Qeios

PEER-APPROVED

v1: 4 July 2023

Research Article

# On Bundles of Varieties V_2^3 in PG(4, q) and Their Codes

Peer-approved: 4 July 2023



Qeios, Vol. 5 (2023)
ISSN: 2632-3834

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**In this note we use the spatial representation in $\Sigma = PG(4,q)$ of the projective plane $\Pi = PG(2,q^2)$, by fixing a hyperplane $\Sigma'$ with a regular spread $\mathcal{S}$ of lines. We consider a bundle $\mathcal{X}$ of varieties $V_2^3$ of $\Sigma$ having in common the $q+1$ points of a conic $\mathcal{C}^2$ of a plane $\pi_0$, $\pi_0 \cap \Sigma' = l_0 \in \mathcal{S}$, thus representing an affine line of $\Pi$, and a further affine point $O \notin \pi_0$. This subset $\mathcal{X}$ of $\Sigma$ represents a bundle of non-affine Baer subplanes of $\Pi$, each of them having one point at infinity (corresponding to a line of $\mathcal{S}$), having in common a subline of affine points of $\Pi$ and a further affine point. Then $\mathcal{X}$ is considered as a projective system of $\Sigma$ and, by using such a representation of $\Pi$, we can calculate the ground parameters of the code $C_{\mathcal{X}}$ arising from it.**

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**Mathematics Subject Classification:** 51E20, 51A05, 94B05, 94B27

## 1. Introduction

It is known that a projective translation plane $\Pi$ of order $n = q^2$ of dimension 2 over its kernel $F = GF(q)$ can be represented by a 4-dimensional projective space $\Sigma = PG(4,q)$ over $F$, fixing a hyperplane $\Sigma' = PG(3,q)$ and a spread $\mathcal{S}$ of lines of $\Sigma'$. The points of $\Pi$ are represented by (i) the points of $\Sigma \setminus \Sigma'$ and (ii) the lines of $\mathcal{S}$. The lines of $\Pi$ are represented by (i) the planes $\alpha$ of $\Sigma \setminus \Sigma'$ such that $\alpha \cap \Sigma'$ belongs to $\mathcal{S}$ and by (ii) the spread $\mathcal{S}$. The translation line $l$ of $\Pi$ is represented by $\mathcal{S}$ (cf. [1]).

A Baer subplane $B$ of $\Pi$ has order $q$ and it is *dense* in the sense that a line of $\Pi$ either is a line of $B$ (that is, meets $B$ in a *subline* of $q+1$ points, such a subplane is *affine*) or it meets $B$ in one point (such a subplane is *non-affine*).

The *affine* Baer subplanes $B$ of $\Pi$ are represented by the *transversal* planes $\beta$ to $\mathcal{S}$, that is, the planes of $\Sigma \setminus \Sigma'$ such that the line $\beta \cap \Sigma' \notin \mathcal{S}$ meets $q+1$ lines of $\mathcal{S}$. In such a way $l$ is a line of $B$ (cf. [2], pp. 68--72). Of course all that holds also in case $\Pi$ is the Desarguesian plane $PG(2,q^2)$ when $\mathcal{S}$ is a regular spread (cf. [3], [2]).

A variety $V_2^3$ of $\Sigma$ with a line $l_\infty$ in $\mathcal{S}$ as the minimum (linear) order directrix, a conic $\mathcal{C}^2$ as a $2nd$ order directrix with $\mathcal{C}^2 \subset \pi_0$, $\pi_0 \cap \Sigma' = l_0 \in \mathcal{S} \setminus l_\infty$ and $\mathcal{C}^2 \cap l_0 = \emptyset$, represents a non-affine Baer subplane of $\Pi$ having one point on the translation line $l$ and the *subline* $\mathcal{C}^2$ of the *line* $\pi_0$ (cf. [3]).

In this paper we consider bundles of $q+1$ varieties $V_2^3$ of $\Sigma = PG(4,q)$ with the linear directrix in $\mathcal{S}$ and having in common a same conic $\mathcal{C}^2$ as a $2nd$ order directrix and one further affine point. By using the spatial representation of $\Pi = PG(2,q^2)$ in $PG(4,q)$, we can characterize such a bundle $\mathcal{X}$ from the intersection point of view, construct a linear code $\mathcal{C}_{\mathcal{X}}$ arising from it and show that its ground parameters allow $\mathcal{C}_{\mathcal{X}}$ to correct an enough large number of errors.

