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Research Article

# Emergent Quantum Mechanics – How the Classical Laws Can Replicate the Quantum Harmonic Oscillator

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Emergent Quantum Mechanics (Em.QM) is defined here as the attempt to bridge the gap between quantum and classical mechanics, and to restore causality to the atomic world. An example described in the paper shows how a conspiracy of the classical laws produces a replica of the quantum harmonic oscillator. This is achieved using a droplet model of electrons, obeying Newton/Einstein dynamics in combination with Maxwell's electromagnetic laws.

A concise sketch of the droplet theory precedes the sections on the harmonic oscillator. This summary also briefly mentions other successes of the droplet theory, such as the derivation of the fundamental quantum laws $p = \frac{h}{\lambda}$ and $E = h.\nu$ with a practically correct numerical value of Planck's constant h. Although these subjects were already addressed in earlier publications about Em.QM, recently simpler and more transparent derivations were found and are reported here.

Finally, the achievements of Em.QM until now are reviewed. Em.QM does not compete with conventional QM: the latter will always be superior for performing actual calculations, whereas Em.QM is concerned with possible interpretations of quantum phenomena. An assessment is given of the present status of Em.QM in this perspective.

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## 1. Introduction

The term Emergent Quantum Mechanics (in the following abbreviated as Em.QM) is here used to indicate the branch of physics aiming to reconcile Quantum Mechanics (hereafter abbreviated as QM) with the classical laws of physics, down to atomic scales. The present–day paradigm in physics is that at atomic

scales, particles obey the laws and rules of QM, whereas, at these scales, the classical laws of physics have completely lost their applicability. In Em.QM, it is investigated whether this is really true. In the paper, it is discussed how some atomic properties until recently were thought to be typical quantum phenomena, but, on closer examination, appeared to be compatible with the classical laws and even could be replicated by a "conspiracy" of these laws. If Em.QM further matures in the future, one may come to a point where the relation between Em.QM and QM becomes analogous to, for instance, the relation between kinetic gas theory and thermodynamics. Thermodynamics is based on the modelling of fluids as continuous media, which leads to very efficient methods to perform calculations, although at the cost of having to introduce a few rather esoteric concepts, remote from intuitive physics. In contrast, kinetic gas theory considers the motion of separate molecules, which, although less suitable for technical calculations, shows how the thermodynamic phenomena are related to classical mechanics. Likewise, conventional QM is very efficient for performing actual calculations, whereas Em.QM can show how quantum phenomena may be explained by classical laws. This could lift some of the magic from QM, and -more importantly- could bring back causality to the atomic world.

Until 2005, it was accepted that two separate physics existed, one applicable to the atomic world, the other one usable in the macroscopic world. However, doubts about this arose when Couder et al. (e.g., refs. 5, 6, and 7) from 2005 onwards published experiments on tiny droplets of oil bouncing on a vibrating oil surface. The macroscopic droplets, obeying the classical laws of physics, nevertheless showed quantum-like behaviour such as discrete levels, tunnelling, and double-slit refraction. This remarkable behaviour was the first indication that the classical laws can conspire in such a way that quantum-like phenomena are mimicked.

After these experiments, the obvious and legitimate question was whether classical analogues of still other -and perhaps of all the known- quantum phenomena might be found. After a theoretical explanation by Anderson and Brady of the oil droplet experiments (ref.8), the same authors published a paper (ref.9) that made it plausible that even the non-local behaviour of entangled quantum systems is amenable to classical analogues.

So far, the above-mentioned investigations have concerned *analogues* of quantum phenomena. The present author took a next step in 2017 (ref.3) by which it was shown that several quantitatively correct *replicas* of the quantum world could be constructed, just using Newton/Einstein dynamics coupled with Maxwell's laws. Examples reported in refs. 3 and 4, also briefly summarised in the present paper, are:

- The classical laws of nature can conspire to reproduce "De Broglie dynamics" (i.e. $p = \frac{h}{\lambda}$) of atomic particles, including a "classical" interpretation of matter waves and a quantitative replication of Planck's Constant $h$ .

- Using the classical mechanical laws and Maxwell's equations, the quantum behaviour of electrons in potential wells (in particular, the motion in boxes and in harmonic oscillators) can be replicated, showing amongst others energy quantisation, violation of the classical motion limits, and tunnelling.

- Planck's energy-frequency relation in radiation ($E = h.\nu$) can be derived from the classical laws.

- A possible answer to the long-standing question can be given how the mass differences of the three charged leptons (the electron, the muon and the tau-particle) can be explained, and why there are three kinds (or: "generations") of these leptons.

The key to obtaining these results was to model the electron as a charged droplet, susceptible to deformations. In contrast to the usual point-like model, the droplet model is mathematically non-singular, and thereby allows the determination of so-called self-forces. Self-forces are the electromagnetic forces exerted by "lumps of charges" on themselves during accelerations and shape deformations. Already a century ago, self-forces played an important role during the studies on non-singular electron models by Lorentz, Abraham, et al. (refs.12 and 13), and they are therefore well-known. However, the droplet model has more degrees of freedom than Lorentz's rigid charge distributions, and consequently, more types of self-forces are found than just Lorentz's electromagnetic mass and radiation resistance force. Cross-couplings between the degrees of freedom cause that shape deformations, such as vibrations of the droplet, do have an influence on the translational motion of the droplet. It is due to these dynamic cross-couplings that, at very small scales, the behaviour of "blobs of charge" becomes quantum-like.

The first part of the present paper gives an exposé of the dynamical theory applicable to the droplet of charge. The central problem in the theory is the determination of the electromagnetic self-forces. The analysis of the self-forces is completely reported in the mathematical appendix of ref.3. The derivations in ref.3 have been peer reviewed, so that this reference can serve as a source that can be consulted for all the details. Unfortunately, the derivations are too lengthy to copy in full in a journal paper.

Another description of the theory with a somewhat different character may be found in ref.4. In the latter reference, the key results of the theory have been stated together with an attempt to explain them

heuristically instead of by rigorous mathematics. The more modest aim was to make these results plausible and understandable.

In the present paper, neither of the above descriptions of the theory has been copied. Rather, a concise sketch will be given by summing up what are the essential steps needed to arrive at the equations of motion of the droplet. Emphasis is thus given to the "general line" in the derivation of the droplet's dynamics, without detailing all the arguments behind the theoretical steps. It is hoped that the present outline of the theory of Em.QM is the most "streamlined" and easy-to-read of all three publications, and may serve as an introduction. More background, in the form of heuristic arguments or even full mathematical derivations and proofs, may be obtained from refs. 3 and 4.

After the introduction to Em.QM's droplet theory in sections 2 to 10, the second part of the paper shows the application of the theory to the harmonic oscillator. The quantum mechanical harmonic oscillator is a popular educational model as it is a "school example" of how radically different QM is compared with classical mechanics. The more surprising is the fact that the same results are here reproduced using the classical laws, without any reference to the laws and rules of quantum mechanics.

The harmonic oscillator consists both macroscopically as well as in the atomic world of a charge moving back and forth within a parabolic potential. Macroscopically, the equation of motion is a simple vibration equation, and the motion is a harmonic oscillation, whereas in QM the motion is governed by Schrödinger's equation, leading to typical quantum phenomena like discrete energy levels and violations of the classical limits of motion. In sec.11, it will be shown how the classical laws of nature can conspire to predict the same quantum behaviour, if the moving charge has been modelled in the form of a deformable droplet instead of a rigid "marble".

An interesting difference between the treatments by QM and by Em.QM must also be pointed out: to find the solution of Schrödinger's equation for this simple configuration is unexpectedly complicated (see e.g. refs.1 or 2). From the point of view of mathematics, the harmonic oscillator is not an "easy piece" in QM. An earlier version of Em. QM's droplet theory (ref.3) required solving a Mathieu equation, and was neither very simple. However, in the present paper, a newer development of the theory will be presented, which is much more transparent than either Schrödinger's or Mathieu's theory. It shows that the "conspiracy of the classical laws" is not necessarily a complex concoction of formulae. Rather, it is straightforward and elegant.

In contrast to secs. 2 to 10, where just an outline of the theory was given, in sec.11 the theory of the harmonic oscillator is developed in all details, starting from the equations of motion up to the derivation

of the discrete energy levels. The more elaborate explanation was chosen because the new, simpler derivations cannot be found in earlier publications like ref.3, where still the Mathieu equation is used as a basis for the solution.

Finally, the frequency observable in the radiation when the droplet exchanges one level for another is discussed. The frequency postulated by Bohr is related to the energy drop of the electron, whereas according to Maxwell's laws, one would expect it to be associated with the vibration of some part of the oscillator. Em.QM offers a solution to this old riddle, even though the results of Em.QM do agree with Planck's energy-frequency relation $E = h.\nu$.

