needed in order for the former (and most

[23] Hess, K., De Raedt, H., and Michielsen, K. (2016). p. 3
[24] Peres, A. (1995). pp. 2006-7; Leggett, A. (1987). pp. 164-5
[25] Hess, K., De Raedt, H., and Michielsen, K. (2016). p. 3

controversial of all) claim to be warranted. In this sense, it seems that their claim that "Bell's choice of mathematical functions and independent variables implicitly includes counterfactual definiteness,"[26] stated nakedly in their paper's abstract, is somewhat unrepresentative of the paper's ensuing logic. In turn, they formulate their very conclusion as the claim that, "The major premise for the derivation of Bell's inequality is counterfactual definiteness,"[27] despite the remarkable fact that the structure of the paper offers it little or no support. So construed, their argument is an argument for the incompleteness of Bell's analysis, in the (at best) narrow sense that there exists at least one pair drawn from an open space of variables which cannot be regarded as independent.

So is any independent justification offered for the first premise in this argument – or are the authors mired in circularity? The only possible such justification is featured subtly – and with the bare minimum of signposts – in their analysis of the probability spaces at work in no-go theorems. The earlier paragraphs of the section "Counterfactual reasoning and EPRB experiments" assign a very heavy explanatory burden to Bell in order to make his theory operate:

> If we perform "calculations where these unknown results are treated as if they were numbers", then we must use the mathematical concept of functions or something equivalent in order to link the imagined but possible tests with numbers. A one to one correspondence of the possible tests and the numbers needs to be established.
>
> In the framework of Boole, we need to be sure that the data can be described by ultimate alternatives (the Boolean variables) and in the framework of Kolmogorov we must be sure to deal with random variables (functions on a Kolmogorov probability space).[28]

These passages assert (albeit obliquely) that no-go theorems must satisfy two demands. Firstly, an isomorphism must hold between parameters ("numbers") and observations from measurements ("tests"). Secondly, Bell's probabilistic reasoning presupposes the applicability of Kolmogorovian space. The first demand leads to the authors' attribution of CFD to Bell: all parameters must correspond to possible observations; parameters corresponding to unperformed measurements are nonetheless determinate; there must be a fact of the matter as to what the results of unperformed measurements would have been. Time-dependence is then introduced as a means to disqualify incompossible measurements (in this instance, non-commuting spin angular momentum operators). This first assertion trivially satisfies one aspect of the definition of CFD mentioned above (the one-to-one correspondence between tests and parameters, connected by functions) although, as respects the second, it says nothing obvious about the independence of these parameters. The second demand, on the other hand, is that Bell's canonical expression for the value of the product of measurement outcomes A and B is accompanied by a proof of the applicability of the Kolmogorov axioms.[29]

$$P(\boldsymbol{a},\boldsymbol{b}) = \int d\lambda \rho(\lambda) A(\boldsymbol{a},\lambda) B(\boldsymbol{b},\lambda)$$

[26] Hess, K., De Raedt, H., and Michielsen, K. (2016). p. 1
[27] Hess, K., De Raedt, H., and Michielsen, K. (2016). p. 4
[28] Hess, K., De Raedt, H., and Michielsen, K. (2016). p. 2
[29] This appears in, for instance, Bell, J. (2004). pp. 15, 36

For Hess, De Raedt and Michielsen, in order to be deployed in reasoning about quantum statistics, all the aspects of the probabilistic formalism are implicitly invoked and must be defended.

The 2016 "Counterfactual Definiteness and Bell's Inequality" paper shares this contentious feature with the account of M. Zukowski and C. Brukner provided in their 2015 analysis of the conception of locality in Bell's theorem.[30] Zukowski and Brukner object to the claim that locality is the only property required to derive the theorem within a hidden variable theory, again focusing on the following formula:

$$p(A,B|x,y) = \int d\lambda\rho(\lambda)\, p(A|x,\lambda)p(B|y,\lambda)$$ [31]

