On bundles of varieties $V_{2^3}$ in $PG(4, q)$ and their codes

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Abstract

In this note we use the spatial representation in $\Sigma = PG(4, q)$ of the projective plane $\Pi = PG(2, q^2)$, by fixing a hyperplane $\Sigma'$ with a regular spread $S$ of lines. We consider a bundle $X$ of varieties $V_2$ of $\Sigma$ having in common the $q + 1$ points of a conic $C_2$ of a plane $\pi_0$, $\pi_0 \cap \Sigma' = l_0 \in S$, thus representing an affine line of $\Pi$, and a further affine point $O \notin \pi_0$. This subset $X$ of $\Sigma$ represents a bundle of non-affine Baer subplanes of $\Pi$, each of them having one point at infinity (corresponding to a line of $S$), having in common a subline of affine points of $\Pi$ and a further affine point. Then $X$ is considered as a projective system of $\Sigma$ and, by using such a representation of $\Pi$, we can calculate the ground parameters of the code $C_X$ arising from it.

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1. Introduction

It is known that a projective translation plane $\Pi$ of order $n = q^2$ of dimension 2 over its kernel $F = GF(q)$ can be represented by a 4-dimensional projective space $\Sigma = PG(4, q)$ over $F$, fixing a hyperplane $\Sigma' = PG(3, q)$ and a spread $S$ of lines of $\Sigma'$. The points of $\Pi$ are represented by (i) the points of $\Sigma \setminus \Sigma'$ and (ii) the lines of $S$. The lines of $\Pi$ are represented by (i) the planes $\alpha$ of $\Sigma \setminus \Sigma'$ such that $\alpha \cap \Sigma'$ belongs to $S$ and by (ii) the spread $S$. The translation line $l$ of $\Pi$ is represented by $S$ (cf. [1]).

A Baer subplane $B$ of $\Pi$ has order $q$ and it is dense in the sense that a line of $\Pi$ either is a line of $B$ (that is, meets $B$ in a
subline of \( q + 1 \) points, such a subplane is affine) or it meets \( B \) in one point (such a subplane is non-affine).

The affine Baer subplanes \( B \) of \( \Pi \) are represented by the transversal planes \( \beta \) to \( S \), that is, the planes of \( \Sigma \setminus \Sigma' \) such that the line \( \beta \cap \Sigma' \neq S \) meets \( q + 1 \) lines of \( S \). In such a way \( l \) is a line of \( B \) (cf. \cite{[2]}, pp. 68--72). Of course all that holds also in case \( \Pi \) is the Desarguesian plane \( PG(2, q^2) \) when \( S \) is a regular spread (cf. \cite{[3]}, \cite{[2]}).

A variety \( V_2^3 \) of \( \Sigma \) with a line \( l_0 \) in \( S \) as the minimum (linear) order directrix, a conic \( C^2 \) as a 2nd order directrix with \( C^2 \subset \pi_0 \), \( \pi_0 \cap l_0 \subset S \setminus l_0 \) and \( C^2 \cap l_0 = \emptyset \), represents a non-affine Baer subplane of \( \Pi \) having one point on the translation line \( l \) and the subline \( C^2 \) of the line \( \pi_0 \) (cf. \cite{[3]}).

In this paper we consider bundles of \( q + 1 \) varieties \( V_2^3 \) of \( \Sigma = PG(4, q) \) with the linear directrix in \( S \) and having in common a same conic \( C^2 \) as a 2nd order directrix and one further affine point. By using the spatial representation of \( \Pi = PG(2, q^2) \) in \( PG(4, q) \), we can characterize such a bundle \( X \) from the intersection point of view, construct a linear code \( C_X \) arising from it and show that its ground parameters allow \( C_X \) to correct an enough large number of errors.

2. Preliminary Notes

Let \( F = GF(q) \) be a finite field, \( q = p^r \), \( p \) prime. Denote \( F^{r+1} \) the \((r+1)\)-dimensional vector space over \( F \),

\[ P^r = PrF^{r+1} = PG(r, q) \]

the \( r \)-dimensional projective space contraction of \( F^{r+1} \) over \( F \). Let \( \overline{F} \) be the algebraic closure of the field \( F = GF(q) \).