In the section "Discussion and conclusions", an assessment is included of what, in view of the successes up to now, is the present state of Em.QM in the perspective of the –very ambitious- goal to find for all the quantum phenomena an interpretation in terms of the classical laws.

## 2. The droplet model

In principle, one is allowed to propose any model, however unrealistic it may seem at first sight, to investigate its properties and compare them with experimental results. The droplet model of electrons could therefore be considered as just an arbitrary working hypothesis, whose usefulness is only determined by its consequences. However, there is another view possible on the plausibility of this model, which will be the subject of the present section.
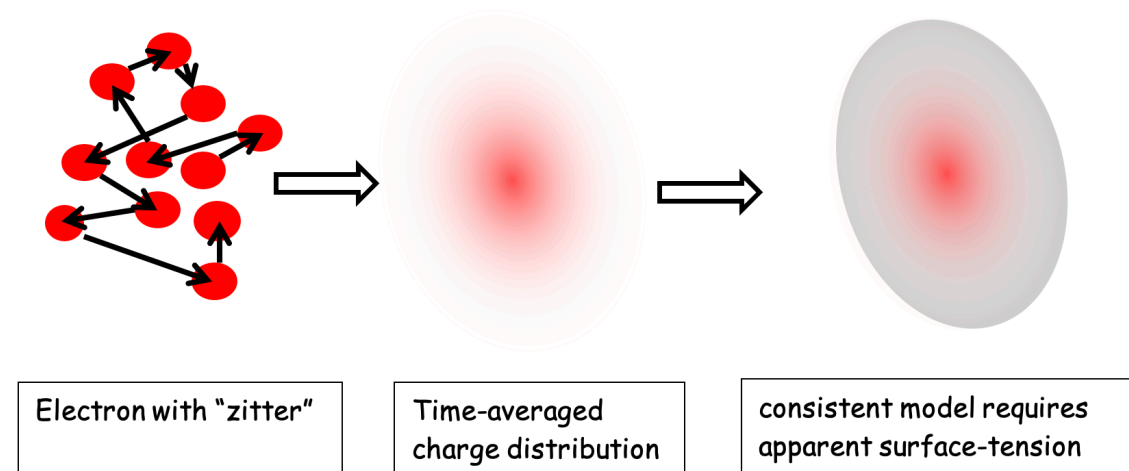


| Electron with "zitter" | Time-averaged charge distribution | consistent model requires apparent surface-tension |

**Figure 1.** Modelling an electron with "zitter"

The droplet model has a connection with the "zitter" motion of electrons. "Zitter" (a German term introduced by Schrödinger in 1930, meaning "shudder" or "tremble") is the phenomenon in which an electron continually performs tiny jumps in all directions (fig.1 left). The basic idea behind the droplet model is that the electron should be modelled *including* this intrinsic motion. In order to obtain a simplified model, one would be tempted to use time-averaging, resulting in a "cloud" of charge (fig.1, middle). Such a deformable cloud with distributed charge is certainly easier to analyse, but unfortunately, it is not a realistic model because it has a tendency to explode, in contrast to the original configuration. The culprit is the time-averaging modelling step itself. In the case of the real zittering electron at any instant of time, there is no more than one charge present, so that there is no tendency for the "zitter volume" to explode. At most, this volume may slowly drift away. On the other hand, the time-averaged model does show an explosion tendency, because the different elements of the resulting cloud co-exist at the same time.

Obviously, merely time-averaging is a wrong modelling tool, or rather, it is too simple. A useful simplified model requires the addition of "apparent" forces as a means of containment, "taming" the explosion tendency, which has purely artificially been introduced by the time averaging. The terminology "apparent force" is here used in the same sense as, for instance, the term "apparent mass" in fluid dynamics. A body immersed in a fluid experiences a force resisting an acceleration, due to a redistribution of pressure forces accompanying the increasing kinetic energy in the flow during an acceleration. When a fluid-*less* dynamical model is made, one may replace the pressure forces with a drag force supplemented by an "apparent mass". Likewise, a time-averaged model of the zittering electron requires, to be consistent, an "apparent" containment force so that the model reflects reality. Using Feynman's terminology (see ref.10), containment forces of whatever origin are often called "Poincaré forces". The form of the Poincaré forces in our case must be some kind of surface-tension-like force, since the explosion tendency is largest in the outer shells of the charged cloud, where the repulsion all comes from within the cloud. The resulting simplified model of the zittering electron is schematically pictured in fig.1 right, and from now on, will be called the "droplet model". The droplet model satisfies the requirement that it affords a simpler analysis than real physics does, whereas it is consistent in the sense that it reflects reality in its essentials and allows an unrestricted application of the electromagnetic laws.

The proposed droplet model may look analogous to the droplet model of nuclei. However, it is essentially different: the surface tension in the electron model is an apparent force existing only in the time-averaged model world. It certainly is *not* a physical force, whereas the strong nuclear force *is*. Rather, the

droplet model might be considered to be somewhat comparable with the concept of the so–called "dressed electron".

The droplet model has been used for a first exploration of the consequences of a non–singular electron model, and to find out whether it could lead to physically realistic conclusions. In view of this limited purpose, drastic further simplifying assumptions were introduced in order to facilitate an easy analysis. The model analysed is shown in fig.2. The droplet is assumed to move along the Z–axis inside a one-dimensional potential well, one half of which is schematically indicated by the barrier at a distance $\frac{a}{2}$ from the well centre at $z = 0$. What is sketched suggests a potential box, but the barrier may be soft as well, such as in a parabolic well. The motion of the droplet, as well as the motion of all its elements, is likewise taken to be one-dimensional. Consequently, the charge elements have a velocity, acceleration and jerk in the Z–direction only. The charge distribution is not specified, except that it is assumed to have rotational symmetry around the Z–axis, and fore–aft symmetry w.r.t. an "equator plane". We can thus define a midpoint, which is the centre of mass as well as the centre of charge.

The model has two degrees of freedom: the variable position of the midpoint $z_m(t)$, and the variable elongation $s(t)$, called the "pulsation" hereafter. A quasi-static deformation mode is assumed, where stretching (i.e. an increase of the total length $s$) takes place as in a spring without dynamic effects. A final assumption is that the droplet has an elongated shape in the Z–direction (i.e. varying from a Zeppelin-shape to a needle or even a long wire or string), so that the analysis of the electromagnetic field and the self-forces can make use of "slender –body" approximations.
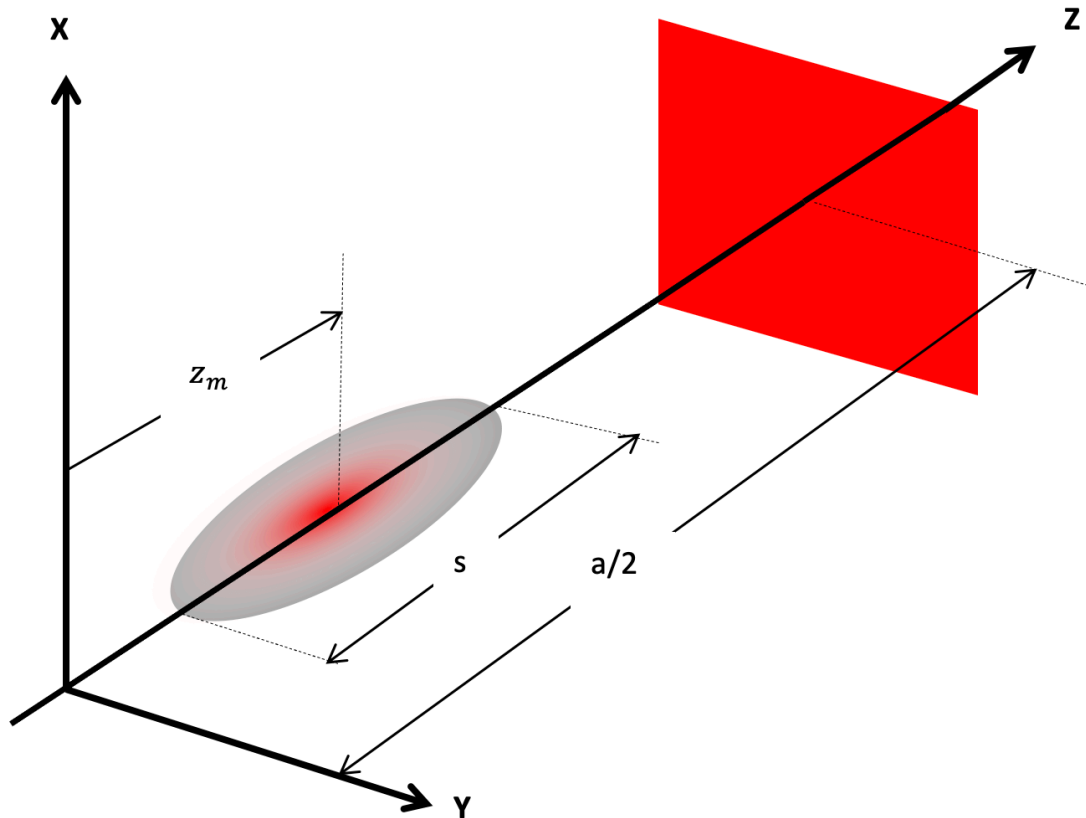
**Figure 2.** Simplified configuration and notations

# 3. Matched asymptotic expansion procedure to derive the instationary electromagnetic field

The mathematical analysis to determine the instationary electromagnetic field and the self-forces caused by this field is described in detail in the appendix of ref.3. A suitable tool for this purpose is the mathematical procedure called "matched asymptotic expansion approximation of singular perturbation problems" (see ref.11, and the short explanation in ref.3). The method (briefly: "max"-method) was popular in fluid dynamics in the sixties and seventies of the last century, before "number crunching" by computers took over. As far as the author knows, the max-technique was in the past never used in connection with Maxwell's equations. Its applicability was verified by comparing several results of the "max"-procedure with Lorentz's results concerning "lumps of charges" (see sec.4).

