Research Article

Field Dynamics in a Unified Differential Forms Framework: From Field Strength Tensor to Compressible Flow, Navier-Stokes Equations, and Vorticity Dynamics

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At a point p in a field, the Lagrangian density can be expressed as the interior product of the tangent velocity vector $\overrightarrow{v} \in T_pM$ and its corresponding metric dual conjugate momentum 1-form $S \in T_p^*M$, i.e., $\mathcal{L} = S\left(\overrightarrow{v}\right)$. Taking the exterior derivative of this 1-form yields a differential 2-form $\omega = dS$, whose components constitute the field strength tensor—an antisymmetric (0,2)-tensor. Contracting this 2-form with the tangent velocity vector gives the dynamic equation of the flow (a 1-form): $\iota_{\overrightarrow{v}}(\omega) = 0$. This formulation is entirely general and does not rely on prior assumptions. In reality, all fields exhibit some degree of compressibility. When this method is applied to a compressible field, it yields the dynamic equations for compressible flow. A singularity arises when the flow velocity is equal to the local wave propagation speed. In the case that the flow velocity is much less than the wave speed, or the wave speed approaches infinity, as an approximation, the dynamic equation degenerates to an incompressible flow. Further, by neglecting local spinning motion and applying Stokes's hypothesis, the equation reduces to the classical Navier-Stokes equations. The second exterior derivative $d^2S = 0$ yields a homogeneous differential 3-form. The coefficients of this 3-form correspond to the dynamic equations governing the vorticity field, providing for the absence of sources, sinks, or singularities at the point under consideration.

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Introduction

Instead of focusing on the motion of a single particle—such as a mass in a gravitational field or a charged particle in an electromagnetic field—a field should be viewed as a collection of massive particles. In such a system, not only do particle interactions occur, but wave propagation also takes place within the field [1]. In reality, all physical fields exhibit some degree of compressibility. Disturbances in the field propagate at finite wave speeds, such as electromagnetic waves traveling through space at the speed of light. In fluid dynamics, pressure (or density) disturbances propagate as mechanical waves at the speed of sound. Consequently, a field carries not only wave energy but also wave momentum.

Unlike a single particle, a complete and accurate description of a field must include both its wave energy and wave momentum. Additionally, the constituent particles may undergo macroscopic motion relative to an observer, contributing kinetic energy to the system from the viewpoint of the observer.

Differential forms offer an elegant and powerful framework for describing fields. When expressed in this formalism, the governing field equations become remarkably compact and transparent. In this paper, we adopt the language of differential forms to formulate the dynamics of physical fields.

The structure of the paper is as follows:

- Section 1 introduces the action 1-form within a Cartesian coordinate system on a manifold. It is defined as the metric dual of the tangent velocity vector, implicitly incorporating the metric tensor field and residing in the cotangent space. The Lagrangian density is then expressed as the interior product of this 1-form with the velocity vector, yielding a scalar energy density field.
- Section 2 derives the field strength tensor as the exterior derivative of the action 1-form, resulting in a differential 2-form—a (0,2)-tensor.
- Section 3 presents the contraction (interior product) of the 2-form with the velocity vector, yielding a 1-form whose components define the dynamical equations of the flow.
- Section 4 develops the dynamic equation for a compressible field, revealing a singularity when the flow velocity matches the local wave propagation speed.
- Section 5 considers the incompressible approximation, where the flow velocity is much less than the wave speed. Under this approximation, the equation simplifies accordingly.
- Section 6 ignores the rotational effects (an antisymmetric part of the velocity gradient) and applies Stokes' hypothesis; the equation reduces further to the formulation of the classical Navier–Stokes equations.

- Section 7 introduces the dynamical equation for the vorticity field. Its mathematical structure mirrors that of the homogeneous Maxwell equations, as it follows automatically from $d\omega = d^2 S = 0$.

Since the full derivations are extensive, detailed step-by-step calculations are provided in the Appendix.

1. The Volumetric Density of the Action 1-Form

Suppose there exists a physical field in space, within which the particles are moving along their actual physical trajectories; in other words, the paths taken by the particles are not arbitrary but are determined by the dynamics of the physical field itself.

To describe the motion of these particles analytically, an inertial coordinate system is required, since particle velocities are reference-frame dependent. In most practical scenarios, we describe the particle motion relative to the laboratory frame, which can be treated as a quasi-inertial frame. For simplification, we adopt a Cartesian coordinate system (t, x, y, z) to describe the motion of particles; see Fig. 1.

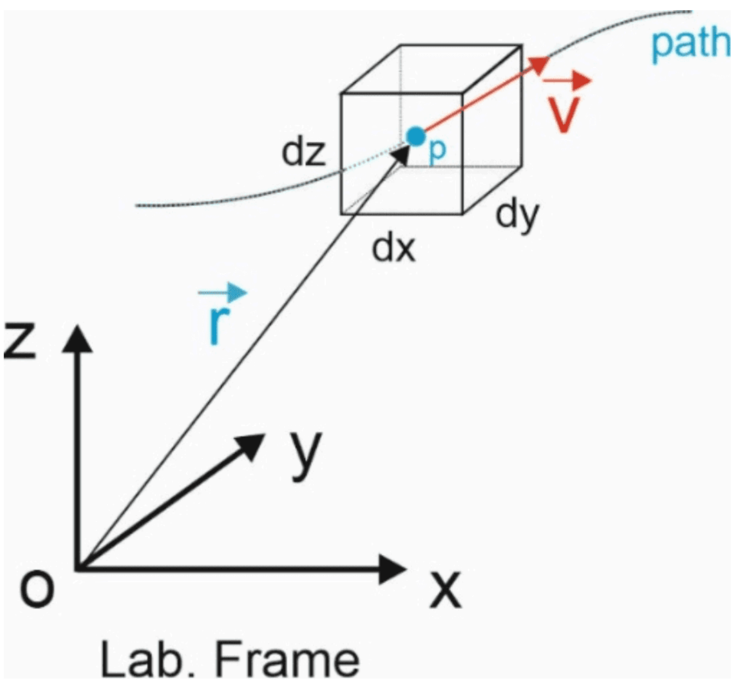

Figure 1. Observing the particle's motion from the lab frame (a pseudo-inertial frame), the velocity vector lives in the tangent space at point p: $T_p M$.

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A Poincaré-Cartan-like differential 1-form S is defined as follows [2][3][4][5][6]:

$$S = p_i dx^i - \left(\frac{V}{c}\right)(cdt). \tag{1}$$

Here, S is the action density 1-form per unit volume. The quantity V denotes the volumetric potential energy density, c represents the wave propagation speed, and p_i is the momentum density per unit volume in the i-direction ($i = x, y, z$). The components p_i are defined as the conjugate momenta, which are the metric dual of the tangent velocity vector $\overrightarrow{v} \in T_pM$, in the tangent space at a point p under consideration. That is, S is a cotangent vector (a differential 1-form), corresponding to the tangent velocity vector at point p, with $p_i \in T_p^*M$. This defines a duality between vectors and covectors via the metric tensor field. The cotangent basis 1-forms are dx^i (and dt). In SI units, the components of p_i and V have dimensions of energy per unit volume, i.e., $\left[\frac{J}{m^3}\right]$ along the time direction. Physically, the 1-form S represents an infinitesimal variation of the momentum and potential energy density along the actual path (dt, dx, dy, dz) at a point p in 3+1-dimensional space.

The 1-form can be expressed explicitly in Cartesian coordinates as:

$$S = p_x dx + p_y dy + p_z dz - V dt. \tag{2}$$

In Section 4, we will see that this formulation offers a significant advantage: the metric tensor is implicitly embedded into the conjugate momenta, p_i.

By the way, it should be mentioned here that if the last term $(-V dt)$ is neglected, the exterior derivative of equation (1) becomes the symplectic 2-form:

$$dS = dp_i \wedge dx^i. \tag{3}$$

2. Differential 2-form is the Field Strength Tensor

The 1-form of eq. (2) is called a potential of a differential 2-form. (We can also call it a vector potential for a 2-form).