Denote \( S_t \) with \( t \geq 2 \) a subspace of \( P^r \) of dimension \( t \). A hyperplane \( S_{r−1} \) will be denoted also by \( H \), a plane by \( \pi \).

The geometry \( P^r \) is considered a sub-geometry of \( \overline{P^r} \), the projective geometry over \( \overline{F} \). We refer to the points of \( P^r \) as the rational points of \( P^r \).

**Definition 2.1.** A variety \( V'_u^r \) of dimension \( u \) and of order \( v \) of \( P^r \) is the set of the rational points of a projective variety \( V_u^r \) of \( \overline{P^r} \) defined by a finite set of polynomials with coefficients in the field \( \overline{F} \).

From \cite{[4]}, p.290, 7.- for \( r \geq 4 \) follows

**Lemma 2.2.** The ruled variety \( V_2^3 \) of \( PG(r, q) \) is generated by the lines connecting the corresponding points of two birationally (or, projectively) equivalent curves in two complementary subspaces, of order \( m \) and \( r−1−m \), respectively. It has order the sum of the orders of the curves as there are no fixed points.

Let \( P^4 \) be the projective geometry \( PG(4, q) \).

**Lemma 2.3.** A variety \( V_2^3 \) of \( PG(4, q) \) is obtained by joining the corresponding points of a directrix line \( l \) and a directrix conic \( C \) in a plane \( \pi \), \( l \) and \( C \) being projectively equivalent and with \( l \cap \pi = \emptyset \).

Choose a coordinate system in $\mathcal{P}^4$ so that it is a coordinate system for $\tilde{\mathcal{P}}^4$ too, denote a point

$$P = (x_1, x_2, y_1, y_2, \tilde{\eta}) := \tilde{F}(x_1, x_2, y_1, y_2, \tilde{\eta}), \tilde{F}^* = \tilde{F} \setminus \{0\}.$$ 

$P$ is a rational point if there exists $(x_1, x_2, y_1, y_2, \tilde{\eta}) \in \tilde{F}^5$ such that $P = (x_1, x_2, y_1, y_2, \tilde{\eta})$. A variety $V$ of $\mathcal{P}^4$ is the set of the rational points of $\mathcal{P}^4$ solutions of a finite set of polynomials of $\tilde{F}[x_1, x_2, y_1, y_2, \tilde{\eta}]$.

**Lemma 2.4.** The variety $V_2^3$ can be represented as the definite intersection of two quadrics of $PG(4, q)$, that is, the cone of planes $Q_1: sx_2^2 - x_1^2 - sx_2 t = 0$ (where $s$ is a non square of $GF(q)$) and the cone of planes $Q_2: x_1 y_1 - x_2 y_2 = 0$. The plane $\pi': x_1 = 0, x_2 = 0$ is contained in both quadrics so that, by Bezout, the order of the intersection variety is $a - 1 = 3$.


Let $\Pi = PG(2, q^2)$ be the Desarguesian plane over $GF(q^2)$. Denote $l$ the line at infinity of $\Pi$. In the spatial representation of $\Pi$ in $\mathcal{P}^4 = PG(4, q)$ fix a hyperplane $\Sigma' = PG(3, q)$ and a regular spread $S$ of lines of $\Sigma'$, where $|S| = q^2 + 1$.

**Lemma 2.5.** The points of $\Pi$ are represented by (i) the points of $\Sigma \setminus \Sigma'$ (the affine points of $\Pi$) and by (ii) the lines of $S$ (the points at infinity of $\Pi$). The lines of $\Pi$ are represented by (i) the planes $\alpha$ of $\Sigma \setminus \Sigma'$ such that $\alpha \cap \Sigma'$ belongs to $S$ and by (ii) the spread $S$, representing the line at infinity $l$.

Proof. See [1] the Bruck and Bose representation and [2], p. 775.

**Definition 2.6.** A Baer subplane of $\Pi = PG(2, q^2)$ is an affine subplane if it meets the line at infinity $l$ of $\Pi$ in a subline $l_1$, it is a non-affine subplane if it meets the line $l$ in one point.

**Lemma 2.7.**

(i) Two affine Baer subplanes of $\Pi$ having in common the subline $l_1$ can meet in at most one further point.

(ii) The Baer subplanes having in common only a subline $l_1$ are $q^2$.

(iii) The Baer subplanes having in common a subline $l_1$ and one further point are $q + 1$.