The asymptotic expansion parameter used is $\frac{v}{c}$ , where $v$ stands for a typical velocity of the charge elements, and $c$ is the speed of light. Asymptotic expansions are developed of the "near field" (where the

retardation times are small and may be approximated) and -separately- of the "far field" (which may be approximated by a superposition of multipoles). There are ambiguities left in both series expansions, since the near field approximation lacks boundary conditions at infinity (so that in principle fields are allowed -and indeed prove to be indispensable- that do not vanish at infinity), whereas the type and number of poles of the far field are undetermined because of the lack of any "inner" boundary conditions. The ambiguities are removed by matching the two fields, through the intermediary of the so-called "common part" of the expansions. Moreover, singularities in either of the expansions are also removed by the matching process, so that the so-called "composite field" satisfies all the physical boundary conditions.

The asymptotic expansions have been continued up to and including the order $O\left(\frac{v}{c}\right)^3$. This level of approximation has been chosen in order to guarantee that the so-called "radiation resistance" of the droplet is in agreement with the energy radiated away to infinity (see sec.7).

Because the analysis rests on Maxwell's laws, relativity effects are automatically included. However, the asymptotic expansions are truncated after the terms of order $O\left(\frac{v}{c}\right)^3$, and so are the relativity effects.

# 4. Self-forces and the translational equation of motion

## 4.1. Definition of "self-force"

Most of the terms occurring in the equations of motion of the droplet are so-called "self-forces". They are the forces an electron exerts on itself, due to the retardation of signals between the separate elements of the "smeared out" charge. Consider, for instance, the electron model sketched in fig.2, and assume that it is subject to a translational acceleration $\ddot{z}_m$ . Let us furthermore assume that external forces affecting the elongation $s$ are absent (although the dimensions of the droplet may be influenced by Lorentz-FitzGerald contraction). The interaction between two different elements of the droplet takes place through electromagnetic waves propagating with the speed of light $c$ . This means that any element will "see" other elements at a slightly earlier time, and, therefore, with a slightly smaller speed than its own speed. This will disturb the balance of internal forces that existed when the charge was at rest, with the effect that so-called self-forces arise.

The calculation of the self-forces makes use of the instationary electromagnetic field determined by the matched asymptotic expansion procedure. The associated electric and magnetic field strengths are integrated over all the elements of the droplet, and then lead to the self-forces summarised below. The

complete calculation has been described in detail in the mathematical appendix of ref.3. These calculations are too lengthy to include in a relatively short article, for which reason, in ref.4, the detailed mathematics were replaced by heuristic arguments. The latter publication had the more modest aim to give the reader some feeling for the physical meaning of the terms in the equations of motion, and to make the results plausible.

The present paper does not copy the earlier derivations of the self-forces as can be found in either ref.3 or ref.4. Rather, the procedure followed during the derivation is described. Secs. 4 to 10, therefore, merely summarise the steps that have been taken to arrive at the equations of motion, mostly without a justification.

## 4.2. Electromagnetic and inertial mass

The most well-known of the self-forces on a "lump of charge" is the *electromagnetic inertia force*, a force resisting accelerations. A result obtained by Lorentz (see ref.12) is that any lump of electrical charge has inertia, where the so-called electromagnetic mass $m_{em}$ is given by (the symbols are explained in the section "notations" at the end of the paper):

$$m_{em} = \frac{1}{c^2} \frac{q^2}{4\pi\varepsilon_0} \frac{1}{s} \alpha \qquad (1)$$

The same result was found by the "max"-procedure used by the author. Lorentz studied -as a candidate electron model- a thin layer of charge deposited on a spherical core of insulator material. He took as the characteristic dimension $s$ the radius of the core, and found $\alpha = \frac{2}{3}$. The value of the form factor $\alpha$ in eq. (1) depends on the shape and distribution of the electric charge and has, in general, an order of magnitude $O(1)$.

One might have had the intuitive feeling that retardation effects would be negligible on the scale of atomic particles. However, the reverse is true, as evidenced by eq. (1), which shows $m_{em} \to \infty$ if $s \to 0$. Basically, this is the cause of the infinities (of mass and energy) associated with singular electron models.

The electromagnetic mass $m_{em}$ given by eq. (1) is *not* the inertia to be used in dynamical equations, as already pointed out by Poincaré (ref.14). The matter arose after Einstein wrote his paper on special relativity, and the realisation by physicists that the electrostatic energy $U_{es}$ (explained below) of a "lump of charge" and its electromagnetic mass $m_{em}$ do not satisfy the generally valid relation $E = mc^2$.

The electrostatic energy $U_{es}$ (or: "formation energy") reflects the fact that an accumulation of electrical charge is equivalent to an energy storage. The stored energy equals the work expended by bringing all

the charge elements together against the mutual repulsion forces between them. The magnitude of this electrostatic energy $U_{es}$ is derived in most textbooks (see e.g. ref.10), and is in the case of the configuration of fig.2:

$$U_{es}\left(s\right) = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{s}\alpha \qquad (2)$$

Comparing eqs. (1) and (2) clearly shows the problem. Feynman relates in his Lectures (ref.10) that eqs. (1) and (2) were developed before relativity, and he says, "when Einstein and others began to realise that it must always be that $U = mc^2$ , there was great confusion".

Poincaré subsequently pointed out (ref.14) that relativity and the theory of electrically charged objects can be reconciled by taking into account the binding energy that keeps the distributed charge contained. Binding energy is equivalent to a mass deficit, so that the inertial mass $m$ to be used in the dynamic equations of motion is smaller than the electromagnetic mass $m_{em}$ . For the droplet configuration, the inertial mass $m\left(s\right)$ has been found to be

$$m\left(s\right) = m_{em}\left(s\right) - \frac{1}{2}m_{em}\left(s_e\right) \qquad (3)$$

where $s_e$ denotes the "static equilibrium elongation", occurring when the droplet is at rest and the electrostatic expansion tendency is balanced by the squeezing due to the surface-tension-like apparent force. The inertial mass under equilibrium conditions ($s = s_e$) is

$$m\left(s_e\right) = \frac{1}{2}m_{em}\left(s_e\right) = \frac{U_{es}\left(s_e\right)}{c^2} \qquad (4)$$

which does satisfy the relativity relations.

## 4.3. Radiation resistance

By the "max"-analysis of ref.3, one more of Lorentz's self-forces is found: the *"radiation resistance"* (also called "*radiation reaction*"):

$$F = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0}\frac{\ddot{v}}{c^3} \qquad (5)$$

It is a small ($c^3$ in the denominator!) translational force depending on the jerk $\ddot{v}$ of the droplet's centre of charge. If $v(t)$ is harmonic, one can demonstrate (see e.g. Feynman in ref.10) that an external force overcoming the radiation resistance injects -on average- exactly as much energy as can be found in the radiation energy leaving the system far from the droplet. Hence the name "radiation resistance" and the interpretation of this self-force as a recoil force accompanying electromagnetic radiation. In the case of

harmonic motion of the charge, it acts as a damping force, dissipating the energy of the droplet in the form of radiation.