This formula is obtained from the more basic $p(A,B|x,y) = \int d\lambda\rho(\lambda)\, p(A,B|x,y,\lambda)$ together with a version of the factorisability condition and laws of conditional probability.[32] It is argued that these integrals have several problematic presuppositions. Principally, they rely on a Kolmogorovian probability space Ω and associated measure with, importantly, indicator functions $\chi_A(\lambda)$ such that the probability of A can be written as the integral $\int_\Omega \chi_A(\lambda)\rho(\lambda)d\lambda$. These indicator functions have the property that for *any* measurable set in the space they equal 1 if λ belongs to A, and 0 otherwise. Thus probabilities for all observables must be assignable by the χ_i. The authors therefore claim that Bell's assumption of local causality, as the physical basis for factorisability, is tantamount to "the assumption of joint probabilities $p(A_1, A_2, B_1, B_2)$"[33] being probabilities for outcomes which are incompatible in the sense that for appropriate choices of angles between spin measurements made at a given end the corresponding operators are non-commuting. For Zukowski and Brukner, then, as for Hess et al., Bell is committed to inherit all of the baggage of Kolmogorovian probability theory, including the existence of joint probabilities, by virtue of his use of integrals over the space of hidden variables.

This common approach suffers from at least two drawbacks. Firstly, the validity of the argument depends on an implausible metaphysical generalization concerning the relation between physical theories and their mathematical foundations. Secondly, the adduction of the properties of Kolmogorovian probability theory lacks physical motivation. In respect of the first criticism, in order to legitimate the inference from Bell's use of probability densities to its dependence on controversial axioms in probability theory, one must assume that theories which draw upon particular and often incomplete mathematical tools thereby take responsibility for the entire infrastructure supporting them, including their attendant unresolved theoretical limitations. Consider, for instance, the relationship between Newtonian mechanics and the Hamiltonian and Lagrangian formulations of classical mechanics. The latter require the postulation of a configuration space in which an action can be defined, as well as a principle of stationary action which generates the relevant equations of motion (whether via the Euler-Lagrange or Hamiltonian equations). Nonetheless, it would be no objection to, for example, Newton's laws to point out the absence of a full account of this mathematical architecture from the *Principia.* The

[30] Zukowski, M. and Brukner, C. (2015).
[31] Zukowski, M. and Brukner, C. (2015). p. 2
[32] More specifically, the assumptions are $p(A,B|x,y,\lambda) = p(A|B,x,y,\lambda)p(B|x,y,\lambda)$, $p(A|B,x,y,\lambda) = p(A|x,\lambda)$ and $p(B|x,y,\lambda) = p(B|y,\lambda)$.
[33] Zukowski, M. and Brukner, C. (2015). p. 3

Hess chain of reasoning applied here would generate just such an onerous demand. By way of a further illustration, in modern reformulations of classical mechanics, such as the shape dynamics developed over recent decades by J. Barbour, the redundant structure present in Newton's dynamics is eliminated in favour of a more minimalistic ontology.[34] In particular, not only the positions of particles referred to in Newton's laws but the separations (from which Newton held the positions to be deducible) are swapped out for "dimensionless ratios" $\tilde{r}_{ab} = \frac{r_{ab}}{R_{rmh}}$ where R_{rmh} represents the root-mean-harmonic separation $R_{rmh}\sqrt{\sum_{a<b} r_{ab}^2}$.[35] This definition replaces separations with numerical factors which relativize these separations to a statistical measure taken over the totality of particles being considered. Barbour's claim is that this framework enables the "elimination" of "every vestige" of "redundant structure"[36] from Newtonian mechanics, an endeavour pioneered by Leibniz's followers in Newton's era but finally consummated only now – at least, according to Barbour. What, then, of the variables thus purged – the separations r_{ab}, for instance, or times t? Presumably these would fail to placate the insistence for a "one-to-one correspondence of the possible tests and the numbers"[37] when, by hypothesis, these are no longer fundamental, nor absolute time observable. The physical insights contained within Newton's laws – for example, the existence of inertial frames in terms of which the motions of force-free bodies pursue uniform rectilinear motion – would be disqualified unless an isomorphism could be established between the variables they contain and outcomes of tests of whatever sort. More generally, the inclement requirements of Hess and like-minded scholars, if affirmed and imposed consistently, would most certainly inhibit the advancement of physics and likely debar a good number of fertile theories – retroactively imposed, perhaps even early quantum mechanics, considering the evolution of its mathematical skeleton in the Dirac formalism and Hilbert space algebra, years after the original empirical discoveries relating to atomic line spectra and photoelectric effect.