Proof. (i) Two Baer subplanes having in common a subline $l_1$ and two further points coincide, because they have in common at least four reference (three by three non collinear) points.

Without losing generality, we can consider two affine Baer subplanes $B$ and $B'$ of $\Pi$ having in common a subline $l_1$ of $l$. In the spatial representation of $\Pi$, they are represented by two planes $B$ and $B'$ of $\mathcal{P}^4$, respectively, such that the lines $B \cap \Sigma' = r$ and $B' \cap \Sigma' = r'$ are transversal lines of the same regulus $R \subset S$. Denote $R'$ the opposite regulus to $R$.

There are two cases:

(ii) If $r = r'$, the planes $B$ and $B'$ have in common the line $r$ meeting the regulus $R$ in its $q + 1$ lines so that the subplanes $B$ and $B'$ have in common the subline $l_1$ (represented by $R$) of the line $l$ (represented by $S$) and no further (affine) points.
Such planes are $\frac{q^4}{q^2} = q^2$ and represent $q^2$ affine Baer subplanes of $\Pi$ having in common only the subset $l_1$ of $q+1$ points of the line at infinity $l$.

(iii) If $r \neq r'$, the planes $B$ and $B'$ have in common an affine point $O \in \Sigma \setminus \Sigma'$ so that the two subplanes $B$ and $B'$ meet along the subline $l_1$ represented by $R$ and in the affine point $O$. The regulus $R$ has $q+1$ transversal lines $\{t_i|i=1,\ldots,q+1\}$ belonging to $R'$. Each space $O \oplus t$ is a transversal plane $\tau_i$ so that $\{t_i|i=1,\ldots,q+1\}$ represent the $q+1$ affine Baer subplanes of $\Pi$ having in common $l_1$ and the affine point $O$.

Choose and fix a line $l_\infty$ of the (regular) spread $S$, a plane $\tau_0$ such that $\tau_0 \cap \Sigma = l_0 \in S \setminus l_\infty$ and a non-degenerate conic $C^2 \subset \tau_0 \setminus l_0$. Let $\Lambda$ be a projectivity between $l_\infty$ and $C^2$. Denote $V^2_{\tau} \Pi$ the variety arising by connecting corresponding points of $l_\infty$ and $C^2$ via $\Lambda$ (cf. [5], p. 90).

**Lemma 2.8.** The variety $V^2_{\tau} \Pi$ represents a non-affine Baer subplane of $\Pi$ meeting the line at infinity $l$ in the point $l_\infty$ and containing the subline $C^2$ of the line represented by $\tau_0$.

Proof. See [3] and [2].

Let $F^n$ be the $n$-dimensional vector space over $F = GF(q)$.

**Definition 2.9.** A linear $[n, k]_q$-code $C$ of length $n$ is a $k$-dimensional subspace of the vector space $F^n$.

**Definition 2.10.** An $[n, k]_q$-projective system $X$ is a set of $n$ non necessarily distinct points of the projective geometry $PrF^k = PG(k-1, q)$. It is non-degenerate if these points are not contained in a hyperplane (cf. [6], p. 2).

Assume that $X$ consists of $n$ distinct points having maximum rank.

Codes and projective systems are linked by a strict connection one can read in [6], so that from subsets $X$ of a projective geometry linear codes $C_X$ can be generated. More precisely, for each point of $X$ choose a generating vector. Denote $M$ the matrix having as rows such $n$ vectors and let $C_X$ be the linear code having $M^t$ as a generator matrix. The code $C_X$ is the $k$-dimensional subspace of $F^n$ which is the image of the mapping from the dual-$k$-dimensional space $(F^k)^* \to F^n$ that calculates every linear form over the points of $X$. Hence the length $n$ of codeword of $C_X$ is the cardinality of $X$, the dimension of $C_X$ being just $k$ (cf. [6], p. 3).

Denote $H$ the set of all hyperplanes of $P^{k-1} = PrF^k$.

There exists a natural 1-1 correspondence between the equivalence classes of a non-degenerate $[n, k]_q$-projective system $X$ and a non-degenerate $[n, k]_q$-code $C_X$ such that if $X$ is an $[n, k]_q$-projective system and $C_X$ is a corresponding code, then the non-zero codewords of $C_X$ correspond to hyperplanes $H \in H$, up to a non-zero factor. The correspondence preserves the ground parameters.