The exterior derivative of the 1-form yields the differential 2-form. We use the Leibniz rule for the exterior derivative; it then reads:

$$\omega = dS = d\left(p_i\right) \wedge dx^i - d(V) \wedge dt, \tag{4}$$

since $d\left(dx^i\right) = d^2\left(x^i\right) = 0$ and $d(dt) = d^2 t = 0$.

The differential parts dp_i and dV are now expressed as

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$$\begin{cases} dp_i = \frac{\partial p_i}{\partial x^\mu} dx^\mu, \\ dV = \frac{\partial V}{\partial x^\mu} dx^\mu, \end{cases} \quad for\ \mu = (ct, x, y, z). \tag{5}$$

Substituting eq. (5) into (4), the exterior derivative (2-form) is

$$\omega = dS = \left(\frac{\partial p_i}{\partial x^\mu} dx^\mu \right) \wedge dx^i + \left(-\frac{\partial V}{\partial x^\mu} dx^\mu \right) \wedge dt. \tag{6}$$

If each term is written out explicitly, e.g.:

$$\begin{cases} dp_x = \frac{\partial p_x}{\partial t} dt + \frac{\partial p_x}{\partial x} dx + \frac{\partial p_x}{\partial y} dy + \frac{\partial p_x}{\partial z} dz \\ -dV = -\frac{\partial V}{\partial t} dt - \frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz \end{cases}. \tag{7}$$

Using the antisymmetric property of the wedge product of

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \ \ dx^\mu \wedge dx^\mu = 0, \tag{8}$$

and by collecting the same basis terms, we finally get the 2-form:

$$\begin{aligned} \omega = & \left(\frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial x} \right) dt \wedge dx + \left(\frac{\partial p_y}{\partial t} + \frac{\partial V}{\partial y} \right) dt \wedge dy + \left(\frac{\partial p_z}{\partial t} + \frac{\partial V}{\partial z} \right) dt \wedge dz \\ & + \left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y} \right) dx \wedge dy + \left(\frac{\partial p_z}{\partial y} - \frac{\partial p_y}{\partial z} \right) dy \wedge dz + \left(\frac{\partial p_x}{\partial z} - \frac{\partial p_z}{\partial x} \right) dz \wedge dx. \end{aligned} \tag{9}$$

There are 6 independent terms; if written more compactly, it can be expressed as a strictly upper triangular matrix:

$$\omega = \sum_{\mu<\nu}^{4} \omega_{\mu\nu} \left(dx^\mu \wedge dx^\nu \right). \tag{10}$$

Since $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$, we can write this symmetrically by antisymmetrizing the indices:

$$\omega = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu, \ \ for\ \mu, \nu = (ct, x, y, z) \tag{11}$$

where $dx^\mu \wedge dx^\nu$ is a differential 2-form basis vector.

Here we define

$$p_t = \frac{V}{c}. \tag{12}$$

The coefficients of the 2-forms can be written more compactly:

$$\omega_{\mu\nu} = \partial_\mu p_\nu - \partial_\nu p_\mu. \tag{13}$$

Working in cotangent space has great advantages because the Christoffel symbols are symmetric in their lower two indices:

$$\Gamma^k_{\mu\nu} = \Gamma^k_{\nu\mu}. \tag{14}$$

Thus, $\omega_{\mu\nu}$ is a covariant derivative; namely, it is a (0,2)-tensor:

$$\omega_{\mu\nu} = \nabla_\mu p_\nu - \nabla_\nu p_\mu = \left(\partial_\mu p_\nu - \Gamma^k_{\mu\nu}\right) - \left(\partial_\nu p_\mu - \Gamma^k_{\nu\mu}\right). \tag{15}$$

The coefficients (with two subscripts, $\mu\nu$) of the 2-form are the field strength tensor; accordingly, it is an antisymmetric (0,2)-tensor. It can "eat" a tangent vector in the first slot, leaving a 1-form. If the coefficients of the differential 2-form (eq. (11)) are arranged as a 4x4 matrix, it reads:

$$\omega = \frac{1}{2}\begin{bmatrix} 0 & \omega_{tx} & \omega_{ty} & \omega_{tz} \\ -\omega_{tx} & 0 & \omega_{xy} & \omega_{xz} \\ -\omega_{ty} & -\omega_{xy} & 0 & \omega_{yz} \\ -\omega_{tz} & -\omega_{xz} & -\omega_{yz} & 0 \end{bmatrix}. \tag{16}$$

This arrangement will keep the antisymmetric structures, e.g.:

$$\begin{cases} \omega_{tx} dt \wedge dx = \left(\frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial x}\right) dt \wedge dx; \quad \omega_{xt} dx \wedge dt = -\omega_{tx} dx \wedge dt \\ \omega_{xy} dx \wedge dy = \left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right) dx \wedge dy; \quad \omega_{yx} dy \wedge dx = -\omega_{xy} dy \wedge dx \end{cases} \tag{17}$$

The antisymmetric 4x4 matrix still keeps the 6 independent terms.

In 3D Cartesian coordinates, there is a relationship between the vector cross product and the wedge product due to the Hodge dual; it maps the oriented bilinear form (area element 2-forms) to corresponding perpendicular (orthogonal complements) 1-forms:

$$* \left(dx^i \wedge dx^j\right) = \epsilon_{ijk} dx^k, \tag{18}$$

where ϵ_{ijk} is the Levi-Civita symbol. Applying the Hodge dual operator to the spatial components:

$$\begin{cases} * \left(\omega_{xy} dx \wedge dy\right) = * \left[\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right) dx \wedge dy\right] = \left(\nabla \times \overrightarrow{p}\right)_z dz = \omega_z dz \\ * \left(\omega_{yz} dy \wedge dz\right) = * \left[\left(\frac{\partial p_z}{\partial y} - \frac{\partial p_y}{\partial z}\right) dy \wedge dz\right] = \left(\nabla \times \overrightarrow{p}\right)_x dx = \omega_x dx \\ * \left(\omega_{zx} dz \wedge dx\right) = * \left[\left(\frac{\partial p_x}{\partial z} - \frac{\partial p_z}{\partial x}\right) dz \wedge dx\right] = \left(\nabla \times \overrightarrow{p}\right)_y dy = \omega_y dy \end{cases} \tag{19}$$

In vector calculus language, this is the curl of the covector field (metric dual momentum) in 3D. In other words, we can define a vorticity field using the cotangent momenta:

$$\overrightarrow{\omega} = \nabla \times \overrightarrow{p}. \tag{20}$$

In index notation, it can be written as a curl operation:

$$\omega_i = \varepsilon_{ijk} \partial_j \left(p_k\right). \tag{21}$$

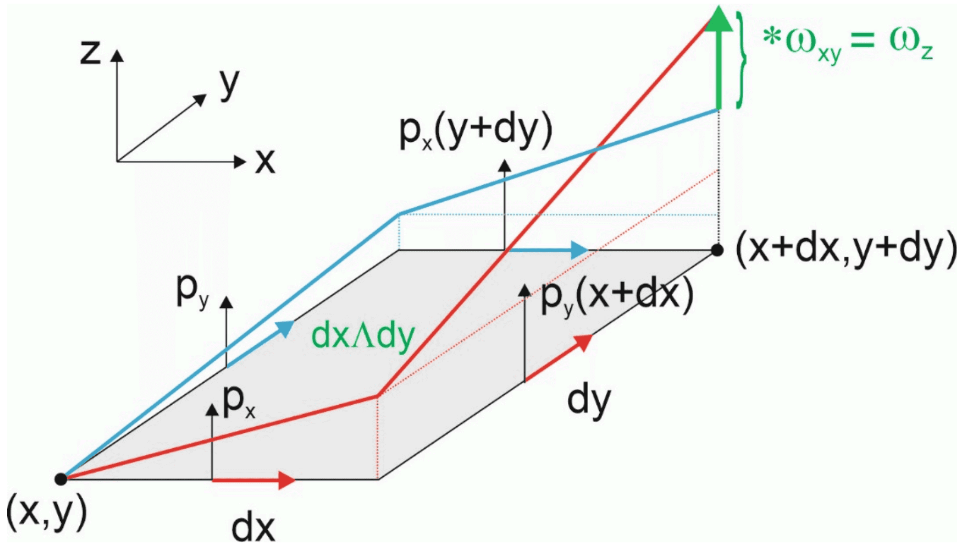

Figure 2. In 3D, the vorticity field is the Hodge dual to the 2-form, e.g.,
$* \left(\omega_{xy} dx \wedge dy\right) = * \left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right) dx \wedge dy = \omega_z dz.$

Here, we use one subscript ($i = 1, 2, 3$) to represent the vorticity field for 3D space in the x-, y-, and z-directions; see Fig. 2.

