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Comment on "On the linearity of the generalized Lorentz transformation"

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In an article in this journal (Am. J. Phys. 90, 425ñ429 (2022)), Frank Verheest presents a proof for the linearity of the Lorentz transformation. We fill in some gaps in his derivation and analyze the role of the light postulate that some physicists, including Verheest, have criticized as a necessary hypothesis for formulating the theory of relativity.

I. INTRODUCTION

In [1], Verheest derives the linearity property of the Lorentz transformation. The reasons Lorentz transformations must be linear are often mentioned only in passing and without due rigor. Although that attitude is justified from a heuristic viewpoint, a rigorous derivation employing only elementary mathematical tools can be educationally fruitful.

Occasionally, linearity is assumed by requiring the fulfillment of the law of inertia [2]. Although linearity preserves motions with constant velocity, the last does not require linearity, as the following example shows [3]. Let $\mathcal{O}(x_0, x_1, x_2, x_3)$ and $\mathcal{O}'(x_0', x_1', x_2', x_3')$ be two inertial frames whose coordinates are related through

$$x_i' = \frac{\sum_k a_{ik}\, x_k + e_i}{\sum_k b_k\, x_k + f}, \qquad i, k = 0, 1, 3, 4 \tag{1}$$

where a_{ik}, b_k, e_i, f are constants and x_0 stands for time. When $x_k(\alpha) = r_k\alpha + h_k$, with α a continuous parameter and r_k, h_k constants, putting $\dot{x}_k = dx_k/d\alpha$ we have

$$v_k = \frac{dx_k}{dx_0} = \frac{\dot{x}_k}{\dot{x}_0} = \frac{r_k}{r_0} = const., \qquad k = 1, 2, 3 \tag{2}$$

so it describes a rectilinear motion with uniform velocity. The student can prove as an exercise that (1) and (2) implies $v_k' = \dot{x}_k'/\dot{x}_0' = const.$ Transformation (1) can be discarded on issues of differentiability or by noting that it takes infinite coordinates values into finite ones. We have presented it only as a counterexample.

On the other hand, that homogeneity and isotropy of space and homogeneity of time require linearity is often mentioned without giving further references or comments, leaving the impression that it is a trivial implication [4, 5].

Rigorous proofs of linearity can be given using different approaches and techniques [6–10]. In particular, Berzi and Gorini [8] proved linearity only from spacetime homogeneity, leaving out isotropy. Whether a detailed proof is necessary or a heuristic justification suffices, as well as the method employed, is a matter of personal taste and philosophical attitude towards a rigorous formulation of the foundations of physics.

In any case, Verheest's approach is a valid contribution for those interested in the foundations of relativity. Although his derivation is generally correct, there are some issues that we consider omissions or gaps in the reasoning rather than mistakes. We explain that problem in section III. But first, in section II, we delve into another fundamental question that Verheest brought about.

Besides the linearity issue, the author of [1] expressed a long-standing concern that some physicists have pointed out regarding the central role that the speed of light seems to play in the principles of relativity theory [10–13]. That uneasiness is justified since relativity constitutes a central pillar in the theories of modern physics. Notwithstanding the importance of electromagnetic theory, it seems odd that a particular type of phenomenon should play such a central role. In the following section, we explain that the crucial role that light purportedly plays in Einstein's formulation is only apparent and is owed to historical and practical reasons.

II. THE LIGHT PRINCIPLE

As observed in [1], Einstein based his special theory of relativity on two principles (i) the laws of physics are invariant in all inertial frames of reference, and (ii) the speed of light in vacuum is the same for all inertial observers.

Principle (i) is an extension of the equivalence of inertial reference frames from mechanics to all physical phenomena [1], while (ii) is also known as the light principle.