## 2. Preliminary Notes

Let $F = GF(q)$ be a finite field, $q = p^s$, $p$ prime. Denote $F^{r+1}$ the $(r+1)$-dimensional vector space over $F$, $P^r = PrF^{r+1} = PG(r,q)$ the $r$-dimensional projective space contraction of $F^{r+1}$ over $F$. Let $\overline{F}$ be the algebraic closure of the field $F = GF(q)$.

Denote $S_t$ with $t \geq 2$ a subspace of $P^r$ of dimension $t$. A hyperplane $S_{r-1}$ will be denoted also by $H$, a plane by

$\pi$.

The geometry $P^r$ is considered a sub-geometry of $\overline{P^r}$, the projective geometry over $\overline{F}$. We refer to the points of $P^r$ as the *rational points* of $\overline{P^r}$.

**Definition 2.1.** *A variety $V_u^v$ of dimension $u$ and of order $v$ of $P^r$ is the set of the rational points of a projective variety $\overline{V}_u^v$ of $\overline{P}^r$ defined by a finite set of polynomials with coefficients in the field F.*

From [4], p.290, 7.- for $r \geq 4$ follows

**Lemma 2.2.** *The ruled variety $V_2^{r-1}$ of $PG(r,q)$ is generated by the lines connecting the corresponding points of two birationally (or, projectively) equivalent curves in two complementary subspaces, of order $m$ and $r-1-m$, respectively. It has order the sum of the orders of the curves as there are no fixed points.*

Let $P^4$ be the projective geometry $PG(4,q)$.

**Lemma 2.3.** A variety $V_2^3$ of $PG(4,q)$ is obtained by joining the corresponding points of a directrix line $l$ and a directrix conic $\mathcal{C}$ in a plane $\pi$, $l$ and $\mathcal{C}$ being projectively equivalent and with $l \cap \pi = \emptyset$.

Proof. See [5] p. 90.

Choose a coordinate system in $P^4$ so that it is a coordinate system for $\overline{P}^4$ too, denote a point $P \approx (x_1, x_2, y_1, y_2, t) := \overline{F}^*(x_1, x_2, y_1, y_2, t)$, $\overline{F}^* = \overline{F} \setminus \{0\}$.

$P$ is a *rational point* if there exists $(x_1, x_2, y_1, y_2, t) \in F^5$ such that $P \approx (x_1, x_2, y_1, y_2, t)$. A variety $V$ of $P^4$ is the set of the rational points of $\overline{P^4}$ solutions of a finite set of polynomials of $F[x_1, x_2, y_1, y_2, t]$.

**Lemma 2.4.** *The variety $V_2^3$ can be represented as the definite intersection of two quadrics of $PG(4,q)$, that is, the cone of planes $\mathcal{Q}_1 : sx_2^2 - x_1^2 - sx_2t = 0$ (where $s$ is a non square of $GF(q)$) and the cone of planes $\mathcal{Q}_2 : x_1y_1 - x_2y_2 = 0$. The plane $\pi' : x_1 = 0, x_2 = 0$ is contained in both quadrics so that, by Bezout, the order of the intersection variety is $4 - 1 = 3$.*

Proof. See [3] Theorem 1.1, [5] p. 92.

Let $\Pi = PG(2,q^2)$ be the Desarguesian plane over $GF(q^2)$. Denote $l$ the line at infinity of $\Pi$. In the spatial representation of $\Pi$ in $P^4 = PG(4,q)$ fix a hyperplane $\Sigma' = PG(3,q)$ and a regular spread $\mathcal{S}$ of lines of $\Sigma'$, where $|\mathcal{S}| = q^2 + 1$.

**Lemma 2.5.** *The points of $\Pi$ are represented by (i) the points of $\Sigma \setminus \Sigma'$ (the affine points of $\Pi$) and by (ii) the lines of $\mathcal{S}$ (the points at infinity of $\Pi$). The lines of $\Pi$ are represented by (i) the planes $\alpha$ of $\Sigma \setminus \Sigma'$ such that $\alpha \cap \Sigma'$ belongs to $\mathcal{S}$ and by (ii) the spread $\mathcal{S}$, representing the line at infinity l.*

Proof. See [1] the Bruck and Bose representation and [2], p. 775.