Thus, the 2-form can be rewritten more compactly in index notation:

$$\omega = \overrightarrow{\omega}_{ti} dt \wedge dx^i + \left(\star_3 \overrightarrow{\omega}_{ij}\right) dx^i \wedge dx^j, \tag{22}$$

or explicitly:

$$\begin{aligned} \omega = {} & \omega_{tx} dt \wedge dx + \omega_{ty} dt \wedge dy + \omega_{tz} dt \wedge dz \\ & + \omega_x dy \wedge dz + \omega_y dz \wedge dx + \omega_z dx \wedge dy. \end{aligned} \tag{23}$$

Here, we use two subscripts ($ti = tx, ty, tz$) to represent the temporal components and one subscript ($i = x, y, z$) to represent the (spatial) vorticity components, e.g.:

$$\begin{cases} \omega_{tx} = \frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial x} \\ \omega_x = \frac{\partial p_z}{\partial y} - \frac{\partial p_y}{\partial z} \end{cases}. \tag{24}$$

In this manner, the field strength (0,2)-tensor can be expressed as

$$\omega = \frac{1}{2} \begin{bmatrix} 0 & \omega_{tx} & \omega_{ty} & \omega_{tz} \\ -\omega_{tx} & 0 & \omega_z & -\omega_y \\ -\omega_{ty} & -\omega_z & 0 & \omega_x \\ -\omega_{tz} & \omega_y & -\omega_x & 0 \end{bmatrix}. \tag{25}$$

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3. Contraction with the Tangent Velocity Yields the Field Dynamic Equation

The velocity vector along the actual path is a linear combination of local coordinate basis vectors of the tangent space at a point p, $\vec{v} \in T_p M$:

$$\vec{v} = v^{\mu}\frac{\partial}{\partial x^{\mu}} = c\frac{\partial}{(c\partial t)} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \tag{26}$$

where the Einstein summation convention is used. For ease, the Cartesian expression is also explicitly written out. Again, c represents the wave propagation speed in the field. (u, v, w) are the particle tangent velocity components along the x-, y-, and z-directions, $\left(\frac{\partial}{\partial x^i}\right)$.

It represents the dynamic flow of the particles in the field along the physical path.

Then, the contraction (or interior product) of the 2-form ω with the tangent velocity vector field yields the dynamic equation of the field; it equals zero, since the orthogonal complements of the Hodge dual operators:

$$\iota_{\vec{v}}(dS) = \iota_{\vec{v}}(\omega) = 0. \tag{27}$$

This means that the 2-form dS is annihilated by the tangent vector $\vec{v}$ via interior contraction.

The interior product of a 2-form with a tangent velocity vector yields a differential 1-form, defined by inserting the tangent velocity vector $\vec{v}$ into the first slot of the 2-form ω. It is also expressed as:

$$\iota_{\vec{v}}(\omega) = \omega(v, -) = 0. \tag{28}$$

Now, we compute and expand the expression $\iota_{\vec{v}}(\omega)$ in Cartesian coordinates.

Using the rule:

$$\iota_{\vec{v}}\left(dx^{\mu} \wedge dx^{\nu}\right) = \iota_{\vec{v}}\left(dx^{\mu}\right) dx^{\nu} - \iota_{\vec{v}}\left(dx^{\nu}\right) dx^{\mu}, \tag{29}$$

and the duality (or natural pairing) between the tangent and cotangent basis vectors:

$$dx^{\mu}\left(\frac{\partial}{\partial x^{\nu}}\right) = \delta^{\mu}_{\nu}, \tag{30}$$

where δ^{μ}_{ν} is the Kronecker delta, and through term-by-term contractions, we have the following expressions for the temporal components:

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$$\begin{cases} \iota_{\vec{v}}\left(\omega_{tx} dt \wedge dx\right) = \omega_{tx} \bullet dx - u\omega_{tx} \bullet dt \\ \iota_{\vec{v}}\left(\omega_{ty} dt \wedge dx\right) = \omega_{ty} \bullet dy - v\omega_{ty} \bullet dt \\ \iota_{\vec{v}}\left(\omega_{tz} dt \wedge dz\right) = \omega_{tz} \bullet dz - w\omega_{tz} \bullet dt \end{cases}. \tag{31}$$

Similarly, the spatial components are

$$\begin{cases} \iota_{\vec{v}}\left(\omega_{xy} dx \wedge dy\right) = u\omega_{xy} \bullet dy - v\omega_{xy} \bullet dx \\ \iota_{\vec{v}}\left(\omega_{yz} dy \wedge dz\right) = v\omega_{yz} \bullet dz - w\omega_{yz} \bullet dy \\ \iota_{\vec{v}}\left(\omega_{zx} dz \wedge dx\right) = w\omega_{zx} \bullet dx - u\omega_{zx} \bullet dz \end{cases}. \tag{32}$$

Grouping the terms for the same 1-form basis:

$$\begin{aligned} \iota_{\vec{v}}(\omega) =& \left[\omega_{tx} + w\omega_{zx} - v\omega_{xy}\right] dx \\ & + \left[\omega_{ty} + u\omega_{xy} - w\omega_{yz}\right] dy \\ & + \left[\omega_{tz} + v\,\omega_{yz} - u\,\omega_{zx}\right] dz \\ & + \left[\omega_{tz} + v\,\omega_{yz} - u\,\omega_{zx}\right] dz \\ & - \left[u\omega_{tx} + v\omega_{ty} + w\omega_{tz}\right] dt = 0. \end{aligned} \tag{33}$$

The differential 1-form is zero; thus, each coefficient should be zero:

$$\begin{cases} \omega_{tx} + w\omega_{zx} - v\omega_{xy} = 0 \\ \omega_{ty} + u\omega_{xy} - w\omega_{yz} = 0 \\ \omega_{tz} + v\,\omega_{yz} - u\,\omega_{zx} = 0 \\ \omega_{tx} u + \omega_{ty} v + \omega_{tz} w = 0 \end{cases}. \tag{34}$$

Using the curl definition of eq. (19) and eq. (20), the first three equations can be written as a vector equation for 3D space:

$$\frac{\partial \vec{p}}{\partial t} + \nabla V - \vec{v} \times \left(\nabla \times \vec{p}\right) = 0. \tag{35}$$

The fourth equation in vector notation reads:

$$\vec{\omega}_t \bullet \vec{v} = 0, \tag{36}$$

or explicitly:

$$\left(\frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial x}\right) u + \left(\frac{\partial p_y}{\partial t} + \frac{\partial V}{\partial y}\right) v + \left(\frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial z}\right) w = 0, \tag{37}$$

The vector $\vec{\omega}_{ti}$ is orthogonal to the tangent vector $\vec{v}$, see Fig. 3 for an illustration of the t-x plane. It indicates that the tangent velocity $\vec{v} = u\partial_x + v\partial_y + w\partial_z$ is the kernel of the 2-form $\vec{\omega}_{ti}$ in 3D space.