In 1905 only two fundamental interactions were known, gravitational and electromagnetic. Newtonian gravity is described by an action at a distance law, i.e., instantaneous interaction. On the other hand, light was known to be an electromagnetic phenomenon with a finite speed, while all attempts to find evidence of a light-carrying medium had failed. That historical prospect explains why Einstein gave light such a central role, notwithstanding that principle (i) encompasses all physical laws.

The tradition of teaching relativity through the light principle continues to this day. As Verheest has observed, from a conceptual viewpoint, it is more compelling to derive the Lorentz transformations without mentioning the speed of light at all. The first to do that was Vladimir Ignatowski, as early as 1910 [11]. Then, many such formulations followed using different approaches and techniques [8, 10, 12, 13, 15].

Regarding the role of light in the formulation of the theory of relativity, it is relevant to observe the following:

(a) The light principle can be replaced by the more general principle, (ii') the principle of finiteness of

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[1] The first to formulate the general "postulate of relativity" was Henri Poincaré [14].

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the speed of propagation of fundamental interactions [4, 16].

(b) As a consequence of (i) and (ii'), we obtain that fundamental interactions must propagate with the same speed in all inertial systems. Therefore, that speed must be a universal constant establishing the limiting speed transmission of any influence.

It is essential to insist that (ii') refers only to "fundamental" interactions, such as electromagnetic and gravitational. For arbitrary interactions, it represents the upper limit for the speed transmission of any other influence. A typical example is that one hears the thunder much later than one sees the lightning, with light being a fundamental interaction while sound is not.

Principles (i) and (ii') can lead us to Lorentz transformations through the usual derivations replacing light speed with a finite universal limiting speed based on the exclusion of unobservable instantaneous interactions.

Also, as done by Verheest, we can hold only to principle (i). That puts Galilean and Lorentz transformations on the same basis. Ironically, such an approach has the conceptual advantage of making more evident the essential difference between Galilean and Einstein's relativity, namely, the existence of a universal finite speed limit and the exclusion of instantaneous interactions.

Indeed, when we assume (i), instantaneous interactions and Newton's absolute time are inextricably related. To express this point formally, we shall consider spacetime transformations between two inertial frames in the so-called standard configuration using the same notation as in [1]

$$x' = F(x, t; v) \tag{3}$$

$$t' = G(x, t; v) \tag{4}$$

where v represents the velocity of frame $\mathcal{O}'$ with respect to $\mathcal{O}$. Let an object A in $\mathcal{O}$, at $x = x_a$, cause an instantaneous effect at time $t = t_1$ through a fundamental interaction on a distant object B, located at $x = x_b$. In the other inertial frame $\mathcal{O}'$, since the laws of physics are the same in $\mathcal{O}$ and $\mathcal{O}'$, that effect must also occur at the same time on A and B, say $t' = t'_1$. The time coordinate transformation (4) gives

$$t'_1 = G(x_1, t_1; v) \tag{5}$$

$$t'_1 = G(x_2, t_1; v) \tag{6}$$

Since x_1 and x_2 are arbitrary, the time variable must be independent of the spatial coordinate, $t' = G(t; v)$. Homogeneity of time requires that the ratio dt'/dt be also independent of time

$$\frac{dt'}{dt} = \frac{\partial}{\partial t} G(t; v) = a(v) \tag{7}$$

then by integration

$$t' = a(v)t + b(v) \tag{8}$$

We can take $b(v) = 0$ by adequate initial conditions, for instance, by choosing $t' = 0$ when $t = 0$

$$t' = a(v)t \tag{9}$$

Space isotropy requires $a(v) = a(-v)$ and, by symmetry, the inverse relation is obtained when changing v by $-v$[2]

$$t = a(-v)t' = a(v)t' \tag{10}$$

Replacing (9) in (10)

$$t = a(v)^2 t \tag{11}$$

The former equation leads us to $a(v)^2 = 1 \rightarrow a(v) = \pm 1$, then conserving the time direction we are left with $t' = t$.