Such an unforgiving set of minimum standards also betrays an extreme and anachronistic logical positivist position in the philosophy of science – consisting in the view that a theory is comprised more or less entirely of variables, each of which corresponds to information harvested from particular tests and their corresponding sense data. It is this position that seems to be at work in the authors' demand for a one to one correspondence between experimental outcomes and the variables of the theory, cited above.[38] Nowhere is there room made in such a presentation for the myriad of other theoretical components – auxiliary hypotheses, conceptual underpinnings, mathematical resources, heuristic aspects, and underlying metaphysical assumptions, of the sort described by Imre Lakatos as constituting the immalleable "hard core"[39] of a theory, as opposed to postulates readily falsifiable and adjustable in the face of shortcomings exposed by experiment. Indeed, the question could even be posed as to the sense in which the quantum state itself, whose associated complex amplitudes stand in a notoriously intractable relation to the statistics which provide their empirical basis, is testable in the narrow sense of Hess, Phillipp and others. As the central object of a theory which has enjoyed unprecedented success in experiments, one might well ask whether this sense is at all relevant in discussions of its philosophical foundations.

[34] Barbour, J. (2011).
[35] Barbour, J. (2011). pp. 3-4
[36] Barbour, J. (2011). pp. 2-3
[37] Hess, K., De Raedt, H., and Michielsen, K. (2016). p. 2
[38] Hess, K., De Raedt, H., and Michielsen, K. (2016). p. 2
[39] Lakatos, I. (1978).

A second drawback of the position of Zukowski, Brukner, Hess et. al. was alluded to above, relating to the lack of physical motivation for the criteria they insist on Bell's theorem meeting in order to take advantage of probabilistic representations such as $P(\boldsymbol{a},\boldsymbol{b}) = \int d\lambda\rho(\lambda)A(\boldsymbol{a},\lambda)B(\boldsymbol{b},\lambda)$. It is well known that a broad class of different no-go results aside from those of Bell, many of which do not display the reliance on Kolmogorovian or other probabilistic concepts highlight similar inconsistencies between hidden variable theory, if conjoined with appropriate metaphysical assumptions (whether locality, contextuality, or others), and the predictions of quantum mechanics. This would appear to undermine the claim that Bell's physical insight is at all undermined by its reliance on probability spaces. The GHZ theorem, published by D. Greenberger, M. Horne, A. Shimony and A. Zeilinger in 1990 considers the case of quadruple of particles propagating away from a source in a single plane with polar and azimuthal angles θ_i, φ_i for the ith particle.[40] Stern-Gerlach magnets are positioned at arbitrary remove from the source and spin measurements carried out just as for the original EPR scenarios. The authors show[41] that the quantum mechanical prediction for the product of outcomes when measurement orientations confined to the xy plane is given by:

$$E^{\varphi}(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3, \hat{\boldsymbol{n}}_4) = -\cos(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)$$

They proceed to contrast this prediction with that obtained in the EPR framework, where a four-fold assumption set (perfect correlation, locality, reality and completeness) is adduced to infer the existence of four functions of hidden variables λ, namely $A_\lambda(\varphi_1), B_\lambda(\varphi_2), C_\lambda(\varphi_3), D_\lambda(\varphi_4)$, describing the resultant spin values. On the premise that the quantum mechanical predictions agree with those generated by these functions, a contradiction can be derived, seemingly refuting the EPR assumption set. Thus, the inconsistency between local hidden variable theory and the predictions of quantum mechanics is teased out without the employment of the probability axioms Hess and Philipp accuse of detracting from Bell's reasoning. In experimental tests of the theorem, such as that proposed by J.-W. Pan et al.[42] the extraction of predictions from the hidden variable framework considers polarization measurement outcomes performed on photon triplets, as opposed to expectation values of spin for pairs. Their reasoning deduces constraints on the products of these outcomes: "Thus from a local realist point of view the only possible results for an xxx experiment are $V'V'V', H'H'V', H'V'H'$, and $V'H'H'$,"[43] a series of possibilities which "predicts none of the terms occurring in the quantum prediction," and is therefore falsifiable (subject to plausible ancillary assumptions) given observation of the quantum-theoretic outcomes.