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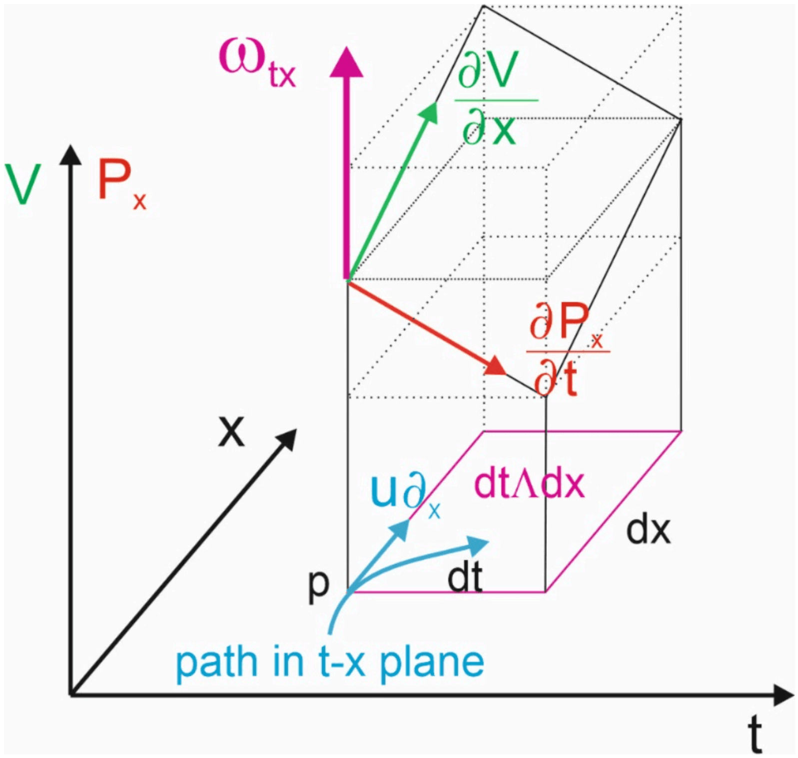

Figure 3. Temporal component of $\omega_{tx}(dt \wedge dx)$ and the tangent velocity component $u\partial_x$ in the t-x plane. In 1D flow, they are “orthogonal” $\omega_{tx}u = 0$.

4. Dynamic Equations for the Compressible Field

Suppose there exists a compressible physical field in space.

The kinetic and potential energy density per unit volume in a compressible field is expressed as follows [7]:

$$V = (\alpha\rho_0)\,c^2; \quad T = (\gamma\rho_0)\,\vec{v}^{\,2}. \tag{38}$$

where ρ_0 is the mass density when the flow velocity is zero, relative to the stationary lab frame. γ is the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1 - \left\|\vec{M}\right\|^2}}. \tag{39}$$

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It represents the mass density increasing effect due to the compression of the volume in the direction of motion, and $\overrightarrow{M}$ is the Mach number vector in fluid dynamics; it is the ratio of the flow velocity of $\overrightarrow{v}$ to the wave propagation speed of c:

$$\overrightarrow{M} = \frac{\overrightarrow{v}}{c}. \tag{40}$$

α is the reciprocal of the Lorentz factor $\alpha = \frac{1}{\gamma}$. It is an expansion factor; physically, it represents the potential energy density decreasing factor due to the relative motion (potential energy changes into kinetic energy).

It is easy to see that there is a reciprocal relation between α and γ:

$$\gamma \bullet \alpha = 1. \tag{41}$$

The Lagrangian density per unit volume is, thus, a quadratic form:

$$\mathcal{L} = T - V = (\gamma\rho_0)\overrightarrow{v}^2 - (\alpha\rho_0)c^2 = \overrightarrow{v}^T g \overrightarrow{v}, \tag{42}$$

where $\overrightarrow{v}$ is the tangent velocity vector in the Cartesian coordinate system:

$$\overrightarrow{v} = v^\mu \partial_\mu = \partial_t + u\partial_x + v\partial_x + w\partial_z. \tag{43}$$

The quadratic form of Eq. (42) can be written out explicitly with the help of a covariant metric tensor:

$$\mathcal{L} = \rho_0 g_{\mu\nu} v^\mu v^\nu = (\rho_0 g_{\mu\nu} v^\mu) v^\nu = p_\nu v^\nu. \tag{44}$$

Here, the metric tensor reads:

$$g_{\mu\nu} = diag(-\alpha, \gamma, \gamma, \gamma) . \tag{45}$$

In this way, we get the conjugate momenta, the metric dual to the tangent velocity vector:

$$p_\nu = \rho_0 g_{\mu\nu} v^\mu = \rho_0(-\alpha c, \gamma u, \gamma v, \gamma w). \tag{46}$$

Now, it is defined as a cotangent vector (1-form) at the point p, $p_\nu \in T_p^* M$. In other words, in contrast to the tangent vector, the conjugate momenta are equipped with the metric tensor of Eq. (45).

Substituting Eq. (46) into Eq. (2), in Cartesian coordinates, the Poincaré–Cartan-like differential 1-form S for the compressible field is defined as:

$$S = (\rho_0\gamma u)\, dx + (\rho_0\gamma v)\, dy + (\rho_0\gamma w)\, dz - \left(\alpha\rho_0 c^2\right) dt. \tag{47}$$

This expression provides a key advantage—namely, that the metric tensor is implicitly incorporated into the definition of the conjugate momenta p_ν. A wave travels in space at the wave speed of c; actually, the last term is the wave conjugate momentum:

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$$\rho_0 \alpha c = \frac{\alpha \rho_0 c^2}{c}. \tag{48}$$

In other words, Eq. (47) can also be written as:

$$S = (\rho_0 \gamma u)\, dx + (\rho_0 \gamma v)\, dy + (\rho_0 \gamma w)\, dz - (\alpha \rho_0 c)\,(cdt). \tag{49}$$

It can be seen that the interior product (contraction) of the differential 1-form S with the tangent velocity $\vec{v}$ yields the Lagrangian density for compressible fields:

$$\mathcal{L} = S\left(\vec{v}\right). \tag{50}$$

Using the rule of duality (or natural pairing) between the tangent and cotangent basis vectors, Eq. (30).

Substituting these into Eq. (35), we finally get the dynamic equation for compressible flow:

$$\frac{\partial\left(\gamma \rho_0 \vec{v}\right)}{\partial t} + \nabla\left(\alpha \rho_0 c^2\right) - \vec{v} \times \left[\nabla \times \left(\gamma \rho_0 \vec{v}\right)\right] = 0. \tag{51}$$

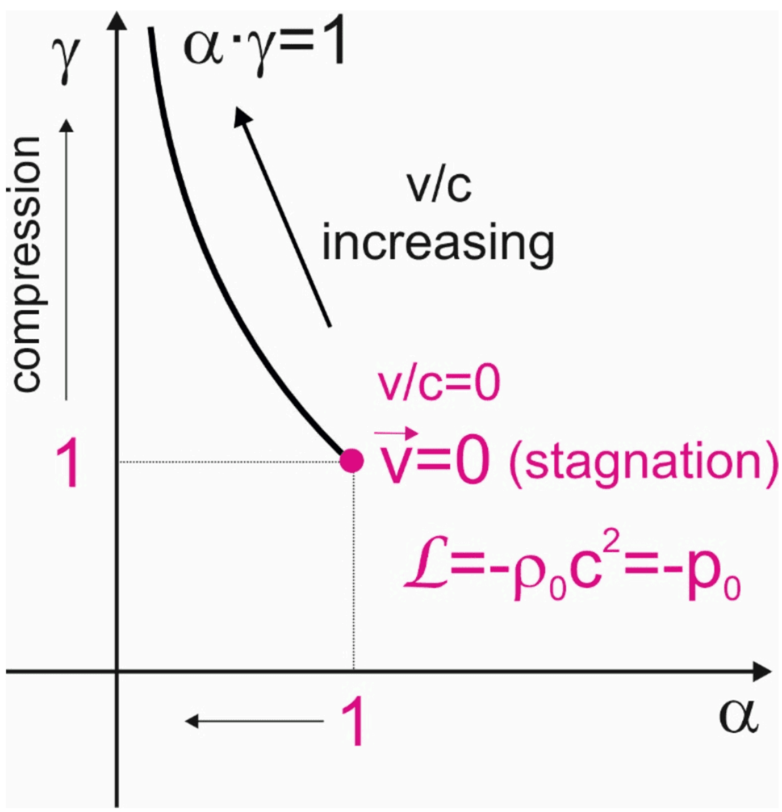

Figure 4. Reciprocal relation between α and γ; when the flow velocity approaches the wave speed c, γ becomes infinitely great; $\alpha \leq 1.0$.