However, for any other type of motion, for instance, the starting-up transient of a harmonic motion, $F$ as given by eq. (5) leads to severe problems: the motion then violates causality. In Lorentz's time, these problems were called the "runaway motion" and "pre-acceleration" of electrons (i.e. start of the motion before the force is being applied), and they remained unexplained for a long time despite many attempts to revise the theory (see the overview given by Feynman in ref.10). It was only towards the end of the $20^{th}$ century that Yaghjian offered a possible solution which eliminates the non-causal dynamical behaviour of "lumps of charges" (ref.13). Yaghjian postulated a "starting-up delay of the radiation resistance" and proved that the assumption of even a tiny delay (of the order of the time span needed by a light ray to pass from one end of an electron to the other) would solve all the problems with causality. A physical interpretation of Yaghjian's mathematical solution was given by the present author in ref.3, and will be explained below. Yaghjian's work, recovering causality, cleared the way to use and extend the theory of lumped charges as developed by Lorentz and contemporaries like Abraham. This proved vital for the present research on non-singular electron models.

The physical explanation of a delay in the radiation resistance during transients is the following. The solution of Maxwell's equations allows both outgoing as well as incoming waves. This corresponds with the dissipation of the droplet's energy, respectively, energy absorption from the field. A superposition of both types of waves is mathematically allowable, the particular mix being determined by the physical boundary conditions. A well-known example of complete cancelling of radiation resistance is the oscillating charge inside an enclosure so that no radiation of energy to infinity can take place. The mathematical description entails the sum of outgoing and incoming waves of equal strength, resulting in a field of standing waves inside the enclosed space. In ref.3 it is shown that, in such a case, the radiation resistance indeed vanishes, without affecting the other self-forces.

In ref.3 it is also shown that the transient situation of a charge which is initially at rest and starts to oscillate requires a mathematical description which, apart from outgoing waves, includes a quickly subsiding field of incoming waves. The result is that at the very first moment of the motion, the radiation resistance is zero, after which it grows to the value given by eq. (5). This is the kind of "resistance delay" that had been postulated by Yaghjian and was proven by him to eliminate the old problems of violated causality in the form of runaway motion or pre-acceleration.

In the equations of motion stated below, the radiation resistance will therefore be written for the most general case as

$$F_{rad} = \delta(t) . \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0} \frac{\ddot{v}}{c^3} \left( -1 \leq \delta \leq +1 \right) \qquad (6)$$

It is shown in ref.3 that all the other self-forces keep the same value, no matter whether one deals with outgoing or incoming waves. The other self-forces, therefore, do not depend on the value of $\delta$.

## 4.4. Cross-coupling between translation and pulsation, translational equation of motion

A third translational self-force is found to be:

$$F_\times = m_{em}\left(s\right) . \frac{\dot{s}}{s} v \qquad (7)$$

It is a force representing a cross-coupling between the two degrees of freedom of the droplet. The existence of this cross-coupling term enables one to write the translational equation of motion in the – deceptively– simple form:

$$\frac{dp}{dt} = F_{external} + F_{rad} \qquad (8)$$

The crux of why this innocuous-looking equation leads to quantum-like behaviour instead of to "normal" motion is the fact that in the momentum $p = mv$, the mass of the droplet $m\left\{s\left(t\right)\right\}$ is variable due to the pulsation of the droplet.

# 5. The pulsation equation

## 5.1. Lagrange's equation and generalised forces

The variable elongation $s(t)$ must be treated as a generalised coordinate in the sense of Lagrangian dynamics. Lagrange's generalisation of Newton's dynamical equation is usually written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad (9)$$

where the generalised coordinates $q_i$ in our case are $z_m$ and $s$ (see fig.2). The system Lagrangian is $L = T - U$, where both the kinetic energy $T$ and the potential energy $U$ are expressed in terms of the generalised coordinates. The notation $Q_i$ stands for the generalised force in $q_i$ -direction. In the following, an explanation is given of how the generalised "explosion" force $Q_s$ must be derived according to Lagrange's theory.

## 5.2. Generalised force associated with the internal repulsion forces of a static droplet

There are several contributions to the generalised force in $s$–direction $Q_s$. The contribution by the electrostatic repulsion between the elements of the droplet is here denoted as $Q_{s,electr}$. In the following, it will sometimes be called the "explosion tendency of the droplet". We assume zero velocity of the droplet and apply a small increase of the length, a so-called virtual displacement $\delta s$. Because of the model assumptions about the symmetries and the elastic type of the deformation (see sec.2), this one quantity $\delta s$ determines how all the elements of the droplet are displaced, and how much all the mutual distances between the elements are increased. Recalling the physical meaning of formation energy (or electrostatic energy $U_{es}$), all these elemental displacements together cause a reduction of $U_{es}$ (if $\delta s$ is positive). The generalised force $Q_{s,electr}$ is defined as the single parameter whose product with $\delta s$ would give the same change $\delta U_{es}$ of the formation energy. Using eq. (2):

$$Q_{s,electr} = -\frac{\delta U_{es}}{\delta s} = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{s^2}\alpha \qquad (10)$$

## 5.3. Relativity correction of the "explosion tendency", energy conservation

When the self-forces of the droplet are determined in the more general situation $v \neq 0$, it appears that the "explosion tendency" is reduced, and the expression in eq. (10) is multiplied by the relativity factor $\left\{1 - \left(\frac{v}{c}\right)^2\right\}$.

The extra term $\left(\frac{v}{c}\right)^2.Q_{s,electr}$ in the pulsation equation represents a cross-coupling between the two degrees of freedom, like the earlier force $F_\times$ in the translation equation. When the work is determined, which is done by these two cross-coupling effects, it is found that energy may be transferred from the translation to the pulsation and back, whereas the total energy of the droplet is conserved.

## 5.4. Generalised force associated with the surface tension

The electrostatic expansion is counteracted by the surface-tension-like apparent force, which is similarly expressed in the form of a Lagrangian generalised force $Q_{s,surf.tension}$ :

$$Q_{s,surf.tension} = A.\left(s - s_{sphere}\right) + C\frac{1}{s} = A.s + B + C\frac{1}{s} \qquad (11)$$

The first part, $A\left(s - s_{sphere}\right)$, reflects the property that surface tension causes a tendency to squeeze the droplet into a spherical shape, which tendency is larger the more the actual shape differs from a sphere. This first term is in the second line written as $A.s + B$. The second term represents a scale effect: the

compression due to surface tension leads to a tendency to minimise the size, which tendency is relatively stronger the smaller the overall size. In the case of spherical droplets, the scale effect can be shown to be inversely proportional to the radius, which scale effect in the present case (one-dimensional variations of the elongation) is represented by the factor $\frac{C}{s}$ .

## 5.5. Squeezing inertia, the pulsation equation of motion

If the matched asymptotic expansions are truncated after terms of the $O\left(\frac{v}{c}\right)^3$ , only one more self-force is found. This is the so-called "squeezing inertia", resisting elongation accelerations $\ddot{s}$ . Including the squeezing inertia term, the equation of motion for the pulsation $s(t)$ reads:

$$m^*\left(s\right)\ddot{s} = \left(1 - \frac{v^2}{c^2}\right)Q_{s,electr} + Q_{s,surf.tension} \qquad (12)$$

where the symbol $m^*(s)$ stands for the "generalised mass in s-direction" or "squeezing inertia", which has the same form as electromagnetic mass, and the same dependency on the elongation $s$ :

$$m^*(s) = m^*_{em}\left(s\right) - \frac{1}{2}m^*_{em}\left(s_e\right) \text{ where } m^*_{em}\left(s\right) = \frac{1}{c^2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{s}\alpha^* \qquad (13)$$

The form factor $\alpha^*$ depends on the precise distribution of charge inside the droplet. In general, it is an order smaller than $\alpha$ in $m_{em}$ , as derived in app.B.

# 6. Static equilibria: the three charged leptons

The static equilibrium condition reads:

$$Q_{s,electr} + Q_{s,surf.tension} = 0 \qquad (14)$$

Substituting eqs. (10) and (11) into eq. (14) leads to a cubic algebraic equation in $s$ , with potentially (if all the roots are real) three different values of the elongation for which equilibrium is possible. In view of eq. (3), which expresses that the inertial mass $m(s)$ depends on the elongation $s$ , we see that three different values of the equilibrium mass are possible. Fig.3 illustrates this conclusion graphically. In the figure, it is also inscribed how one - tentatively- could try to identify the three possible equilibria of the droplet model with the three charged leptons (electron, muon and tau-particle).

These three "guises" of the electron are identical, except for their mass. The mass of elementary particles is bound to be determined in some way by the laws of physics. Were it not so, it would be inexplicable why all the particles of a kind everywhere have exactly the same mass. It has, therefore, been a long-

standing question how it is possible that the three charged leptons (why three?) have different masses, in spite of the fact that there are no other discernable differences between them. A more comprehensive discussion about the "enigma of the leptons" may be found in the introduction of ref.4.

The assumptions about the nature of the three leptons, as inscribed in fig.3, would answer both questions, viz. where the mass differences come from as well as why there are three of these electron variants.