**Definition 2.6.** *A Baer subplane of $\Pi = PG(2,q^2)$ is an affine subplane if it meets the line at infinity $l$ of $\Pi$ in a subline $l_1$, it is a non-affine subplane if it meets the line $l$ in one point.*

**Lemma 2.7.**
$(i)$ *Two affine Baer subplanes of $\Pi$ having in common the subline $l_1$ can meet in at most one further point.*
$(ii)$ *The Baer subplanes having in common only a subline $l_1$ are $q^2$.*
$(iii)$ *The Baer subplanes having in common a subline $l_1$ and one further point are $q+1$.*

Proof. $(i)$ Two Baer subplanes having in common a subline $l_1$ and two further points coincide, because they have in common at least four *reference* (three by three non collinear) points.

Without loosing generality, we can consider two affine Baer subplanes $\mathcal{B}$ and $\mathcal{B}'$ of $\Pi$ having in common a subline $l_1$ of $l$. In the spatial representation of $\Pi$, they are represented by two planes $B$ and $B'$ of $P^4$, respectively, such that the lines $B \cap \Sigma' = r$ and $B' \cap \Sigma' = r'$ are transversal lines of the same regulus $\mathcal{R} \subset \mathcal{S}$. Denote $\mathcal{R}'$ the opposite regulus to $\mathcal{R}$.

There are two cases:

$(ii)$ If $r = r'$, the planes $B$ and $B'$ have in common the line $r$ meeting the regulus $\mathcal{R}$ in its $q+1$ lines so that the subplanes $\mathcal{B}$ and $\mathcal{B}'$ have in common the *subline* $l_1$ (represented by $\mathcal{R}$) of the *line* $l$ (represented by $\mathcal{S}$) and no further (affine) points. Such planes are $\frac{q^4}{q^2} = q^2$ and represent $q^2$ affine Baer subplanes of $\Pi$ having in common only the subset $l_1$ of $q+1$ points of the line at infinity $l$.

$(iii)$ If $r \neq r'$, the planes $B$ and $B'$ have in common an affine point $O \in \Sigma \setminus \Sigma'$ so that the two subplanes $\mathcal{B}$ and $\mathcal{B}'$ meet along the subline $l_1$ represented by $\mathcal{R}$ and in the affine point $O$. The regulus $\mathcal{R}$ has $q+1$ transversal lines $\{t_i | i = 1, \ldots, q+1\}$ belonging to $\mathcal{R}'$. Each space $O \oplus t_i$ is a transversal plane $\tau_i$, so that $\{\tau_i | i = 1, \ldots, q+1\}$ represent the $q+1$ affine Baer subplanes of $\Pi$ having in common $l_1$ and the affine point $O$.

Choose and fix a line $l_\infty$ of the (regular) spread $\mathcal{S}$, a plane $\pi_0$ such that $\pi_0 \cap \Sigma' = l_0 \in \mathcal{S} \setminus l_\infty$ and a non-degenerate conic $\mathcal{C}^2 \subset \pi_0 \setminus l_0$. Let $\Lambda$ be a projectivity between $l_\infty$ and $\mathcal{C}^2$. Denote $V_2^3$ the variety arising by

 doi.org/10.32388/XLWZT8 

connecting corresponding points of $l_\infty$ and $\mathcal{C}^2$ via $\Lambda$ (cf. [5], p. 90).

**Lemma 2.8.** *The variety $V_2^3$ represents a non-affine Baer subplane of $\Pi$ meeting the line at infinity $l$ in the point $l_\infty$ and containing the subline $\mathcal{C}^2$ of the line represented by $\pi_0$.*

Proof. See [3] and [2].

Let $F^n$ be the $n$-dimensional vector space over $F = GF(q)$.

**Definition 2.9.** *A linear $[n,k]_q$-code $C$ of length $n$ is a $k$-dimensional subspace of the vector space $F^n$.*

**Definition 2.10.** *An $[n,k]_q$-projective system $\mathcal{X}$ is a set of $n$ non necessarily distinct points of the projective geometry $PrF^k = PG(k-1,q)$. It is non-degenerate if these points are not contained in a hyperplane (cf. [6], p. 2).*

Assume that $\mathcal{X}$ consists of $n$ distinct points having maximum rank.