When the flow velocity approaches the wave speed c, the Lorentz factor γ becomes infinitely great, while the expansion factor α approaches zero; see Fig. 4. For compressible fluids, this exhibits a singularity; namely, when the flow velocity equals the wave speed, the equations become undefined.

When the velocity is zero, both the Lorentz factor and the expansion factor are equal to one, and the Lagrangian density [8], Eq. (42), becomes:

$$\mathcal{L} = S\left(\overrightarrow{v}\right) = -\rho_0 c^2 = -p_0. \tag{52}$$

This is the total energy density per unit volume stored inside the system. In fluid dynamics, it is called stagnation pressure (it is a scalar function), also known as total pressure, which is the pressure a fluid possesses when isentropically brought to rest in a lab frame without any loss of mechanical energy (that is to say, the flow velocity is zero relative to the observer).

The fluid is at a stagnation state; the flow velocity is zero relative to the lab frame (to the observer). Thus, the tangent velocity vector $\overrightarrow{v} \in T_p M$ at the point p is:

$$\overrightarrow{v} = c\partial_{ct} = \partial_t. \tag{53}$$

The metric dual action 1-form becomes:

$$S = -\rho_0 c(cdt) = -\rho_0 c^2 dt. \tag{54}$$

Physically, any small disturbance at a point p (an infinitesimal oscillation of potential energy density about its equilibrium point) will propagate across the field in the form of a wave at a wave speed of c, and the wave momentum amounts to $\rho_0 c$, as long as there is no macroscopic motion relative to the observer.

5. Approximations to the Incompressible Flow Model

When the flow velocity is moderate but still much smaller than the wave speed, or the wave speed approaches infinity (the elastic compression bulk modulus of the field material is very high, and the medium is difficult to compress):

$$\overrightarrow{M} = \frac{\overrightarrow{v}}{c} \ll 1 \;\; or \;\; c \to \infty. \tag{55}$$

Under this approximation, both the mass density compressing factor γ and the potential energy decreasing factor α approach one.

$$\gamma \to 1 \;\; and \;\; \alpha \to 1. \tag{56}$$

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This condition allows for certain approximations: it simplifies dynamics problems. First of all, the equation exhibits no singularity anymore; secondly, the potential energy density can be approximated as a scalar function. Here, we use the thermodynamic pressure in the field to represent the potential energy density, as commonly used in the literature. It depends merely on the position and time:

$$V=\alpha\rho_0 c^2 \ \approx p(x,y,z,t). \tag{57}$$

It is no longer explicitly defined as a function of the wave propagation speed c. With this definition, any small disturbances of the pressure (potential energy) in the field propagate instantaneously through the whole field to the boundary without any time lag, regardless of how big the field is. In other words, an incompressible assumption does not treat the pressure oscillation as traveling at a finite wave speed in the field, but rather at an infinitely great wave speed.

In this case, the differential action 1-form degenerates to:

$$S=\rho_0 udx+\rho_0 vdy+\rho_0 wdz-\left(\frac{p}{c}\right)d(ct). \tag{58}$$

Under this approximation, the dynamic equation (50) degenerates to

$$\frac{\partial\left(\rho_0\overrightarrow{v}\right)}{\partial t}+\nabla p=\overrightarrow{v}\times\left[\nabla\times\left(\rho_0\overrightarrow{v}\right)\right]. \tag{59}$$

This equation shows that the particle's motion is a combination of a translational flow (the LHS of the equation) and a rotational motion (the RHS of the equation). It reveals that the particle moves along a helical (spiral) path. The flow field exhibits eddies and turbulence.

Rearranging the pressure gradient to the RHS of the equation:

$$\frac{\partial\left(\rho_0\overrightarrow{v}\right)}{\partial t}=-\nabla p+\overrightarrow{v}\times\left(\nabla\times\left(\rho_0\overrightarrow{v}\right)\right). \tag{60}$$

If both sides have a convective term added, it reads:

$$\frac{\partial\left(\rho_0\overrightarrow{v}\right)}{\partial t}+\left(\overrightarrow{v}\bullet\nabla\right)\left(\rho_0\overrightarrow{v}\right)=-\nabla p+\left(\overrightarrow{v}\bullet\nabla\right)\left(\rho_0\overrightarrow{v}\right)+\overrightarrow{v}\times\left[\nabla\times\left(\rho_0\overrightarrow{v}\right)\right]. \tag{61}$$

Using the vector calculus identity:

$$\nabla\left(\frac{1}{2}\rho_0\overrightarrow{v}^2\right)=\left(\overrightarrow{v}\bullet\nabla\right)\left(\rho_0\overrightarrow{v}\right)+\overrightarrow{v}\times\left[\nabla\times\left(\rho_0\overrightarrow{v}\right)\right], \tag{62}$$

eq. (61) can be rewritten compactly as

$$\frac{D\left(\rho_0 \overrightarrow{v}\right)}{Dt} = -\nabla p + \nabla T. \tag{63}$$

where T is the volumetric density of the kinetic energy:

$$T = \frac{1}{2}\rho_0 \overrightarrow{v}^2. \tag{64}$$

It is recognized that the conjugate momenta in Cartesian coordinates can be expressed as

$$\left(\rho_0 \overrightarrow{v}\right) = \frac{\partial T}{\partial \overrightarrow{v}}, \tag{65}$$

and the Lagrangian density per unit volume (namely, the contraction of the metric dual 1-form and tangent velocity vector) is

$$\mathcal{L} = S\left(\overrightarrow{v}\right) = T - V = \frac{1}{2}\rho_0 \overrightarrow{v}^2 - p. \tag{66}$$

Then, eq. (63) can be expressed more compactly as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \overrightarrow{v}}\right) - \nabla \mathcal{L} = 0. \tag{67}$$

The Euler-Lagrange equation is recovered because we assume the tangent velocity vector of eq. (43) is not an arbitrary vector but rather is along the true path, following a legitimate physical trajectory in the velocity tangent space. The corresponding conjugate momentum is the metric dual to the tangent velocity at the point p in question. Namely, the tangent velocity is the kernel of dS, or we can say that the 2-form is "orthogonal" to the tangent vector (similar to the property that the cross product of two vectors is always orthogonal to both of the original vectors in 3-dimensional space):

$$\iota_{\overrightarrow{v}}(dS) = \iota_{\overrightarrow{v}}(\omega) = 0. \tag{68}$$

If another particle, like a leaf with a density of q, drifts within this field, the drifting velocity of the leaf is $\overrightarrow{v}_q$, relative to the field velocity at point p. In this case, the leaf, or more precisely, $\overrightarrow{v}_q$, is not in the kernel of the 2-form $\omega = dS$. The interaction between the field strength tensor ω and the leaf produces a force on the leaf:

$$\overrightarrow{F}_q = q\left(\overrightarrow{\omega}_t + \overrightarrow{v}_q \times \overrightarrow{\omega}\right). \tag{69}$$

This is similar to the Lorentz force expression; namely, the field strength tensor exerts a force on the flowing particle, similar to the effect that a charged particle experiences when it moves in an electromagnetic field.

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In this case, the space has two components. Like multi-phase, multi-component flow, a two-fluid model has to be used; each component has its own dynamic equation, but with an extra interaction force term of eq. (69) between the two fields [9]. This is beyond the content of this article.