Under the assumptions inscribed in fig.3, the relative position of the two curves in the figure is fixed by the known value of $m_{electron}$ and the known ratios $\mu = \frac{m_{muon}}{m_{electron}}$ and $\tau = \frac{m_{tau}}{m_{electron}}$ .
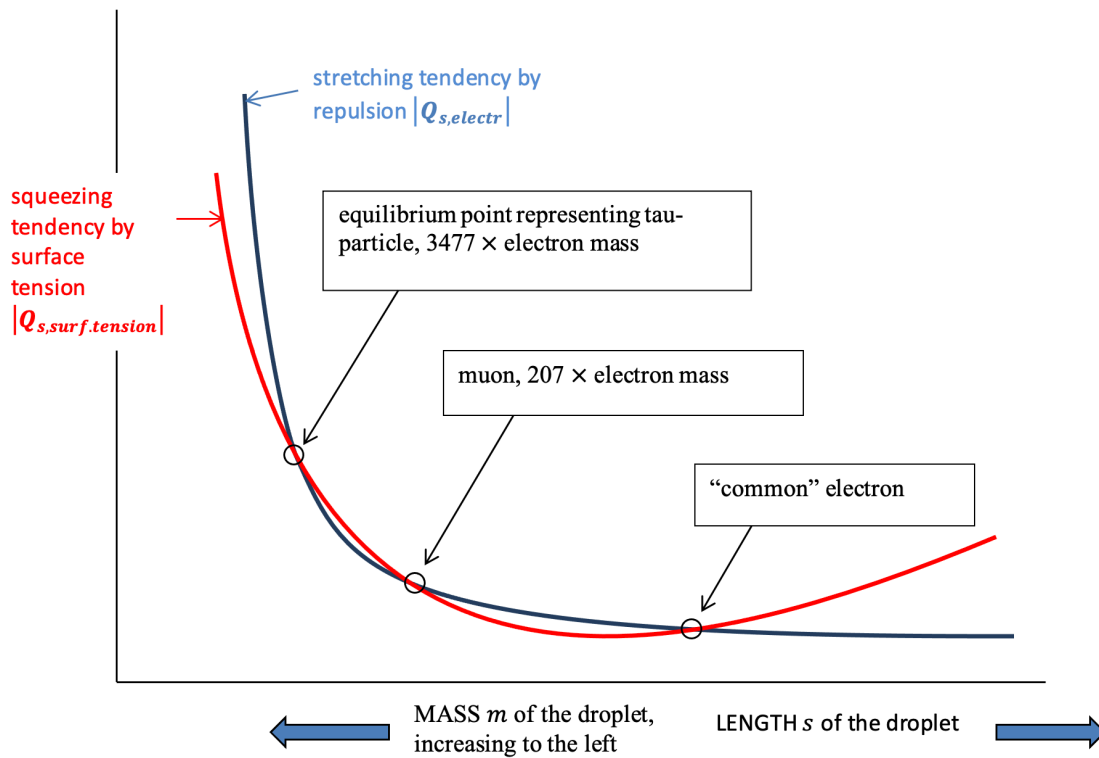


**Figure 3.** The equilibria between electrostatic expansion and surface tension

The hitherto unknown coefficients of $Q_{s,surf.tension}$ are then given by:

$$\frac{B}{A} = -s_e \left( 1 + \frac{1}{\mu} + \frac{1}{\tau} \right), \ \frac{C}{A} = s_e^{\,2} \left( \frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\mu\tau} \right), \ A = -\frac{1}{2} \mu\tau \frac{q^2}{4\pi\varepsilon_0} \frac{\alpha}{s_e^{\,3}} \qquad (15)$$

The equations of motion are hereby fully quantified. As will be shown in sec.10, the so quantified dynamic equations are able to reproduce Planck's Constant $h$ and De Broglie's relation $p = \frac{h}{\lambda}$ . This

makes it plausible that the three equilibria in fig.3 indeed do coincide with the lepton masses. It should be emphasised that this conclusion is strictly valid only within the scope of the model world of Em.QM, and one could argue about its real physical meaning (whatever is meant by "real").

# 7. Larmor's law and energy conservation in the electromagnetic field

Larmor's law specifies the rate of energy radiated outward by an oscillating charge:

$$\frac{dW}{dt} = \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0} \frac{\dot{v}^2}{c^3} \qquad (16)$$

The matched asymptotic expansion method (or: "max" procedure) used in the present study to determine the instationary electromagnetic field leads to the same expression. However, there is one difference between the "max"-result and Larmor's law: Larmor derived eq. (16) for the case of harmonic motion of the charge, whereas according to the asymptotic solution eq. (16) is more generally valid and may be applied to the instantaneous acceleration $\dot{v}$ (with due allowance for the retardation), irrespective of the time history of the motion.

Obviously, eq. (16) is based on outgoing waves and on an outward energy flow. If, as in sec.4.3, incoming waves are allowed too, as well as a mix of outgoing and incoming waves with different strengths, we must write the loss of energy $\frac{dW}{dt}$ from the system in the more general form

$$\frac{dW}{dt} = \delta. \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0} \frac{\dot{v}^2}{c^3} \qquad (-1 \le \delta \le +1) \qquad (17)$$

The radiation resistance $F_{rad}$ does work on the droplet at a rate $v. F_{rad}$ . In reverse, the corresponding power injected into the near field is $-v. F_{rad}$ or, according to eq. (6):

$$-v. F_{rad} = \delta. \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0} \frac{-v.\ddot{v}}{c^3} \qquad (18)$$

The harmonic motion assumed by Larmor, for instance, $v = v_0 \, cos \, (\omega t)$, indeed satisfies the requirement that –averaged over a cycle- the power input by the droplet into the field balances the power lost by radiation to infinity. The fact that this equality only applies to a cycle-average implies that the electromagnetic field must have some capacity to temporarily store energy.

Feynman, in ref. (10), broadened the conclusion somewhat, by showing that compatibility of eqs. (17) and (18) requires $v(t)$ to be periodic, and not necessarily harmonic.

A class of suitable functions $v(t)$ is, for instance, the ensemble of solutions of the following linear differential equation with a variable coefficient:

$$\ddot{v} + f(t).v = 0 \qquad \text{if } f(t) \geq 0 \text{ is periodic} \qquad (19)$$

According to Floquet–Lyapunov theory, which pertains to linear differential equations with variable coefficients, the solutions $v(t)$ of this equation have the same periodicity as the function $f(t)$ , so that Feynman's condition for the compatibility of eqs. (17) and (18) is met. Furthermore, $v(t)$ determined by eq. (19) satisfies a second, rather essential requirement not yet mentioned, viz. that the direction of the energy flow injected into or absorbed from the field should at all times correspond with the direction of the electromagnetic waves themselves. Functions $v(t)$ determined by eq. (19) agree with this condition because of the requirement $f(t) \geq 0$ .

Later, in sec.11 dealing with the harmonic oscillator, it will be seen that for small pulsation amplitudes of the droplet, the velocity $v(t)$ indeed satisfies an equation of the type described by eq. (19). The linearised equations of motion of the droplet thus automatically ensure compatibility of the radiation resistance and the energy loss due to radiation.

However, other types of motion do occur, for instance, when the pulsation amplitude is so large that linearisation of the equations of motion is not allowed. The non-linear equations of motion (further detailed in sec.8) lead to chaotic pulsation and chaotic velocity fluctuations. Such a motion is non-periodic, and neither can it at all times satisfy the above-mentioned criterion of co-directionality of energy flow and wave propagation. In order to restore the conservation of energy in the field, we must then take $\delta = 0$ , i.e., zero radiation resistance and zero radiation to infinity. For this kind of motion, the radiation environment then acts like a "closed" system, with outgoing and incoming waves compensating each other. This is the same situation as described in sec.4.3, when during the first instant of the start-up of an oscillation from rest, the radiation resistance is zero and subsequently delayed.

Ref.3 explains the zero radiation loss during chaotic motion in physical terms, by considering the auto-correlation function of a fully developed chaotic signal, which closely approaches a delta function. The almost complete lack of correlation from one moment in time to the next can physically be interpreted as "an almost continuous sequence of starting-up transients". It was seen that any transient in the motion of the charged droplet causes a starting-up delay of radiation. The lack of correlation in the chaotic motion will therefore lead to a state where a regular radiation field cannot develop.