Thus, when we assume there is no limit to the speed of the transmission of interactions, Newton's absolute time is not optional but a necessary imposition. Vice versa, if we assume time is absolute, velocities are added according to classical Newtonian mechanics. Hence, only an infinite speed of fundamental interactions can make different inertial frames physically equivalent [3].

On the other hand, elementary considerations between inertial observers in relative motion (such as Einstein's train and embankment example [18]) prove that Newton's absolute time has to be abandoned if a finite speed remains invariant in all inertial frames.

The former considerations about absolute time and its abandonment are related only to principle (i) and the existence of a finite universal speed limit for all inertial observers without mentioning light or electromagnetism. Admittedly, if electromagnetism constitutes a fundamental interaction, then the universal speed limit must coincide with that of light in a vacuum.

III. LINEARITY

Here we address two issues that were not sufficiently clarified in section **B** of [1]. In the following, F and G refer to the spacetime transformations (3) and (4), which we shall assume constitute a twice differentiable bijection. We include equation numbers in [1] with an asterisk. From (3) and (4)

$$p' = \frac{dx'}{dt'} = \frac{p\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t}}{p\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t}}, \qquad (6^*) \tag{12}$$

where p and p' are the velocities in $\mathcal{O}$ and $\mathcal{O}'$ respectively. When $p = 0$ we must have $p' = -v$ and from (12) we obtain

$$\frac{\partial F}{\partial t} + v\frac{\partial G}{\partial t} = 0 \qquad (7^*) \tag{13}$$

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[2] Sometimes this symmetry is called reciprocity [8].

[3] This relation between absolute time and instantaneous interactions is remarked, for instance, in ref. [17].

Similarly, when $p' = 0$ we have $p = v$ and (12) reduces to

$$v\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} = 0 \qquad (8^*) \tag{14}$$

Equations (12), (13), and (14) form the basis of Verheest's formulation.

A. First issue

The first issue arises after equation (8*). There Verheest asserts, "This implies that F is a function of the combined argument $x - vt$ as well as of v." without further explanation. Note that v enters the equation as as parameter, F being a function of the variables x and t.

It is clear that if F has the functional form $F(x-vt; v)$, (14) is satisfied. However, the former argument constitutes only a sufficient condition, and Verheest's derivation requires F to have that functional form necessarily.

Luckily that has an elegant solution. As observed in [1], from (13) and (14) we obtain

$$\frac{\partial G}{\partial t} = \frac{\partial F}{\partial x} \qquad (9^*) \tag{15}$$

Taking derivatives with respect to t in (13) and (15)

$$\frac{\partial^2 F}{\partial t^2} + v\frac{\partial^2 G}{\partial^2 t} = 0 \tag{16}$$

$$\frac{\partial^2 G}{\partial^2 t} = \frac{\partial^2 F}{\partial x \partial t} \tag{17}$$

eliminating from the former two equations $\partial^2 G/\partial^2 t$

$$\frac{\partial^2 F}{\partial t^2} + v\frac{\partial^2 F}{\partial t \partial x} = 0 \tag{18}$$

Taking derivative with respect to x in (14)

$$v\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x \partial t} = 0 \tag{19}$$

Eliminating the cross derivatives in (18) and (19)

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0 \tag{20}$$

The former wave equation and its general solution is well-known to physics students

$$F(x, t; v) = f_v(x - vt) + g_v(x + vt) \tag{21}$$

The g_v part of the solution does not satisfy (14) so we must have $g_v = 0$ and we obtain the necessary solution

$$F(x, t; v) = f_v(x - vt) \tag{22}$$

B. Second issue

The second issue arises when solving a homogeneous linear system that Verheest's method requires

$$\begin{aligned} \frac{\partial^2 F}{\partial t \partial x} &+ v\frac{\partial^2 G}{\partial t \partial x} = 0 \\ C(v)\frac{\partial^2 F}{\partial t \partial x} &- \frac{\partial^2 G}{\partial t \partial x} = 0 \end{aligned} \qquad (12^*) \tag{23}$$

where

$$C(v) = \frac{\partial G/\partial x}{\partial F/\partial x} \qquad (11^*) \tag{24}$$