Relatedly, N. Mermin, in his 1985[44] and 2002[45] contributions, provides a qualitative and minimally mathematical account of a sister result of Bell's – involving two particles with three measurement setting possibilities at the relevant detectors – which refuses a local hidden variable interpretation (that is, that the measurement outcomes unfold based on properties, determined before measurement directions are selected, and inscribed into the particles as instruction sets). It is not at all obvious where the entry point of Kolmogorovian assumptions is within these

[40] Greenberger, D. et al. (1990).
[41] Greenberger, D. et al. (1990). pp. 1141
[42] Pan, J.-W. et al. (2000).
[43] Pan, J.-W. et al. (2000). p. 517
[44] Mermin, N. (1985).
[45] Mermin, N. (2002).

alternatives, or where it would be appropriate to place the burden of justifying the use of indicator functions. The arguments given in no way rely on them. Thus, even acceptance of this objection to Bell in the context of his 1964 theorem (and the aforementioned are reasons to be sceptical about this) fails to undermine the broader family of no-go results which highlight serious discrepancies between quantum mechanics and local hidden variables, *i.a.*. Accordingly, it shows that the fundamental physical insights of these theorems are in no way married to supposedly controversial assumptions in the foundations of probability theory: the requisite incompability between local hidden variables and quantum mechanics can be illuminated without recourse to them.

These counter-arguments aim to show that the first premise of Hess' argument (B), that Bell's theorem requires counter-factual definiteness, lacks justification for the following reasons. The support offered for this premise in Hess' writings (among others') in many cases presupposes rather than justifies this relationship. Where justifications are offered, these impose unreasonably strict requirements: the constraint that all variables in the formalism weaponised by Bell must be treated as real in order for the operations performed to be meaningful is both (i) implausible and (ii) sets a bar for the architecture of physical theories which is unreasonably high, leaving it vulnerable to counter-examples.

However, this was not the only issue noted above with Hess' position in his argument B. Aside from the first premise attributing CFD to Bell, the third premise and conclusion also introduced sufficient conditions for the completeness or incompleteness of a theory. These conditions insist that the variables in the space used in Bell's theorem are independent only if this space is incomplete. The authors here accuse Bell of neglecting the possibility of time-correlations in the parameter space used to define the relevant probability measures. Neglecting these time-correlations enables Bell to avoid invoking the assumption that parameters such as apparatus settings are independent of time. This omission, they claim, marks the incompleteness of his description; a fully rigorous no-go proof would require these variables to be considered too. However, there are good reasons to suspect this objection to be too weak to support their case. At no point does any aspect of the models purported as counter-examples to Bell refer to time. Nor do they tease out *why* the absence of this among however many other absent variables is decisive for the proof. Were the omission of t critical to the demonstration, some such illustration of its role in the dynamics of no-go gedankenexperiments would surely be forthcoming, and accounts without it unsuccessful. As a result, the insistence on an analysis of time falls into arbitrariness: it can be motivated only by factors which are in no way integral to the theorem, being, rather, precepts in broader metaphysics and the philosophy of science. Indeed, this principle would then have to be applied consistently across other physical theories, and its relevance justified – a dauntingly burdensome task.

Accordingly, proposition 4. is detrimental to Bell's work only to the extent that completeness in the limited sense defined above (namely, as relates to the structure of the Bell parameter space) is an unexceptionable desideratum of no-go theorems or its absence obviously problematic. It is unclear however that completeness of this sort is or ought to be an aspiration for no-go theorems. It can hardly be predicated of all legitimate physical theories that any choosable pair of variables must be independent and explicitly included in the parameter space involved. Such a criterion threatens to disqualify any number of bona fide and productive theories, as an arbitrary and metaphysical constraint on the nature of legitimate theories.[46]

[46] In this vein, Lambare and Franco allege these attributions of CFD to Bell to be "devoid of physical and logical sense." Lambare, J. and Franco, R. (2021). p. 2