# 8. The non-linear equations of motion, chaotic motion

It is convenient to introduce the parameter $\omega_0$ (as will be shown later, the physical meaning of $\omega_0$ is the "zero-speed frequency" of the pulsation), where:

$$\omega_0{}^2 = \frac{m(s_e)}{m^*(s_e)}\frac{c^2}{s_e{}^2}\mu\tau = \frac{\alpha}{\alpha^*}\frac{c^2}{s_e{}^2}\mu\tau \qquad (20)$$

which transforms the pulsation equation (12) into:

$$\frac{\ddot{s}}{s_e}\left[2\frac{s_e}{s} - 1\right] = \left[1 - \left(\frac{v}{c}\right)^2\right]\frac{\omega_0{}^2}{\mu\tau}\left(\frac{s_e}{s}\right)^2 +$$

$$-\omega_0{}^2\frac{s}{s_e}\left[1 - \frac{s_e}{s}\left(1 + \frac{1}{\mu} + \frac{1}{\tau}\right) + \left(\frac{s_e}{s}\right)^2\left(\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\mu\tau}\right)\right] \qquad (21)$$

The character of the non-linearities in this equation can be inferred by expanding eq. (21) in terms of the relative perturbation $\frac{\Delta s}{s_e}$ , where $\Delta s$ denotes a perturbation of the elongation w.r.t. the equilibrium value $s_e$ :

$$s(t) = s_e + \Delta s(t) = s_e\left(1 + \frac{\Delta s}{s_e}\right) \qquad (22)$$

The resulting equation in $\frac{\Delta s}{s_e}$ reads, after subtracting the equilibrium state and with the approximation $\left(1 - \frac{1}{\mu} - \frac{1}{\tau} + \frac{1}{\mu\tau}\right) \cong 1$ :

$$\frac{\ddot{\Delta s}}{s_e}\cdot\left(1 - 2\frac{\Delta s}{s_e}\right) + \omega_0^2\left[1 - \frac{2}{\mu\tau}\left(\frac{v}{c}\right)^2\right]\frac{\Delta s}{s_e} = -\left(\frac{v}{c}\right)^2\frac{\omega_0^2}{\mu\tau} + \cdots O\left(\frac{\Delta s}{s_e}\right)^2 \qquad (23)$$

For very small perturbations $\frac{\Delta s}{s_e}$ "1 , when complete linearisation would be allowed, eq. (23) reduces to a simple vibration equation. For increasing amplitudes of $\frac{\Delta s}{s_e}$ the non-linearities will gradually become more important, and will lead to a chaotic response. Multiplying by a factor $\left(1 - 2\frac{\Delta s}{s_e}\right)^{-1} = \left(1 + 2\frac{\Delta s}{s_e}\right) + O\left(\frac{\Delta s}{s_e}\right)^2$ shows that the type of non-linearity is comparable to a mass-spring system with one–sided spring softening. Ref.16 discusses the properties of a system with double-sided spring softening. In such a case, the chaotic response coexists with a periodic limit cycle, which occurs for small amplitudes *at the same control parameter values*. The same kind of behaviour may be expected in the case of eq. (23), since it also reduces to a normal vibration equation for small amplitudes.
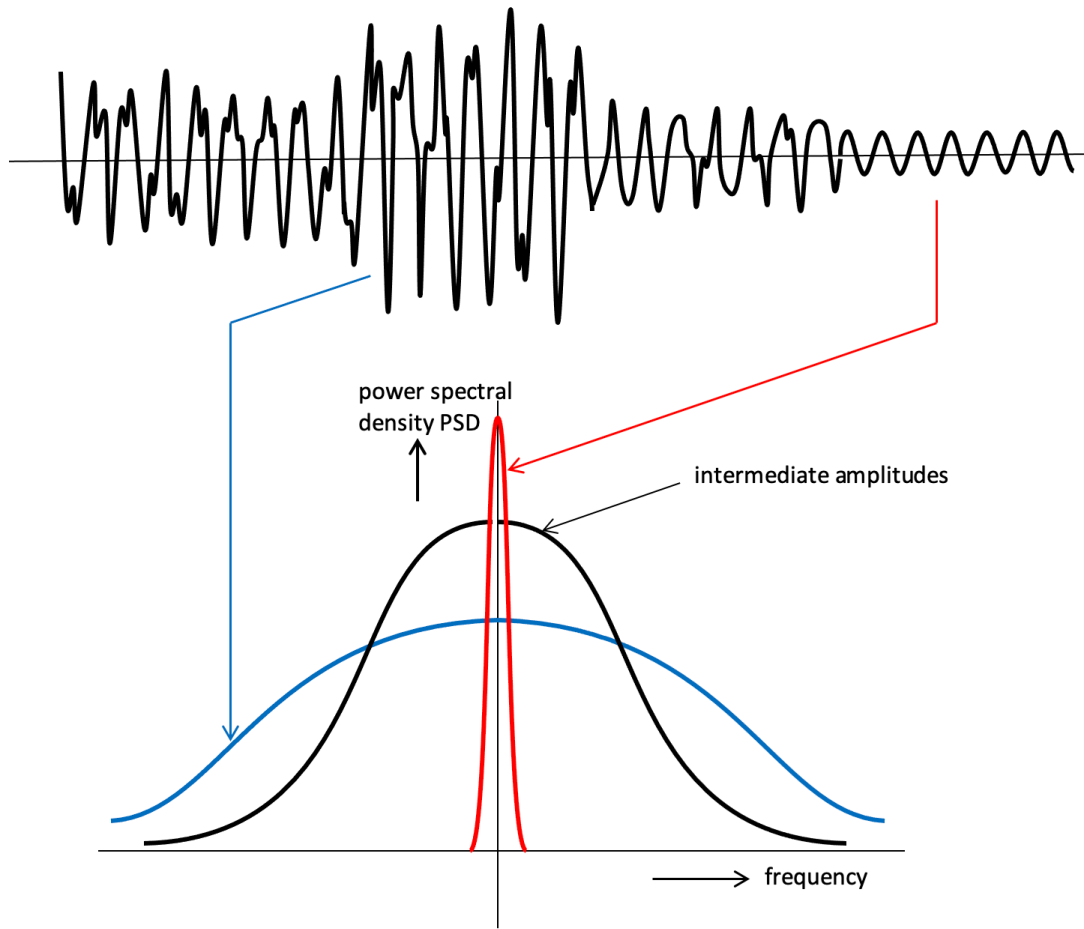
**Figure 4.** Power Spectral Density of chaotic signal (blue) and near–harmonic signal (red)

A characteristic of the non–linear behaviour is that slow alternations between small and large amplitudes of the chaotic motion randomly occur, thus accompanied by changes in the character of the signal from near–harmonic to fully chaotic. This property of the equation can be seen when all the non–linear terms in (23) are written in the r.h.s., and are then considered as "pseudo–forces" (somewhat comparable to d'Alembert's concept of inertia–forces), acting on the vibration equation as excitations. The "pseudo–force" $2 \frac{\Delta s}{s_e} \frac{\ddot{\Delta s}}{s_e}$ in the r.h.s. is non–conservative, so that it may have the effect of either an energy source or energy sink, thereby affecting the amplitude of the vibrations. As earlier remarked in sec.5.3, the total energy in the droplet is nevertheless constant thanks to energy transfers between the different dynamic modes (translation, pulsation and the elongation "spring").

The slow variations in amplitude of the chaotic pulsation $\frac{\Delta s}{s_e}(t)$ , spanning many short cycles of the pulsation, are schematically depicted in the upper part of fig.4. The variations are accompanied by

changes in the character of the signal. As a further illustration of the latter feature, the lower part of fig.4 shows – again very schematically– the Power Spectral Density (PSD) of the signal. Small amplitudes of the pulsation, associated with near–harmonic behaviour, correspond in the PSD–diagram with a sharp "spike", practically a delta–function. Large amplitudes, on the other hand, leading to chaotic motion, in the PSD–diagram show up as a broadening of the "spike" and a more or less wide spread of frequencies.

## 9. Linearised equations of motion

The frequency of the "spike" in fig.4 will hereafter be referred to as the "central frequency" of the pulsation. The characteristics of this central frequency are easily determined, because they follow from the completely linearised pulsation equation:

$$\frac{\ddot{\Delta}s}{s_e} + \omega_0^2 \left[ 1 - \frac{2}{\mu\tau} \left( \frac{v}{c} \right)^2 \right] \frac{\Delta s}{s_e} = - \left( \frac{v}{c} \right)^2 \frac{\omega_0^2}{\mu\tau} \qquad (24)$$

This equation will be combined with the linearised expression for the momentum $p_0$ , which is constant during "free flight" of the droplet, i.e. when in eq. (8) both $F_{external} = 0$ and $F_{rad} = 0$ (the latter due to chaotic motion and $\delta = 0$) :

$$v(t) = \frac{p_0}{m(s)} = v_{av} \left( 1 + 2 \frac{\Delta s}{s_e} \right) \qquad (25)$$

with $v_{av}$ , the so-called "average velocity" where the velocity "ripples" induced by the pulsation are ignored, defined as

$$v_{av} = \frac{p_0}{m(s_e)} \qquad (26)$$

Furthermore, the following abbreviation will be used:

$$k = \frac{1}{c^2} \frac{1}{\mu\tau} \qquad (27)$$

Combining eqs. (24) and (25) eliminates the variable $v\left(t\right)$ so that $\frac{\Delta s}{s_e}(t)$ is left as the only variable:

$$\frac{\ddot{\Delta}s}{s_e} + \omega_0^2 \left( 1 + 2k.\,{v_{av}}^2 \right) \frac{\Delta s}{s_e} = -k\omega_0^2 {v_{av}}^2 \qquad (28)$$

Note the change of sign in the second term, compared with eq. (24).