Verheest's derivation is based on the vanishing of the second order derivatives in (23). That is justified assuming a nonzero determinant of the coefficients, $[1+vC(v)] \neq 0$. However, a rigorous treatment requires that we also analyze the case when $1 + vC(v) = 0$ giving

$$C(v) = -\frac{1}{v} \tag{25}$$

When this happens, we cannot assume that both cross derivatives vanish. In this case the system (23) reduces to a single equation

$$\frac{\partial^2 G}{\partial t \partial x} = -\frac{1}{v}\frac{\partial^2 F}{\partial t \partial x} \tag{26}$$

so we cannot assume that both sides vanish. To solve this case, from (13), (25), and (24)

$$\frac{\partial G}{\partial t} = -\frac{1}{v}\frac{\partial F}{\partial t} \tag{27}$$

$$\frac{\partial G}{\partial x} = -\frac{1}{v}\frac{\partial F}{\partial x} \tag{28}$$

Replacing (22) in the former two equations

$$\frac{\partial G}{\partial t} = f'_v \tag{29}$$

$$\frac{\partial G}{\partial x} = -\frac{1}{v}f'_v \tag{30}$$

By integration we have

$$G(x, t; v) = -\frac{1}{v}f_v + h(x) \tag{31}$$

$$G(x, t; v) = -\frac{1}{v}f_v + l(t) \tag{32}$$

Therefore $h(x) = l(t) = k = const.$ and we are left with the following spacetime transformation

$$t' = -\frac{1}{v}f_v(x - vt) + k \tag{33}$$

$$x' = f_v(x - vt) \tag{34}$$

However, this transformation is inadmissible because it does not have an inverse. Really, when $t' \neq -(1/v)x' + k$ it does not have solution in (x, t).

IV. CONCLUSIONS

We have completed Verheets's linearity proof by solving two overlooked mathematical issues. However, the foundationally relevant points were discussed in section II. From a conceptual viewpoint, we have stressed that it is better to base the derivation of Lorentz transformation using axioms (i) and (ii'), replacing the light principle with a more physically compelling one. In this respect, we highlight two authoritative references, Landau & Lifshitz [16] and J. D. Jackson [4]; both postulate (ii') instead of the light principle. In particular, J.D. Jackson explicitly spells out, *Because special relativity applies to everything, not jut light, it is desirable to express the second postulate in terms that convey its generality:*

> In every inertial frame, there is a finite universal limiting speed C for physical entities.

Regarding the question asked in the introduction of [1]:

> The question is: How and where to start a derivation of the generalized Lorentz transformation if the invariant speed of light is not explicitly required? How do we incorporate the transformation of time from one inertial observer to the next? Of course, we all know the Lorentz transformation, but how to get there?

That question was duly responded to in section II. The form of incorporating time into the transformation is necessarily subject to our assumption about the existence of instantaneous interactions or their impossibility. Instantaneous interactions necessarily imply absolute Newtonian time. So, the time transformation cannot include spatial variables.

On the other hand, the existence of a universal finite limiting speed for physical interactions requires abandoning absolute time (train-embankment example [4]). That demands that time enter the transformation as a fourth coordinate depending on the spatial variables.

The former considerations prove that Lorentz transformation reduction to the Galilean case when $c \to \infty$ is not circumstantial but a consistency requirement. It does not mean the speed of light has to be infinite for this to occur because electromagnetic phenomena (including light) are excluded from the considerations. As is well known, it is possible to maintain a finite speed of light notwithstanding instantaneous interactions if we assume light is a sort of "sound" phenomenon, taking place in a substance called ether by nineteenth-century physicists.

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[4] To turn the train-embankment example into a generic one, we must replace light signals with signals through an unspecified fundamental interaction.