Codes and projective systems are linked by a strict connection one can read in [6], so that from subsets $\mathcal{X}$ of a projective geometry linear codes $C_\mathcal{X}$ can be generated. More precisely, for each point of $\mathcal{X}$ choose a generating vector. Denote $\mathcal{M}$ the matrix having as rows such $n$ vectors and let $C_\mathcal{X}$ be the linear code having $\mathcal{M}^t$ as a generator matrix. The code $C_\mathcal{X}$ *is the $k$-dimensional subspace of* $F^n$ which is the *image of the mapping from the dual $k$-dimensional space $(F^k)^*$ onto $F^n$ that calculates every linear form over the points of $\mathcal{X}$.* Hence the length $n$ of codeword of $C_\mathcal{X}$ is the cardinality of $\mathcal{X}$, the dimension of $C_\mathcal{X}$ being just $k$ (cf. [6], p. 3).

Denote $\mathcal{H}$ the set of all hyperplanes of $P^{k-1} = PrF^k$.

There exists a natural 1-1 correspondence between the equivalence classes of a non-degenerate $[n,k]_q$-*projective system* $\mathcal{X}$ and a non-degenerate $[n,k]_q$-*code* $C_\mathcal{X}$ such that if $\mathcal{X}$ is an $[n,k]_q$-projective system and $C_\mathcal{X}$ is a corresponding code, then the non-zero codewords of $C_\mathcal{X}$ correspond to hyperplanes $H \in \mathcal{H}$, up to a non-zero factor. The correspondence preserves the ground parameters.

The weight of a codeword $c$ corresponding to the hyperplane $H_c$ is the number of points of $\mathcal{X} \setminus H_c$, thus the minimum weight (or, the minimum distance) $d$ of the code $C_\mathcal{X}$ is $d = |\mathcal{X}| - max\{|\mathcal{X} \cap H| \mid H \in \mathcal{H}\}$. Therefore in order to find the minimum distance of the code $C_\mathcal{X}$ it needs to calculate the maximum intersection of $\mathcal{X}$ with the hyperplanes of $\mathcal{H}$.

A linear code with length $n$, dimension $k$ and minimum distance $d$ over the field $F = GF(q)$ can be denoted also as an $[n,k,d]_q$-code.

If $C$ is an $[n,k,d]_q$-code, then $C$ is an $s$-error-correcting code for all $s \leq \lfloor \frac{d-1}{2} \rfloor$. We call $t = \lfloor \frac{d-1}{2} \rfloor$ the *error-correcting capability of $C$* (cf.[6], p.3).

# 3. Main Results

With the notations of the previous section, choose and fix the line $l_0 \in \mathcal{S}$, the plane $\pi_0$ such that $\pi_0 \cap \Sigma' = l_0 \in \mathcal{S}$ and the non-degenerate conic $\mathcal{C}^2 \subset \pi_0 \setminus l_0$.

Denote $\Sigma''$ a hyperplane of $\Sigma = PG(4,q)$ containing the plane $\pi_0$. Let $\pi = \Sigma'' \cap \Sigma'$. The plane $\pi$ contains the line $l_0$ and each of the $q^2$ points of $\pi \setminus l_0$ belongs to one of the $q^2$ lines of $\mathcal{S} \setminus \{l_0\}$. Let $O$ be a point, $O \in \Sigma'' \setminus \{\pi_0 \cup \pi\}$. Denote $\mathcal{Q}$ the quadric cone having vertex the point $O$ and directrix the conic $\mathcal{C}^2$. Let $\mathcal{C}'^2 = \mathcal{Q} \cap \pi$. Obviously $\mathcal{C}'^2$ is a non-degenerate conic with $\mathcal{C}'^2 \cap l_0 = \emptyset$.