The particular solution of eq. (28), associated with the non-zero r.h.s., does not concern us in the following. From now on, we concentrate on the general solution associated with the homogeneous equation. The homogeneous pulsation equation is a simple vibration equation and has harmonic

functions as solution. The factor $k$ is very small, so that a good approximation of the pulsation frequency $\omega$ (i.e. the central frequency of the PSD-curve of fig.4) is:

$$\omega = \omega_0 \left(1 + k . v_{av}{}^2\right) \qquad (29)$$

The velocity ripples $\Delta v(t)$ , defined by $v = v_{av} + \Delta v$ , have − according to eq. (25) − the same frequency $\omega$ . The fact that the frequency increases with speed at first sight would appear to be in conflict with the predictions of special relativity (the slowing of clocks at speed). However, in the asymptotic expansions of the electromagnetic field, time dilation effects will show up as terms of the fourth asymptotic order, and are therefore neglected. The speed dependency in eq. (29) is caused solely by the cross-couplings between translational motion and pulsation of the droplet.

## 10. De Broglie's relation and Planck's constant

If we still assume free-flight conditions (i.e. constant momentum) with chaotic pulsation so that $\delta = 0$ (constant energy), and focussing on the central frequency of the PSD, eq. (28) has the solution:

$$\frac{\Delta s}{s_e} = A \cos(\omega t + \varphi_0) - k v_{av}{}^2 \qquad (30)$$

with $\omega$ given by equ. (29). Because of the small factor $k$ in eq. (28), this solution might be looked upon as a slightly perturbed form of the solution for $k = 0$ , which is $A \cos\left(\omega_0 t + \varphi_0\right)$ . The interpretation of (30) as a "perturbed harmonic function" is useful when eq. (28) (where $v_{av} = constant$ ) is considered as a degenerated case of a more general equation where $v_{av}(t)$ is a -slow- variable. With $v_{av}(t)$ variable, the equation is of the type "linear differential equation with variable coefficients", and belongs to another category than the straightforward vibration equation. Even with the $v_{av}(t)$ variable, the solution can still be seen as a "perturbed harmonic function". What will therefore be done in sec.11 dealing with the harmonic oscillator is to find an approximated solution of the differential equation in the form of an asymptotic series:

$$\frac{\Delta s}{s_e}(t) = \left[\frac{\Delta s}{s_e}(t)\right]_0 + k . \left[\frac{\Delta s}{s_e}(t)\right]_1 + O\left(k^2\right)(k \to 0) \qquad (31)$$

where the first term is the solution for $k = 0$ . If, in the present section, the same approach is chosen as in sec.11, we can immediately write down:

$$\left[\frac{\Delta s}{s_e}(t)\right]_0 = A \cos(\omega_0 t + \varphi_0) \qquad (32)$$

$$k.\left[\frac{\Delta s}{s_e}(t)\right]_1 = A\cos(\omega t + \varphi_0) - kv_{av}{}^2 - A\cos(\omega_0 t + \varphi_0) \qquad (33)$$

The two cosines in (33) with their slightly differing frequencies lead to a slow beat phenomenon, meaning that $\left[\frac{\Delta s}{s_e}(t)\right]_1$ is a fast oscillating function of time with a slow variation of its amplitude. What will be called here the "beat pattern" is the enveloping curve of the fast oscillation. In the case at hand, the beat pattern is not sinusoidal, but it is periodic so that we can still define a beat frequency $\omega_{beat} = \frac{2\pi}{T}$ with $T$ the repetition period. The frequency of the beat equals the frequency difference of the cosines in eq. (33), which is $\omega - \omega_0$ , so that according to eq. (29):

$$\omega_{beat} = \omega - \omega_0 = k\omega_0.v_{av}{}^2 \qquad (34)$$

Another interpretation of $k\omega_0.v_{av}{}^2$ follows directly from eq. (29): it is the frequency-*shift* due to velocity, which explains the terminology that $\omega_0$ is the so-called "zero-speed frequency". Therefore, $\omega_{beat}$ can also be interpreted as the beat frequency observed when the field of a moving and a static droplet interfere.

If we define the wavelength $\lambda_{beat}$ of the beat pattern as the distance travelled by the droplet between two consecutive beats, then:

$$\lambda_{beat} = v_{av}T_{beat} = v_{av}\frac{2\pi}{\omega_{beat}} = \frac{2\pi}{k\omega_0}\frac{1}{v_{av}} = 2\pi\frac{m\left(s_e\right)}{k\omega_0}\frac{1}{p_0} \qquad (35)$$

Using Eqs. (4) and (2) to express $m\left(s_e\right)$ , and substituting eqs. (20) and (27) for $\omega_0$ and $k$ , eq. (35) can be cast in the form

$$p_0 = \frac{K}{\lambda_{beat}} \qquad (36)$$

$$K = \frac{2\pi}{k\omega_0}m\left(s_e\right) = \frac{1}{c}\frac{q^2}{4\pi\varepsilon_0}\sqrt{\mu\tau}.\beta \qquad (37)$$

$$\beta = \sqrt{\pi^2\alpha\alpha^*} \qquad (38)$$

Eq. (36) looks remarkably similar in form to De Broglie's equation $p = h/\lambda$ (with $h$ Planck's Constant and $\lambda$ the wavelength of the matter wave "accompanying" the particle). We now compare the numerical value of the constant $K$ of the droplet theory with the value of Planck's Constant $h = 6.6256 \times 10^{-34}\,J.s$ . For this purpose, the following values are substituted into eq. (37): $c = 2.9979 \times 10^8\frac{m}{s}$ ; $q = 1.6021 \times 10^{-19}\,C$ ; $\varepsilon_0 = 8.8544 \times 10^{-12}N^{-1}m^{-2}C^2$ ; $\mu = 206.85$ ; $\tau = 3477.1$

Apart from these constants of nature, the constant $K$ also depends on the factor $\beta$ , so that it depends on how the charge is distributed within the droplet. The precise value of the factor $\beta$ is thus unknown: in theory, as it stands now, no assumptions were introduced (nor were they needed) about the variation of

the electrical charge density throughout the droplet. The only thing known is that the factor is of order unity: $\beta = O(1)$ (see the remarks about the orders of $\alpha$ and $\alpha^*$ in secs. 4.2, 5.5 and app.B). The best estimate that presently can be made is therefore $\beta = 1$. We then find

$$K = 6.52 \times 10^{-34} \; J.s \; \; for \; \; \beta = 1 \qquad (39)$$

The difference with Planck's constant is 1.5 %, which, if it is interpreted as an error, is quite acceptable in view of the fact that the present droplet theory is just of an exploratory nature (one-dimensional, matched asymptotic expansions truncated after three terms, linearisations, etc.). This result lends credibility to the supposition that the three leptons correspond to the three equilibria of a deformable droplet of charge. The result should be understood to belong strictly to the model world of Em.QM. Conventional QM is based on different kinds of models and laws, so that the above result is meaningless within the scope of QM. The dichotomy between the two theories shows up very clearly in the interpretation of the wavelength in De Broglie's formula: in QM, this concerns the wavelength of so-called "matter waves", in Em.QM it is associated with a beat phenomenon buried in the pulsation and the velocity fluctuations of the droplet model.