Conclusion

The fecundity of Bell's original no-go theorem and the frugality of his assumptions has driven critics of his work to increasingly remote conceptual terrain, compelled by the hope of exposing some or other controversial hidden assumption in the proof and evading its uniquely troubling conclusions. Over the decades since its articulation, suspected loopholes in the corresponding quantum tests have been addressed one-by-one, yet, at its extreme, this tendency has driven some to question even the statistical independence of the preparation procedure used to generate the pairs of spin-correlated or anti-correlated particles which are the subject of the proof, in favour of a "superdeterminist" picture, or to increasingly conspiratorial alternatives. The assumption of counter-factual definiteness is just such an ostensible defect which, having aroused particular scholarly interest in recent years, is supposedly to be found concealed in a variety of whereabouts, from Bell's choice of parameter space to his use of probability theory.

This essay argues for three conclusions. Firstly, the logical structure of the papers which claim to establish CFD as a postulate of Bell's reasoning establish no such thing, but rather presuppose the same as an aid to more general attacks on the theorem in its many forms. The only justifications given rest on metaphysical generalisations which fail to universalize across legitimate scientific theories and impose on them gratuitous and unphysical constraints. Moreover, they have no natural application to alternative no-go theorems with similar consequences to the original expressions of Bell's inequality, suggesting that whatever limitations they pick out in Bell's logic are non-fundamental. Secondly, these arguments do not succeed even when restructured in a valid form: they do not meet the crucial requirement that parameter independence holds for the models offered by way of justification. These conclusions militate against the perfunctory ascription of CFD to Bell and suggest the need for a fuller defence of this move if invoked in the analysis of no-go results.

Bibliography

Barbour, J. (2011). SHAPE DYNAMICS. An Introduction. Available at: https://arxiv.org/abs/1105.0183

Bell, J. (2004). *Speakable and Unspeakable in Quantum Mechanics*, 2nd Ed. Cambridge: Cambridge University Press

Blaylock, G. (2009). "The EPR paradox, Bell's inequality, and the question of locality." Available at: https://arxiv.org/abs/0902.3827

Clauser, J., Holt, R., Horne, M. and Shimony, A. (1969). "Proposed Experiment to Test Local Hidden-Variable Theories," Phys. Rev. Lett., Vol. 23, 880.

Greenberger, A. et al. (1990). "Bell's theorem without inequalities." *American Journal of Physics, 58(12)*. pp. 1131-1143

Hess, K., De Raedt, H. and Michielsen, K. (2016). "Counterfactual Definiteness and Bell's Inequality" Available at: https://arxiv.org/abs/1605.04889

Hess, K and Philipp, W. (2001a). "Einstein-separability, Time Related Hidden Parameters for Correlated Spins, and the Theorem of Bell." https://arxiv.org/abs/quant-ph/0103028

Hess, K and Philipp, W. (2001b). "A possible loophole in the theorem of Bell." Proc. Natl. Acad. Sci. USA, Vol. 98, 25.

Lakatos, I. (1978). *The Methodology of Scientific Research Programmes (Philosophical Papers:* Volume I*)*. Edited by J. Worrall and G. Currie. Cambridge: Cambridge University Press

Lambare, J. and Franco, R. (2021). "Note on Bell's Theorem Logical Consistency." Available at: https://arxiv.org/abs/2012.10238

Leggett, A. (1987). *The Problems of Physics*. Oxford: Oxford University Press

Mermin, N. (1985). "Is the Moon There When Nobody Looks? Reality and the Quantum Theory." *Physics Today, 38(4)*. pp. 38-47

Mermin, N. (2002). "Shedding (red and green) light on "time related hidden parameters." Available at: https://arxiv.org/abs/quant-ph/0206118

Pan, J.-W. et al. (2000). "Experimental test of quantum nonlocality in three-photon Greenberger-Horne-Zeilinger entanglement." In: *Nature, 403(6769)* , pp. 515-519

Peres, A. (1995). *Quantum Theory: Concepts and Methods.* Dordrecht: Kluwer.

Rubin, M. (2001). "Locality in the Everett Interpretation of Heisenberg-Picture Quantum Mechanics." Available at: https://arxiv.org/abs/quant-ph/0103079

Zukowski, M. and Brukner, C. (2015). "Quantum Non-locality – It Ain't Necessarily So…" Available at: https://arxiv.org/abs/1501.04618