Let $\{R_i | i = 1, \dots, q+1\}$ be the set of the $q+1$ points of $\mathcal{C}^2$, $\{r_i | i = 1, \dots, q+1\}$ the $q+1$ lines of the cone $\mathcal{Q}$ with $R_i \in r_i$, $\{R'_i = r_i \cap \mathcal{C}'^2 | i = 1, \dots, q+1\}$ the *corresponding* set of $q+1$ points of $\mathcal{C}'^2$ with $R'_i \in r_i$, $\{s_i | i = 1, \dots, q+1\}$ the $q+1$ lines of $\mathcal{S}$ with $\{R'_i \in s_i | i = 1, \dots, q+1\}$.

For each line $s_i$ let $\lambda_i$ be a projectivity between $s_i$ and $\mathcal{C}^2$ such that $\lambda_i(R'_i) = R_i$

Denote $S_i$ the point at infinity of the plane $\Pi$ represented by the line $s_i \in \mathcal{S}$, $p_0$ the line of $\Pi$ represented by the plane $\pi_0$ and $c_2$ the subline of $p_0$ corresponding to $\mathcal{C}^2$.

Let $\mathcal{V}_i$ be the variety $V_2^3$ having the conic $\mathcal{C}^2$ and the line $s_i$ as directrices constructed via $\lambda_i$. Note that, by construction, the line $r_i$ is one of the $q+1$ generatrix lines of $\mathcal{V}_i$.

From Lemma 2.8 follows that each of the $q+1$ variety $\mathcal{V}_i$ is a non-affine Baer subplane of $\Pi$ meeting the line $l$ in the point $S_i$, containing $c_2 \subset p_0$ and the point $O$.

Define $\mathcal{V} := \bigcup_i \mathcal{V}_i$ the union of the points of all varieties $\mathcal{V}_i$ for all $i = 1, \dots, q+1$ .

**Lemma 3.1.** *$\mathcal{V}$ represents the bundle of the full set of $q+1$ non-affine Baer subplanes having in common the subline $c_2$ and the point $O$.*

Proof. See $(iii)$ of Lemma 2.7 and [3].

**Proposition 3.2.** $\Sigma'' \cap \mathcal{V} = \mathcal{Q}$.

Proof. By construction the hyperplane $\Sigma''$ contains $\mathcal{Q}$. As for any variety $\mathcal{V}_i$, $\Sigma'' \cap \mathcal{V}_i$ cannot contain the directrix line $s_i$ (otherwise $\Sigma'' = \Sigma'$), then $\Sigma''$ meets $\mathcal{V}_i$ at most in a cubic curve $\mathcal{C}^2 \cup r_i$ (cf. [5], $(ii)$, p. 93).

Assume $\Sigma'' \cap \mathcal{V}$ contains $\mathcal{C}^2 \cup r_i \subset \mathcal{V}_i$ and a further point $P_j \in V_j$ with $j \neq i$. Hence $\Sigma''$ contains the line $r = P_jR_j \in \mathcal{V}_j$ with $R_j \in \mathcal{C}^2$. If $r \neq r_j$, then $\Sigma''$ should meet $\mathcal{V}_j$ in $\mathcal{C}^2 \cup r_j \cup r$ where $r_j$ and $r$ are two generatrix lines of $\mathcal{V}_j$, then the line $s_j$ should belong to $\Sigma''$, a contradiction (cf. [5], $(ii)$, p. 93). Hence $\Sigma'' \cap \mathcal{V} = \mathcal{Q}$.

Denote $\mathcal{V}_{aff} = \mathcal{V} \setminus \Sigma'$.

**Proposition 3.3.**
$(i)$ A hyperplane of $\Sigma$ having maximum intersection with $\mathcal{V}$ is $\Sigma'$, and $\Sigma' \cap \mathcal{V}$ consists of the points of the lines $\{s_i | i = 1, \ldots, q+1\} \subset \mathcal{S}$.
$(ii)$ A hyperplane of $\Sigma$ having maximum intersection with $\mathcal{V}_{aff}$ is $\Sigma''$ and $\Sigma'' \cap \mathcal{V}_{aff}$ consists of the points of $\mathcal{Q} \setminus \mathcal{C}'^2$.

Proof. $(i)$ Let $H \in \mathcal{H}$ a hyperplane. If $H = \Sigma'$ then $H \cap \mathcal{V}$ is the set of the $(q+1)^2$ points of $\{s_i | i = 1, \ldots, q+1\} \subset \mathcal{S}$. If $H = \Sigma''$ then $H \cap \mathcal{V}$ is the set of the $q^2 + q + 1$ points of $\mathcal{Q}$.