The weight of a codeword $c$ corresponding to the hyperplane $H_c$ is the number of points of $X \setminus H_c$; thus the minimum weight (or, the minimum distance) $d$ of the code $C_X$ is $d = |X| - \max\{|X \cap H| | H \in H\}$. Therefore in order to find the
minimum distance of the code $C_X$ it needs to calculate the maximum intersection of $X$ with the hyperplanes of $H$.

A linear code with length $n$, dimension $k$ and minimum distance $d$ over the field $F = GF(q)$ can be denoted also as an $[n, k, d]_q$-code.

If $C$ is an $[n, k, d]_q$-code, then $C$ is an $s$-error-correcting code for all $s \leq \lfloor \frac{d-1}{2} \rfloor$. We call $t = \lfloor \frac{d-1}{2} \rfloor$ the error-correcting capability of $C$ (cf.[6], p.3).

3. Main Results

With the notations of the previous section, choose and fix the line $l_0 \in S$, the plane $\pi_0$ such that $\pi_0 \cap \Sigma^* = l_0 \in S$ and the non-degenerate conic $C^2 \subset \pi_0 \setminus l_0$.

Denote $\Sigma^*$ a hyperplane of $\Sigma = PG(4, q)$ containing the plane $\pi_0$. Let $\pi = \Sigma^* \cap \Sigma^*$. The plane $\pi$ contains the line $l_0$ and each of the $q^2$ points of $\pi \setminus l_0$ belongs to one of the $q^2$ lines of $S \setminus \{l_0\}$. Let $O$ be a point, $O \in \Sigma^* \setminus \{\pi_0 \cup \pi\}$. Denote $Q$ the quadric cone having vertex the point $O$ and directrix the conic $C^2$. Let $C^2 = Q \cap \pi$. Obviously $C^2$ is a non-degenerate conic with $C^2 \setminus l_0 = \emptyset$.

Let $\{R_i | i = 1, \ldots, q + 1\}$ be the set of the $q + 1$ points of $C^2$, $\{r_i | i = 1, \ldots, q + 1\}$ the $q + 1$ lines of the cone $Q$ with $R_i \in r_i$, $\{R'_i = r_i \cap C^2 | i = 1, \ldots, q + 1\}$ the corresponding set of $q + 1$ points of $C^2$ with $R'_i \in r_i$, $\{s_i | i = 1, \ldots, q + 1\}$ the $q + 1$ lines of $S$ with $\{R_i \in s_i | i = 1, \ldots, q + 1\}$.

For each line $s_i$ let $\lambda_i$ be a projectivity between $s_i$ and $C^2$ such that $\lambda_i(R'_i) = R_i$

Denote $S_i$ the point at infinity of the plane $\Pi$ represented by the line $s_i \in S$, $p_0$ the line of $\Pi$ represented by the plane $\pi_0$ and $c_2$ the subline of $p_0$ corresponding to $C^2$.

Let $V_i$ be the variety $V_2^i$ having the conic $C^2$ and the line $s_i$ as directrices constructed via $\lambda_i$. Note that, by construction, the line $r_i$ is one of the $q + 1$ generatrix lines of $V_i$.

From Lemma 2.8 follows that each of the $q + 1$ variety $V_i$ is a non-affine Baer subplane of $\Pi$ meeting the line $l$ in the point $S_i$, containing $c_2 \subset p_0$ and the point $O$.

Define $V := \bigcup V_i$ the union of the points of all varieties $V_i$ for all $i = 1, \ldots, q + 1$.

**Lemma 3.1.** $V$ represents the bundle of the full set of $q + 1$ non-affine Baer subplanes having in common the subline $c_2$ and the point $O$.

**Proof.** See (iii) of Lemma 2.7 and [3].

**Proposition 3.2.** $\Sigma^* \cap V = Q$. 
Proof. By construction the hyperplane $\Sigma^*$ contains $Q$. As for any variety $V_i$, $\Sigma^* \cap V_i$ cannot contain the directrix line $s_i$ (otherwise $\Sigma^* = \Sigma$), then $\Sigma^*$ meets $V_i$ at most in a cubic curve $C^2 \cup r_i$ (cf. [5], (ii), p. 93).