6. Approximations to the Navier-Stokes Equations

For simplification, in order to explore the mathematical structure cleanly, eq. (61) can be rewritten as

$$\frac{\partial \overrightarrow{v}}{\partial t} + \left(\overrightarrow{v} \bullet \nabla\right)\left(\overrightarrow{v}\right) = -\nabla \tilde{p} + \left(\overrightarrow{v} \bullet \nabla\right)\overrightarrow{v} + \overrightarrow{v} \times \left(\nabla \times \overrightarrow{v}\right). \tag{70}$$

Here

$$\tilde{p} = \frac{p}{\rho_0}. \tag{71}$$

Likewise, eq. (70) can be rewritten as

$$\frac{\partial \overrightarrow{v}}{\partial t} + \left(\overrightarrow{v} \bullet \nabla\right)\left(\overrightarrow{v}\right) = -\nabla \tilde{p} + \nabla\left(\frac{1}{2}\overrightarrow{v}^2\right). \tag{72}$$

In Cartesian coordinates, the kinetic density per unit density is

$$\nabla\left(\frac{1}{2}\overrightarrow{v}^2\right) = \frac{1}{2}\nabla\left(u^2 + v^2 + w^2\right). \tag{73}$$

It can be expressed as a matrix-vector multiplication:

$$\nabla\left(\frac{1}{2}\overrightarrow{v}^2\right) = \begin{bmatrix} \partial_x u & \partial_x v & \partial_x w \\ \partial_y u & \partial_y v & \partial_y w \\ \partial_z u & \partial_z v & \partial_z w \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{74}$$

This matrix is the transpose of the gradient of the velocity vector:

$$J^T = \begin{bmatrix} \partial_x u & \partial_x v & \partial_x w \\ \partial_y u & \partial_y v & \partial_y w \\ \partial_z u & \partial_z v & \partial_z w \end{bmatrix}. \tag{75}$$

It can be decomposed into the sum of a symmetric matrix part and an antisymmetric part:

$$J^T = S + A, \tag{76}$$

where

$$S = \frac{1}{2}\left(J^T + J\right)\ ;\ A = \frac{1}{2}\left(J^T - J\right) \tag{77}$$

Thus, the kinetic energy gradient can be written as the sum of multiplying the tangent velocity vector by a symmetric and an antisymmetric matrix.

$$\nabla\left(\frac{1}{2}\overrightarrow{v}^2\right)=S\overrightarrow{v}+A\overrightarrow{v}. \tag{78}$$

The symmetric part represents the local stretch or shrink in the eigen-basis, while the antisymmetric part represents the local spinning motion.

Recalling the Stokes hypothesis, the viscous stress tensor in the Navier-Stokes equation for a Newtonian fluid is modeled as

$$\tau=2\mu S=2\mu\left(J^T+J\right). \tag{79}$$

Comparing this with the Navier-Stokes equations, we can see that the Navier-Stokes equations have modeled the symmetric matrix-vector multiplication as a divergence of a symmetric viscous stress tensor:

$$\nabla\bullet\tau\approx S\overrightarrow{v}, \tag{80}$$

through an arithmetic mean of the velocity gradient and its transpose, retaining only the symmetric part and neglecting the spinning or turning of the particles (the antisymmetric part).

Furthermore, from eq. (79) we can see that the symmetric viscous stress tensor model is independent of the magnitude of the velocity and dependent only on the velocity gradient by Stokes's hypothesis. The divergence of this symmetric viscous stress tensor is modeled as a net force per unit volume due to viscous stresses in the Navier-Stokes equations:

$$\frac{\partial\overrightarrow{v}}{\partial t}+\left(\overrightarrow{v}\bullet\nabla\right)\left(\overrightarrow{v}\right)=-\nabla\tilde{p}+\nabla\bullet\tau. \tag{81}$$

7. Dynamic Equations of the Vorticity Field

In 3+1 dimensions, the differential 2-form of the field strength tensor (just like the Faraday tensor, also called the electromagnetic field strength tensor in electromagnetic theory) can be expressed by eqs. (23) and (25).

A foundational structure of differential forms is that the second exterior derivative is always zero, as long as dS is locally smooth (the Poincaré Lemma), i.e.,

$$d\omega=d^2(S)=0 \tag{82}$$

If it is fully expanded and written out, it reads:

$$d\omega=\left(\nabla\bullet\overrightarrow{\omega}\right)dx\wedge dy\wedge dz+\left(\partial_t\overrightarrow{\omega}_k-\nabla\times\overrightarrow{\omega}_{ti}\right)\left(dt\wedge dx^i\wedge dx^j\right)=0 \tag{83}$$

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This is a differential 3-form; it is equal to zero, and each coefficient is thus zero:

$$\begin{cases} \nabla \bullet \overrightarrow{\omega} = 0 \\ \frac{\partial \overrightarrow{\omega}}{\partial t} = \nabla \times \overrightarrow{\omega}_t \end{cases} \tag{84}$$

where $\overrightarrow{\omega}$ is the vorticity field, defined by eq. (20):

$$\overrightarrow{\omega} = \nabla \times \left(\gamma \rho_0 \overrightarrow{v}\right) \tag{85}$$

The first part of eq. (84) says that the vorticity field is divergence-free (similar to the idea that magnetic monopoles do not exist, as in Gauss's law for magnetism).

Similar to the electromagnetism theory that moving charged particles will produce magnetic fields, the motion of mass particles will produce a vorticity field, as expressed by eq. (85). From the viewpoint of mathematics, it can be hypothesized that if charged particles are brought to rest, they will not produce a magnetic field, and when the velocity of the mass particles is zero, there is no vorticity field.

$\overrightarrow{\omega}_t$ is the vorticity vector in the temporal direction; its components are

$$\begin{cases} \omega_{tx} = \frac{\partial(\gamma \rho_0 u)}{\partial t} + \frac{\partial(\alpha \rho_0 c^2)}{\partial x} \\ \omega_{ty} = \frac{\partial(\gamma \rho_0 v)}{\partial t} + \frac{\partial(\alpha \rho_0 c^2)}{\partial y} \\ \omega_{tz} = \frac{\partial(\gamma \rho_0 w)}{\partial t} + \frac{\partial(\alpha \rho_0 c^2)}{\partial z} \end{cases} \tag{86}$$

Locally, in the field, the macroscopic motion (an infinitesimal change of the momentum along the time direction) is driven by the spatial gradient of the potential energy density, $V = \alpha \rho_0 c^2 \leq \rho_0 c^2$.

As mentioned before, it is not defined at $v = c$. For a compressible fluid, when the flow velocity is equal to the wave speed, the vorticity field also exhibits a singularity. When the flow velocity approaches the wave speed, the mass density increasing factor γ becomes great (see eq. (39)), and the vorticity will become very strong; locally, a hurricane will form.

Conclusions

Differential forms provide an elegant and powerful framework for describing physical fields, revealing their fundamental geometric and essential dynamic structure. The differential action 1-form $S = p_\mu dx^\mu$, defined as the conjugate momentum, is the metric dual to the tangent velocity vector $\overrightarrow{v} = v^\mu \partial_\mu$ and lives in the cotangent space $T_p^* M$. The Lagrangian volumetric density—represented as a scalar field—is given by the interior product (or natural pairing) of S and $\overrightarrow{v}$, i.e., $\mathcal{L} = S\left(\overrightarrow{v}\right)$. The field strength tensor arises from the exterior derivative of the 1-form, $\omega = dS$, forming a differential 2-form whose coefficients constitute an antisymmetric (0,2)-tensor. The dynamic equation of the field is then the interior product of the tangent velocity vector and the 2-form: $\iota_{\overrightarrow{v}}(\omega) = 0$. In reality, fields are more or less compressible, and wave propagation in the field occurs at a finite wave speed c, which depends on the properties of the field (materials) under consideration. The dynamic equation exhibits a singularity when the flow velocity is equal to the wave speed. When the flow velocity is much smaller than the wave speed, or the wave speed is infinitely great, then the disturbances in the field are instantaneously propagated across the whole field to the field boundary without any time lag. Under this approximation, the field can be approximated as incompressible, the dynamic equation will degenerate to an incompressible flow, and the equation exhibits no singularity. Furthermore, by neglecting the spinning motion (i.e., the antisymmetric part of the velocity gradient), retaining only the symmetric part, and applying the Stokes hypothesis, the dynamic equation can be modeled as the Navier-Stokes equations. Finally, using the fundamental identity from differential geometry that the second exterior derivative always vanishes ($d^2 = 0$), a homogeneous dynamic equation for the vorticity field is obtained: $d\omega = 0$, assuming no sources, sinks, or singularities at the point in question.