Let $H \neq \Sigma', \Sigma''$.

Denote $H \cap \Sigma' = \pi'$, $H \cap \Sigma'' = \pi''$.

For $H$ there are two possibilities: 1) $H$ contains $\pi_0$, 2) $H$ does not contain $\pi_0$.

1) It is $\pi'' = \pi_0$ so that it contains $\mathcal{C}^2$. Moreover $\pi' \neq \pi$ otherwise $H = \Sigma''$. The plane $\pi'$ forms bundle with axis the line $l_0$ with $\pi_0$ and $\pi$. Each point of $\pi'$ belongs to one line of $\mathcal{S} \setminus l_0$ then it meets the $q+1$ points $\{P_i = \pi' \cap s_i | i = 1, \ldots, q+1\}$. Therefore $H \cap \mathcal{V}$ contains at least the $q+1$ points $P_i$ and the points of $\mathcal{C}^2$. Then $|H \cap \mathcal{V}| \geq 2(q+1)$. The maximum intersection is reached if each line $P_iR_i$ coincides with one generatrix line of the variety $\mathcal{V}_i$ for every $i$, In such a case $|H \cap \mathcal{V}| = (q+1)^2$.

2) Let $\pi'' \cap \Sigma' = l$. Then $l$ is a line of $\pi'$ too.

Let $l = l_0$. The plane $\pi''$ contains no generatrix line of the varieties $\mathcal{V}_i$ otherwise $l_0$ would meet some line $s_i$, it meets $\mathcal{V}$ in at most a conic $\mathcal{C}_Q$ of $\mathcal{Q}$. Set $\{P_i \in \mathcal{C}_Q | i = 1, \ldots, q+1\}$.

If $\pi' = \pi$, then $\pi' \cap \mathcal{V} = \mathcal{C}'^2$. If $\pi' \neq \pi$, then it contains no line $s_i$ (otherwise $l_0 \cap s_i \neq \emptyset$), it can meet at most $q+1$ lines $s_i$ in points $T_i$. In both cases the maximum intersection is reached if the $q+1$ lines $P_iR'_i$, or $P_iT_i$, respectively, coincide with the generatrix lines of the varieties $\mathcal{V}_i$. Hence $|H \cap \mathcal{V}| \leq (q+1)^2$.

Let $l \neq l_0$. Denote $r' = \pi'' \cap \pi_0$. Then $l = s_i$ for some $i$ or $l$ meets at most $q+1$ lines $s_i$.

If $\pi' = \pi$, it contains the $q+1$ points of $\mathcal{C}'^2$ and according to $r'$ is secant, tangent or external to the conic $\mathcal{C}^2$, $|H \cap \mathcal{V}|$ is less or equal to $(q+1) + 2q = 3q+1$, $(q+1) + q = 2q+1$ or $q+1$, respectively.

Assume $\pi' \neq \pi$. The plane $\pi'$ must contain one line $t$ of $\mathcal{S}$ and the $q^2$ points of the remaining lines of $\mathcal{S}$. Then the plane $\pi'$ contains the $q+1$ points of $t = s_i$ for some $i$, or the $q+1$ points of the set $\{s_i \cap \pi' | i = 1, \ldots q+1\} \subset \mathcal{V}$.
According to $r'$ is secant, tangent or external to the conic $\mathcal{C}^2$, $H$ meets $\mathcal{V}$ in 2 generatrix lines, in 1 generatrix line or in no generatrix line. Therefore $|H \cap \mathcal{V}|$ is less or equal to $(q+1) + 2q = 3q+1$, $(q+1) + q = 2q+1$ or $q+1$.

Hence the maximum intersection a hyperplane can have with $\mathcal{V}$ consists of $(q+1)^2$ points. $\Sigma'$ is one of such hyperplanes.