Assume $\Sigma^* \cap V$ contains $C^2 \cup r_i \subset V_i$ and a further point $P_j \in V_j$ with $j \neq i$. Hence $\Sigma^*$ contains the line $r = P_j R_j \in V_j$ with $R_j \in C^2$. If $r \neq r_i$, then $\Sigma^*$ should meet $V_j$ in $C^2 \cup r_j \cup r$ where $r_j$ and $r$ are two generatrix lines of $V_j$, then the line $s_j$ should belong to $\Sigma^*$, a contradiction (cf. [5], (ii), p. 93). Hence $\Sigma^* \cap V = Q$.

Denote $V_{aff} = V \setminus \Sigma^*$.

**Proposition 3.3.**

(i) A hyperplane of $\Sigma$ having maximum intersection with $V$ is $\Sigma^*$, and $\Sigma^* \cap V$ consists of the points of the lines $\{s_i\}_{i=1}^{q+1} \subset S$.

(ii) A hyperplane of $\Sigma$ having maximum intersection with $V_{aff}$ is $\Sigma^*$ and $\Sigma^* \cap V_{aff}$ consists of the points of $Q \setminus C^2$.

Proof. (i) Let $H \in H$ a hyperplane. If $H = \Sigma^*$ then $H \cap V$ is the set of the $(q+1)^2$ points of $\{s_i\}_{i=1}^{q+1} \subset S$. If $H = \Sigma^*$ then $H \cap V$ is the set of the $q^2 + q + 1$ points of $Q$.

Let $H \neq \Sigma^*, \Sigma^*$.

Denote $H \cap \Sigma^* = P_i, H \cap \Sigma^* = P_f$.

For $H$ there are two possibilities: 1) $H$ contains $P_0$, 2) $H$ does not contain $P_0$.

1) It is $P^* = P_0$ so that it contains $C^2$. Moreover $P^* \neq P$, otherwise $H = \Sigma^*$. The plane $P^*$ forms bundle with axis the line $l_0$ with $P_0$ and $P$. Each point of $P^*$ belongs to one line of $S \setminus l_0$ then it meets the $q + 1$ points $\{P_i = P^* \cap s_i\}_{i=1}^{q+1}$.

Therefore $H \cap V$ contains at least the $q + 1$ points $P_i$ and the points of $C^2$. Then $|H \cap V| \geq 2(q + 1)$. The maximum intersection is reached if each line $P_i R_i$ coincides with one generatrix line of the variety $V_i$ for every $i$. In such a case $|H \cap V| = (q + 1)^2$.

2) Let $P^* \cap P^* = l$. Then $l$ is a line of $P^*$. Let $l = l_0$. The plane $P^*$ contains no generatrix line of the varieties $V_i$; otherwise $l_0$ would meet some line $s_i$, it meets $V$ in at most a conic $C_{O}$ of $Q$. Set $\{P_i \subset C_{O}\}_{i=1}^{q+1}$.

If $P^* = P$, then $P^* \cap V = C^2$. If $P^* \neq P$, then it contains no line $s_j$ (otherwise $l_0 \cap s_i \neq \emptyset$), it can meet at most $q + 1$ lines $s_j$ in points $T_j$. In both cases the maximum intersection is reached if the $q + 1$ lines $P_i R_i$ or $P_i T_i$ respectively, coincide with the generatrix lines of the varieties $V_i$. Hence $|H \cap V| \leq (q + 1)^2$.

Let $l \neq l_0$. Denote $P^* = P^* \cup P_0$. Then $l = s_i$ for some $i$ or $l$ meets at most $q + 1$ lines $s_k$.

If $P^* = P$, it contains the $q + 1$ points of $C^2$ and according to $l$ is secant, tangent or external to the conic $C^2$, $|H \cap V|$ is less or equal to $(q + 1) + 2q = 3q + 1, (q + 1) + q = 2q + 1$ or $q + 1$, respectively.

Assume $P^* \neq P$. The plane $P^*$ must contain one line $l$ of $S$ and the $q^2$ points of the remaining lines of $S$. Then the plane $P^*$...
contains the $q + 1$ points of $t = s_i$ for some $i$, or the $q + 1$ points of the set $\{ s_i \cap \pi' \mid i = 1, \ldots, q + 1 \} \subset V$.