Appendix: Mathematical Derivations in Detail

A1. The differential 2-form: $\omega = \mathbf{dS}$

The Lagrangian density is the interior product of the action 1-form and the tangent velocity vector:

$$\mathcal{L} = S(\overrightarrow{v}). \tag{A1}$$

where the differential action 1-form is given:

$$S = p_x dx + p_y dy + p_z dz - \left(\frac{V}{c}\right) cdt. \tag{A2}$$

The differential 1-form lives in the cotangent space $p_i \in T_p^* M$ with the cotangent basis dx^i (and dt). It defines a duality between tangent vectors ($\overrightarrow{v}$) and conjugate momenta ($\overrightarrow{p}$) by the metric tensor.

The exterior derivative of the 1-form yields the differential 2-form

$$dS = \omega = dp_x \wedge dx + dp_y \wedge dy + dp_z \wedge dz - dV \wedge dt. \tag{A3}$$

The RHS has four terms; we expand each term step-by-step in detail.

For the first term, the differential of p_x reads:

$$dp_x = \frac{\partial p_x}{\partial t} dt + \frac{\partial p_x}{\partial x} dx + \frac{\partial p_x}{\partial y} dy + \frac{\partial p_x}{\partial z} dz. \tag{A4}$$

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Thus:

$$dp_x \wedge dx = \frac{\partial p_x}{\partial t} dt \wedge dx + \frac{\partial p_x}{\partial y} dy \wedge dx + \frac{\partial p_x}{\partial z} dz \wedge dx. \tag{A5}$$

Since

$$dx \wedge dx = 0. \tag{A6}$$

The wedge product leaves three terms.

Similarly, the second term is:

$$dp_y \wedge dy = \frac{\partial p_y}{\partial t} dt \wedge dy + \frac{\partial p_y}{\partial x} dx \wedge dy + \frac{\partial p_y}{\partial z} dz \wedge dy. \tag{A7}$$

The third term is:

$$dp_z \wedge dz = \frac{\partial p_z}{\partial t} dt \wedge dz + \frac{\partial p_z}{\partial x} dx \wedge dz + \frac{\partial p_z}{\partial y} dy \wedge dz. \tag{A8}$$

And the fourth (potential energy density) term is:

$$-dV \wedge dt = -\frac{\partial V}{\partial x} dx \wedge dt - \frac{\partial V}{\partial y} dy \wedge dt - \frac{\partial V}{\partial z} dz \wedge dt. \tag{A9}$$

Adding all terms together:

$$dS = \begin{cases} \frac{\partial p_x}{\partial t} dt \wedge dx + \frac{\partial p_x}{\partial y} dy \wedge dx + \frac{\partial p_x}{\partial z} dz \wedge dx \\ \frac{\partial p_y}{\partial t} dt \wedge dy + \frac{\partial p_y}{\partial x} dx \wedge dy + \frac{\partial p_y}{\partial z} dz \wedge dy \\ \frac{\partial p_z}{\partial t} dt \wedge dz + \frac{\partial p_z}{\partial x} dx \wedge dz + \frac{\partial p_z}{\partial y} dy \wedge dz \\ -\frac{\partial V}{\partial x} dx \wedge dt - \frac{\partial V}{\partial y} dy \wedge dt - \frac{\partial V}{\partial z} dz \wedge dt \end{cases}. \tag{A10}$$

Applying the antisymmetric properties of the 2-form basis (e.g., $dt \wedge dx = -dx \wedge dt$) and collecting the same basis:

$$\begin{aligned} \omega = & \left(\frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial x}\right) dt \wedge dx + \left(\frac{\partial p_y}{\partial t} + \frac{\partial V}{\partial y}\right) dt \wedge dy + \left(\frac{\partial p_z}{\partial t} + \frac{\partial V}{\partial z}\right) dt \wedge dz \\ & + \left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial p_z}{\partial y} - \frac{\partial p_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial p_x}{\partial z} - \frac{\partial p_z}{\partial x}\right) dz \wedge dx \end{aligned} \tag{A11}$$

Now we have six independent terms for this 2-form.

We can define:

$$\omega_{ij} = \frac{\partial p_j}{\partial x^i} - \frac{\partial p_i}{\partial x^j} \tag{A12}$$

where

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$$V = (\alpha\rho_0)\,c^2; \quad p_t = \frac{V}{c} = (\alpha\rho_0)\,c \tag{A13}$$

Then, the 2-form can be expressed as

$$\begin{aligned} \omega = \ & \omega_{tx} dt \wedge dx + \omega_{ty} dt \wedge dy + \omega_{tz} dt \wedge dz \\ + \ & \omega_{xy} dx \wedge dy + \omega_{yz} dy \wedge dz + \omega_{zx} dz \wedge dx \end{aligned} \tag{A14}$$

A2. Interior Product of 2-Form with Tangent Velocity $\iota_{\overrightarrow{\mathbf{v}}}(\omega) = \mathbf{0}$

Given the particle velocity at point p in the field

$$\overrightarrow{v} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + c\frac{\partial}{c\partial t} \tag{A15}$$

Inputting this tangent vector into the first slot of the 2-form (A14), the contraction then becomes a 1-form. Using the following rule:

$$dx^i \wedge \ dx^j \left(\overrightarrow{v}, -\right) = dx^i \left(\overrightarrow{v}\right) dx^j - dx^j \left(\overrightarrow{v}\right) dx^i \tag{A16}$$

and the duality relationship between the tangent and cotangent basis vectors:

$$dx^i \left(\frac{\partial}{\partial x^j}\right) = \delta^i_j \tag{A17}$$

The first term

$$\omega_{tx} dt \wedge dx \left(\overrightarrow{v}\right) = \omega_{tx} dt \left(\overrightarrow{v}\right) dx - \omega_{tx} dx \left(\overrightarrow{v}\right) dt = \omega_{tx} dx - \omega_{tx} u dt \tag{A18}$$

Similarly, for the second and third terms:

$$\begin{cases} \omega_{ty} dt \wedge dy \left(\overrightarrow{v}\right) = \omega_{ty} dt \left(\overrightarrow{v}\right) dy - \omega_{ty} dy \left(\overrightarrow{v}\right) dt = \omega_{ty} dy - \omega_{ty} v dt \\ \omega_{tz} dt \wedge dz \left(\overrightarrow{v}\right) = \omega_{tz} dt \left(\overrightarrow{v}\right) dz - \omega_{tz} dz \left(\overrightarrow{v}\right) dt = \omega_{tz} dz - \omega_{tz} w dt \end{cases} \tag{A19}$$

Applying the same contraction rule for the spatial terms, we have

$$\begin{cases} \omega_{xy} dx \wedge dy \left(\overrightarrow{v}\right) = \omega_{xy} dx \left(\overrightarrow{v}\right) dy - \omega_{xy} dy \left(\overrightarrow{v}\right) dx = \omega_{xy} u dy - \omega_{xy} v dx \\ \omega_{yz} dy \wedge dz \left(\overrightarrow{v}\right) = \omega_{yz} dy \left(\overrightarrow{v}\right) dz - \omega_{yz} dz \left(\overrightarrow{v}\right) dy = \omega_{yz} v dz - \omega_{yz} w dy \\ \omega_{zx} dz \wedge dx \left(\overrightarrow{v}\right) = \omega_{zx} dz \left(\overrightarrow{v}\right) dx - \omega_{zx} dx \left(\overrightarrow{v}\right) dz = \omega_{zx} w dx - \omega_{zx} u dz \end{cases} \tag{A20}$$

