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A CONVERGENCE NOT METRIZABLE

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Abstract. Certain notions of convergence of sequences of functions such as convergence pointwise, uniform and parts (compact sets or bounded sets) come from suitable topological functional spaces [1]. Under certain conditions these topologies involved are metrizable, which in an advantage since there is an extensive theory on convergence in metric spaces. However, the case of pointwise convergence is delicate, since it is shown that under certain hypothesis this form of convergence of sequences of functions is not equivalent to convergence in metric.

The set of real numbers is denoted as $\mathbb{R}$. The symbols $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ denote the set of rational numbers and the set of irrational numbers, respectively. For (M, d_M) and (N, d_N) metric spaces, the set of functions from M to N is denoted as $\mathcal{F}(M, N)$ and the set of continuous functions from M to N is denoted as $C(M, N)$.

For the purpose of only to fix terminology, we considered the following definition.

Definition 0.1. *A metric space (M, d) is said to be* strongly second numerable *if there exists a dense and numerable subset $D \subset M$ such that $M\backslash D$ is also dense.*

An immediate example of a *strongly second numerable* metric space is $\mathbb{R}$ with usual topology, since $\mathbb{Q}$ is a dense numerable subset whose complement $\mathbb{R}\backslash\mathbb{Q}$ is dense.

Proposition 0.2. *Let (M, d_M) be a complete, nonempty, strongly second numerable metric space and let (N, d_N) be a metric space having a non-unitary path-component. Then, there does not exists a metric d on $\mathcal{F}(M, N)$ such that the convergence of sequences in the space $(\mathcal{F}(M, N), d)$ is equivalent to the pointwise convergence of sequences of functions in $\mathcal{F}(M, N)$.*

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Proof. By reductio ad absurdum, it is assumed that there exists a metric d on $\mathcal{F}(M,N)$ such that the convergence of sequences in the space $(\mathcal{F}(M,N),d)$ is equivalent to the pointwise convergence of sequences of functions in $\mathcal{F}(M,N)$. Since (M,d_M) is strongly second numerable, there exists a dense numerable subset $D\subset M$ such that $M\backslash D$ is dense. Let $a,b\in N$, with $a\neq b$, and a path $\psi:[0,1]\to N$ where $\psi(0)=b$ y $\psi(1)=a$. We consider the function

$$\begin{aligned}\varphi:M &\longrightarrow N\\ x &\longmapsto \varphi(x):=\begin{cases}a, & \text{if } x\in D,\\ b, & \text{if } x\in M\backslash D.\end{cases}\end{aligned}$$

It follows that $\varphi:M\longrightarrow N$ is discontinuous in M. Let $D=\{x_1,x_2,\dots,x_n,\dots\}$ be an enumeration of D. For all $n\in\mathbb{N}$, let

$$\begin{aligned}\varphi_n:M &\longrightarrow N\\ x &\longmapsto \varphi_n(x):=\begin{cases}a, & \text{if } x\in D_n:=\{x_1,\dots,x_n\},\\ b, & \text{if } x\in M\backslash D_n.\end{cases}\end{aligned}$$

This sequence of functions in $\mathcal{F}(M,N)$ converges pointwise to $\varphi:M\longrightarrow N$. Now, for all $n\in\mathbb{N}$, it will be shown that φ_n can be pointwise approximated by an sequence of continuous functions. In fact, let $n\in\mathbb{N}$.

- First, we consider $n=1$. For all $m\in\mathbb{N}$, let
$$F^1_m:=B\left[x_1,\frac{1}{m+1}\right] \quad \text{y} \quad G^1_m:=M\backslash B\left(x_1,\frac{1}{m}\right).$$
Using Urysohn's Lemma, for all $m\in\mathbb{N}$, there exists a continuous function $f^1_{U,m}:M\to[0,1]$ such that $f^1_{U,m}(F^1_m)\subset\{1\}$ y $f^1_{u,m}(G^1_m)\subset\{0\}$. So, for all $m\in\mathbb{N}$, we take $f^1_m:=\psi\circ f^1_{U,m}$. It follows that $(f^1_m)_{m\in\mathbb{N}}$ converges pointwise to φ_1.
- We now assume that $n>1$. Let $\delta>0$ be such that for all $i,j\in\{1,\dots,n\}$, if $i\neq j$, then
$$B(x_i;\delta)\cap B(x_j;\delta)=\varnothing.$$
Let $p\in\mathbb{N}$ be such that $\frac{1}{p}<\delta$. For all $m\in\mathbb{N}$, we consider
$$F^n_m:=\bigcup_{i=1}^{n}B\left[x_i;\frac{1}{m+1+p}\right] \quad \text{y} \quad G^n_m:=M\backslash\bigcup_{i=1}^{n}B\left(x_i;\frac{1}{m+p}\right).$$
Since M is a normal space, by Urysohn's Lemma, for all $m\in\mathbb{N}$, there exists a continuous function $f^n_{U,m}:M\to[0,1]$ such that $f^n_{U,m}(F^n_m)\subset\{1\}$ y $f^n_{U,m}(G^n_m)\subset\{0\}$. Therefore, for all $m\in\mathbb{N}$, we take $f^n_m:=\psi\circ f^n_{U,m}$. Then, $(f^n_m)_{m\in\mathbb{N}}$ converges pointwise φ_n.

Thus, by hypothesis, $\varphi_n \in \overline{C(M,N)}$. It follows that $(\varphi_m)_{m\in\mathbb{N}}$ is a sequence in the space $\overline{C(M,N)}$. Since $\varphi_n \xrightarrow[n\to+\infty]{} \varphi$ in $(\mathcal{F}(M,N),d)$, we have $\varphi \in \overline{C(M,N)}$. So, there exists a sequence of continuous functions that converges to φ in $(\mathcal{F}(M,N),d)$. Then, φ con be approximated pointwise by a sequence of continuous functions. But by Proposition 15 of Chapter VI [1], it follows that the set of discontinuity points of φ is meagre in M. Since φ is discontinuous in M, we have that M is meagre in M. By Baire's Theorem, it follows that $M = \varnothing$, which is an absurdity. □

References

[1] E. L. Lima, *Elementos de Topologia Geral*, Ao Livro Técnico S.A., Rio de Janerio, Brazil, 1970.