According to $\pi'$ is secant, tangent or external to the conic $C^2$, $H$ meets $V$ in 2 generatrix lines, in 1 generatrix line or in no generatrix line. Therefore $|H \cap V|$ is less or equal to $(q + 1) + 2q = 3q + 1$, $(q + 1) + q = 2q + 1$ or $q + 1$.

Hence the maximum intersection a hyperplane can have with $V$ consists of $(q + 1)^2$ points. $\Sigma'$ is one of such hyperplanes.

(ii) Let $H$ be a hyperplane, $H \neq \Sigma'$. From [7], Lemma 11, it is known the maximum intersection a hyperplane of $\Sigma$ has with a variety $V_2^2$ consists of two generatrix lines and the directrix line. Of course $H$ cannot meet two different varieties in such a way otherwise $H$, containing two lines of $S$ would coincides with $\Sigma'$. Therefore $H$ can meet at least $q$ varieties along the conic $C^2$ and one generatrix line for each variety, then $q$ points of the conic $C^2$. In any case $H$ contains $O$ then the cone $Q$. Therefore $H = \Sigma'$. Hence the maximum intersection a hyperplane can have with $V_{aff}$ is $Q \setminus C^2$ with $|Q \setminus C^2| = q^2$.

Denote $X := V$ the projective system defined by $V$, $C_X$ the linear code arising from $X$, $X_{aff} := V_{aff}$ the projective system defined by $V_{aff}$ $C_{X_{aff}}$ the linear code arising from $X_{aff}$.

**Theorem 3.4.**

(i) $C_X$ is an $[n, k, d]_q$-code with $n = q^3 + 2q^2 + q + 1$, $k = 5$, $d = q^3 + q^2 - q$.

(ii) $C_{X_{aff}}$ is an $[n', k, d]'_q$-code with $n' = q^3 + q^2 - q$, $k = 5$, $d = q^3 - q$.

Proof. (i) Each variety $V_i$ consists of $q + 1$ skew lines, hence it has $(q + 1)^2$ points. Every two varieties $V_i$ and $V_j$ have in common the conic $C^2$ and the point $O$ so that for each variety remain $q^2 + 2q + 1 - (q + 1) - 1 = q^2 + q - 1$ points. The varieties are $q + 1$ so that the cardinality of $X$ is $(q^2 + q - 1)(q + 1) = q^3 + 2q^2 - 1$ plus the point $O$ and the $(q + 1)$ points of the conic $C^2$. Hence $|X| = q^3 + 2q^2 + q + 1$. The length of the code $C_X$ is therefore $n = q^3 + 2q^2 + q + 1$.

The dimension of $C_X$ is obviously 5, that is, the vector dimension of $\Sigma$.

From Proposition 3.3, (i), follows the distance of $C_X$ is $d = n - |\{P \in s_i \mid i = 1, \ldots, q + 1\}|$ that is,

$$d = q^3 + 2q^2 + q + 1 - (q^2 + 2q + 1) = q^3 + q^2 - q.$$ 

(ii) The length of the code $C_{X_{aff}}$ equals

$$n' = |X| - |\{P \in s_i \mid i = 1, \ldots, q + 1\}| = q^3 + 2q^2 + q + 1 - (q^2 + 2q + 1) = q^3 + q^2 - q.$$ Its dimension is $k = 5$. From Proposition 3.3, (ii), follows the distance is $d' = n' - |Q \setminus C^2|$ that is, $d' = q^3 + q^2 - q - q^2 = q^3 - q$.

**Examples**

For $q = 2$, $C_X$ is a $[19, 5, 10]_2$-code and it can correct at most $\left\lfloor \frac{10 - 1}{2} \right\rfloor = 4$ errors. For $q = 3$, $C_X$ is a $[49, 5, 33]_3$-code and it can correct at most $\left\lfloor \frac{33 - 1}{2} \right\rfloor = 16$ errors.

For $q = 2$, $C_{X_{aff}}$ is a $[10, 5, 6]_2$-code and it can correct at most $\left\lfloor \frac{6 - 1}{2} \right\rfloor = 2$ errors. For $q = 3$, $C_{X_{aff}}$ is a $[33, 5, 24]_3$-code and it can correct at most $\left\lfloor \frac{24 - 1}{2} \right\rfloor = 11$ errors.
Other References

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