Adding all terms together and collecting the same basis, finally, the contraction reads:

$$\begin{aligned} &(\omega_{tx} - \omega_{xy} v + \omega_{zx} w)\, dx + (\omega_{ty} + \omega_{xy} u - \omega_{yz} w)\, dy \\ &+ (\omega_{tz} + \omega_{yz} v - \omega_{zx} u)\, dz - (\omega_{tx} u + \omega_{ty} v + \omega_{tz} w)\, dt = 0 \end{aligned} \tag{A21}$$

The coefficients for the basis dx, dy, and dz are zero; thus, we have three equations:

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$$\begin{cases} \omega_{tx} - \omega_{xy}v + \omega_{zx}w = 0 \\ \omega_{ty} + \omega_{xy}u - \omega_{yz}w = 0 \\ \omega_{tz} + \omega_{yz}v - \omega_{zx}u = 0 \end{cases} \tag{A22}$$

or to be explicitly written as:

$$\begin{cases} \frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial x} + w\left(\frac{\partial p_x}{\partial z} - \frac{\partial p_z}{\partial x}\right) - v\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right) = 0 \\ \frac{\partial p_y}{\partial t} + \frac{\partial V}{\partial y} + u\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right) - w\left(\frac{\partial p_z}{\partial y} - \frac{\partial p_y}{\partial z}\right) = 0 \\ \frac{\partial p_z}{\partial t} + \frac{\partial V}{\partial z} + v\left(\frac{\partial p_z}{\partial y} - \frac{\partial p_y}{\partial z}\right) - u\left(\frac{\partial p_x}{\partial z} - \frac{\partial p_z}{\partial x}\right) = 0 \end{cases} \tag{A23}$$

According to the vorticity field definition of Eq. (20), we finally get the dynamic equation of Eq. (35).

$$\frac{\partial \overrightarrow{p}}{\partial t} + \nabla V - \overrightarrow{v} \times \overrightarrow{\omega} = 0 \tag{A24}$$

A3. Dynamic equation for the vorticity field $\mathbf{d}\omega = \mathbf{0}$

When the flow velocity is not equal to the wave propagation speed, the Lagrangian density defines a smooth function, or the differential 1-form is on a smooth manifold; then, locally,

$$d\omega = d^2 S = 0 \tag{A25}$$

This is a general property of exterior derivatives.

Using the Hodge dual operator, we have the 2-form as Eq. (23):

$$\begin{aligned} \omega = \omega_{tx} dt \wedge dx + \omega_{ty} dt \wedge dy + \omega_{tz} dt \wedge dz \\ + \omega_z dx \wedge dy + \omega_x dy \wedge dz + \omega_y dz \wedge dx \end{aligned} \tag{A26}$$

First, we take the exterior derivatives for the first three terms:

Taking the exterior derivative for the first term of Eq. (A26):

$$d\left(\omega_{tx} dt \wedge dx\right) = d\omega_{tx} \wedge dt \wedge dx \tag{A27}$$

Now, we have

$$d\omega_{tx} = \partial_t \omega_{tx} dt + \partial_x \omega_{tx} dx + \partial_y \omega_{tx} dy + \partial_z \omega_{tx} dz \tag{A28}$$

Substituting (A28) into (A27):

$$\begin{aligned} d\omega_{tx} \wedge dt \wedge dx &= \left(\partial_y \omega_{tx} dy + \partial_z \omega_{tx} dz\right) \wedge dt \wedge dx \\ &= \partial_y \left(\omega_{tx}\right) dy \wedge dt \wedge dx + \partial_z \left(\omega_{tx}\right) dz \wedge dt \wedge dx \end{aligned} \tag{A29}$$

Only the dy and dz terms survive because $dt \wedge dt = dx \wedge dx = 0$.

Similarly, we perform the same procedures for the second and third terms, $\omega_{ty} dt \wedge dy$ and $\omega_{tz} dt \wedge dz$:

$$
\begin{aligned}
&\partial_y\left(\omega_{tx}\right) dy\wedge dt\wedge dx+\partial_z\left(\omega_{tx}\right) dz\wedge dt\wedge dx\\
&+\partial_z\left(\omega_{ty}\right) dz\wedge dt\wedge dy+\partial_x\left(\omega_{ty}\right) dx\wedge dt\wedge dy\\
&+\partial_x\left(\omega_{tz}\right) dx\wedge dt\wedge dz+\partial_y\left(\omega_{tz}\right) dy\wedge dt\wedge dz
\end{aligned} \tag{A30}
$$

We can reorder the wedge products using antisymmetric properties (e.g., $dt\wedge dx\wedge dy=-dx\wedge dt\wedge dy$, etc.) and collect the same basis:

$$
\begin{aligned}
&\left[\partial_y\left(\omega_{tx}\right)-\partial_x\left(\omega_{ty}\right)\right] dt\wedge dx\wedge dy\\
&+\left[\partial_z\left(\omega_{ty}\right)-\partial_y\left(\omega_{tz}\right)\right] dt\wedge dy\wedge dz\\
&+\left[\partial_x\left(\omega_{tz}\right)-\partial_z\left(\omega_{tx}\right)\right] dt\wedge dz\wedge dx
\end{aligned} \tag{A31}
$$

Now we take exterior derivatives for the spatial terms (the fourth, fifth, and sixth terms of Eq. (A26)):

The exterior derivative of the fourth term:

$$
d\left(\omega_x dy\wedge dz\right)=d\omega_x\wedge dy\wedge dz \tag{A32}
$$

The differential of $d\omega_x$ is thus

$$
d\omega_x=\partial_t\omega_x dt+\partial_x\omega_x dx+\partial_y\omega_x dy+\partial_z\omega_x dz \tag{A33}
$$

Substituting (A33) into (A32), thus

$$
d\left(\omega_x dy\wedge dz\right)=\partial_t\omega_x dt\wedge dy\wedge dz+\partial_x\omega_x dx\wedge dy\wedge dz \tag{A34}
$$

Similarly, only the dt and dx terms survive since $dy\wedge dy=dz\wedge dz=0$.

Repeat this procedure for $\omega_z dx\wedge dy$ and $\omega_y dz\wedge dx$. Adding all terms together and collecting the same basis, we finally have:

$$
\begin{aligned}
d\omega=&\left(\partial_x\omega_x+\partial_y\omega_y+\partial_z\omega_z\right) dx\wedge dy\wedge dz\\
&+\left(\partial_t\omega_x+\partial_z\omega_{ty}-\partial_y\omega_{tz}\right) dt\wedge dy\wedge dz\\
&+\left(\partial_t\omega_y+\partial_x\omega_{tz}-\partial_z\omega_{tx}\right) dt\wedge dz\wedge dx\\
&+\left[\partial_t\omega_z+\partial_y\omega_{tx}-\partial_x\omega_{ty}\right] dt\wedge dx\wedge dy
\end{aligned} \tag{A35}
$$

The coefficients are zero; eventually, we get the dynamic equations for the vorticity field:

$$
\begin{aligned}
&\nabla\bullet\vec{\omega}=0\\
&\frac{\partial\vec{\omega}}{\partial t}=\nabla\times\vec{\omega}_t
\end{aligned} \tag{A36}
$$

By the way, if we define the tangent velocity vector for charged particles by Eq. (A15) and the metric dual 1-form by Eq. (A2):

$$
p_\nu=\rho_0 g_{\mu\nu}v^\mu=\frac{\mu_0\rho_0}{4\pi}(-\alpha c,\gamma u,\gamma v,\gamma w)=\frac{\mu_0}{4\pi}\left(-\alpha\rho_0 c,\ \vec{J}\right). \tag{A37}
$$

Here, ρ_0 is defined to be the charge volumetric density when it is at rest relative to the observer, μ_0 is the permeability in space, and c is the photon propagation speed. With the same procedure, we can get the electromagnetic field strength tensor and the dynamic equation; thus, they are unified in a differential form framework. This is beyond the content of this article.

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Potential competing interests: No potential competing interests to declare.