$(ii)$ Let $H$ be a hyperplane, $H \neq \Sigma'$. From [7], Lemma 11, it is known the maximum intersection a hyperplane of $\Sigma$ has with a variety $V_2^3$ consists of two generatrix lines and the directrix line. Of course $H$ cannot meet two different varieties in such a way otherwise $H$, containing two lines of $\mathcal{S}$ would coincides with $\Sigma'$. Therefore $H$ can meet at least $q$ varieties along the conic $\mathcal{C}^2$ and one generatrix line for each variety, then $q$ points of the conic $\mathcal{C}'^2$. In any case $H$ contains $O$ then the cone $\mathcal{Q}$. Therefore $H = \Sigma''$. Hence the maximum intersection a hyperplane can have with $\mathcal{V}_{aff}$ is $\mathcal{Q} \setminus \mathcal{C}'^2$ with $|\mathcal{Q} \setminus \mathcal{C}'^2| = q^2$.

Denote $\mathcal{X} := \mathcal{V}$ the projective system defined by $\mathcal{V}$, $C_{\mathcal{X}}$ the linear code arising from $\mathcal{X}$, $\mathcal{X}_{aff} := \mathcal{V}_{aff}$ the projective system defined by $\mathcal{V}_{aff}$, $C_{\mathcal{X}_{aff}}$ the linear code arising from $\mathcal{X}_{aff}$.

**Theorem 3.4.**
*$(i)$ $C_{\mathcal{X}}$ is an $[n,k,d]_q$-code with $n = q^3 + 2q^2 + q + 1$, $k = 5$, $d = q^3 + q^2 - q$.*
*$(ii)$ $C_{\mathcal{X}_{aff}}$ is an $[n',k,d']_q$-code with $n' = q^3 + q^2 - q$, $k = 5$, $d' = q^3 - q$.*

Proof. $(i)$ Each variety $\mathcal{V}_i$ consists of $q+1$ skew lines, hence it has $(q+1)^2$ points. Every two varieties $\mathcal{V}_i$ and $\mathcal{V}_j$ have in common the conic $\mathcal{C}^2$ and the point $O$ so that for each variety remain $q^2 + 2q + 1 - (q+1) - 1 = q^2 + q - 1$ points. The varieties are $q+1$ so that the cardinality of $\mathcal{X}$ is $(q^2 + q - 1)(q+1) = q^3 + 2q^2 - 1$ plus the point $O$ and the $(q+1)$ points of the conic $\mathcal{C}^2$. Hence $|\mathcal{X}| = q^3 + 2q^2 + q + 1$. The length of the code $C_{\mathcal{X}}$ is therefore $n = q^3 + 2q^2 + q + 1$.


doi.org/10.32388/XLWZT8

The dimension of $C_{\mathcal{X}}$ is obviously 5, that is, the vector dimension of $\Sigma$.

From Proposition 3.3, $(i)$, follows the distance of $C_{\mathcal{X}}$ is $d = n - |\{P \in s_i | i = 1, \ldots, q+1\}|$ that is, $d = q^3 + 2q^2 + q + 1 - (q^2 + 2q + 1) = q^3 + q^2 - q$.

$(ii)$ The length of the code $C_{\mathcal{X}_{aff}}$ equals $n' = |\mathcal{X}| - |\{P \in s_i | i = 1, \ldots, q+1\}| = q^3 + 2q^2 + q + 1$ $- (q^2 + 2q + 1) = q^3 + q^2 - q$.
Its dimension is $k = 5$. From Proposition 3.3, $(ii)$, follows the distance is $d' = n' - |\mathcal{Q} \setminus \mathcal{C}'^2|$ that is, $d' = q^3 + q^2 - q - q^2 = q^3 - q$.

**Examples**

For $q = 2$, $C_{\mathcal{X}}$ is a $[19, 5, 10]_2$-code and it can correct at most $\lfloor \frac{10-1}{2} \rfloor = 4$ errors. For $q = 3$, $C_{\mathcal{X}}$ is a $[49, 5, 33]_3$-code and it can correct at most $\lfloor \frac{33-1}{2} \rfloor = 16$ errors.

For $q = 2$, $C_{\mathcal{X}_{aff}}$ is a $[10, 5, 6]_2$-code and it can correct at most $\lfloor \frac{6-1}{2} \rfloor = 2$ errors. For $q = 3$, $C_{\mathcal{X}_{aff}}$ is a $[33, 5, 24]_3$-code and it can correct at most $\lfloor \frac{24-1}{2} \rfloor = 11$ errors.

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## Declarations

**Funding:** No specific funding was received for this work.
**Potential competing interests:** No potential competing interests to declare.