# 11. The harmonic oscillator

## 11.1. Preliminaries

The case considered in the present section is a droplet of charge moving inside a parabolic potential well. If the droplet did *not* have pulsation freedom, its translational motion would be the same as that of a macroscopic "marble" moving inside a parabolically shaped bowl:

$$v_{av}\left(t\right) = V \; sin\left(\Omega t\right) \qquad (40)$$

where the notations $V$ and $\Omega$ are defined in fig.5, and $v_{av}\left(t\right)$ here means the –variable– droplet speed when the velocity ripples induced by the pulsation are ignored. The maximum velocity $V$ of the "marble" occurs in the lowest point of the well, where all the potential energy has been exchanged for kinetic energy. Therefore, during the complete cycle of the motion, the marble's total energy $E$ is constant and equals $E = \frac{1}{2}m(s_e)V^2$ . Note that the angular velocity $\Omega$ does not depend on the energy $E$ in the case of the parabolic well.
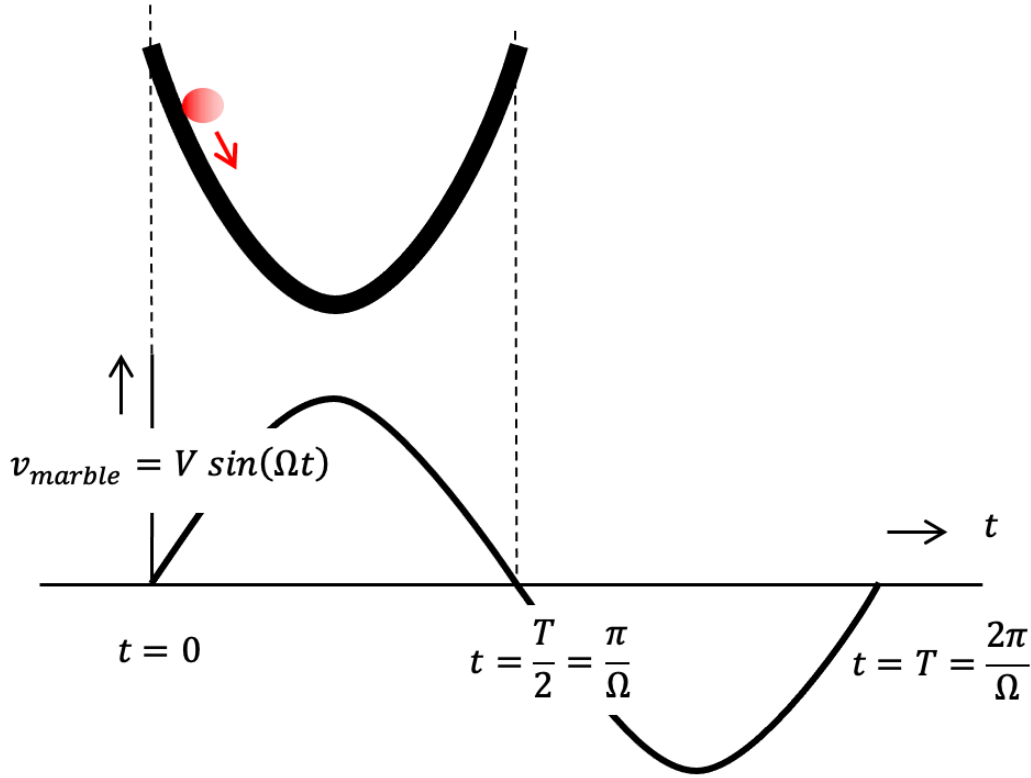
**Figure 5.** Notations classical harmonic oscillator

The mode of pulsation of the deformable droplet is assumed to be chaotic, so that its total energy (not to be confused with the energy $E$ of the corresponding "marble") is constant because of the lack of dissipation by radiation. The pulsation at the central frequency of the PSD-curve of fig.4 is, apart from a constant, described by the homogeneous differential equation

$$\frac{\ddot{\Delta s}}{s_e} + \omega_0^2 \left[1 + 2kV^2 \sin^2(\Omega t)\right] \frac{\Delta s}{s_e} = 0 \qquad (41)$$

If the value of $k$ were exactly zero, the pulsation would be $\frac{\Delta s}{s_e} = A\cos(\omega_0 t + \varphi_0)$ . The so-called "zero-speed pulsation frequency" $\omega_0$ is much larger (as shown in ref.3) than the perturbation frequency $\Omega$ so that $\frac{\Omega}{\omega_0}$ "1 . Using this property, the rather meaningless factor $k$ in the equation of motion eq. (41) can be replaced by an alternative, physically more transparent factor. In view of the results from sec.10, eq. (37) and $K \approx h = 2\pi \hbar$ , the following approximation may be used:

$$\frac{\omega_0 k}{m(s_e)} \approx \frac{1}{\hbar} \qquad (42)$$

which, substituted into (41), leads to the following modified form of the pulsation equation:

$$\frac{\ddot{\Delta s}}{s_e} + \omega_0^2 \left[ 1 + 4\frac{E}{\hbar\Omega}\frac{\Omega}{\omega_0}\sin^2(\Omega t) \right] \frac{\Delta s}{s_e} = 0 \qquad (43)$$

## 11.2. Approximate solution of the pulsation equation

Eq. (43) can be transformed into a Mathieu equation, so that exact solutions (the so-called Mathieu functions) are available. In ref.3, this approach has been chosen to find the energy quantisation of the harmonic oscillator. However, we can avoid the complexities of Mathieu-functions, by using the property $\frac{\Omega}{\omega_0}''1$ and linearising by neglecting terms of the order $O\left(\frac{\Omega}{\omega_0}\right)^2$ . Later, it will be seen that the factor $\frac{E}{\hbar\Omega}$ is of the order $O(1)$ , in agreement with the results of the usual QM theory. Therefore, the second term within the square brackets of eq. (43), which causes the perturbation w.r.t. the exact harmonic solution, is of the order $O(\frac{\Omega}{\omega_0})$ . Following the same procedure as in sec.10, we again try to determine the solution in the form of an asymptotic series:

$$\frac{\Delta s}{s_e}(t) = \left[ \frac{\Delta s}{s_e}(t) \right]_0 + \left( \frac{\Omega}{\omega_0} \right) \cdot \left[ \frac{\Delta s}{s_e}(t) \right]_1 + O\left( \frac{\Omega}{\omega_0} \right)^2 \left( \frac{\Omega}{\omega_0} \to 0 \right) \qquad (44)$$

$$\left[ \frac{\Delta s}{s_e}(t) \right]_0 = A\cos(\omega_0 t + \varphi_0) \qquad (45)$$

In order to determine the next term of the asymptotic series, an approximation of $\frac{\Delta s}{s_e}(t)$ is constructed by heuristic means, which can afterwards (in App.A) be proven to be accurate up to and including the order $O\left( \frac{\Omega}{\omega_0} \right)$ .

The small value of $\frac{\Omega}{\omega_0}$ suggests that the solution of eq. (43) might be approximated by a near-harmonic function with a variable frequency, in other words, by an FM (Frequency Modulated) signal. Eq. (43) suggests that the variable frequency $\omega_{FM}(t)$ of this FM-signal is:

$$\omega_{FM}(t) = \omega_0 \left[ 1 + 2\frac{E}{\hbar\Omega}\frac{\Omega}{\omega_0}sin^2(\Omega t) \right] \qquad (46)$$

The FM-signal itself may then be written in the form of a sinusoidal function with a non-uniformly increasing phase angle $\varphi(t)$ :

$$\left( \frac{\Delta s}{s_e} \right)_{FM} = A\cos\varphi(t) \qquad (47)$$

where the phase-angle $\varphi(t)$ , i.e. the integral of the frequency, is given by:

$$\varphi(t) = \int_0^t \omega_{FM}(\tau).d\tau = \omega_0 t + \frac{E}{\hbar\Omega}\left\{ \Omega t - \frac{1}{2}sin(2\Omega t) \right\} + \varphi_0 \qquad (48)$$

In appendix A, this tentative solution is back–substituted into the differential equation (43), and an ordering scheme w.r.t. the relative magnitude of the terms is applied. As expected, it then appears that $\left(\frac{\Delta s}{s_e}\right)_{FM}$, as given by eqs. (47) and (48), does not satisfy the differential equation exactly; in other words, the r.h.s. is not exactly zero after the substitution. However, the residue in the r.h.s. is of the order $O\left(\frac{\Omega}{\omega_0}\right)^2$. It may be concluded that the near harmonic FM-approximation satisfies eq. (43) with *sufficient* accuracy, and that the solution of eq. (43), to be inspected further, is thus given by

$$\frac{\Delta s}{s_e} = A \cos\left[\omega_0 t + \frac{E}{\hbar\Omega}\left\{\Omega t - \frac{1}{2}\sin(2\Omega t)\right\} + \varphi_0\right] \qquad (49)$$

## 11.3. Quantisation of energy

A physical requirement is that the pulsation pattern as given by eq. (49) repeats itself with a period $T = \frac{2\pi}{\Omega}$ , i.e. the same period as the rolling of the "marble" inside the parabolic bowl (see fig.5). Consequently, substituting eq. (49) into the requirement $\frac{\Delta s}{s_e}\left(t + \frac{2\pi}{\Omega}\right) = \frac{\Delta s}{s_e}(t)$ gives a constraint to guarantee the correct repetition of the physical phenomena:

$$\left(\frac{\omega_0}{\Omega} + \frac{E}{\hbar\Omega}\right) = n \quad (n \text{ integer}) \qquad (50)$$

From eq. (50) it follows that $\frac{E}{\hbar\Omega}$ is quantised. Though $\frac{\omega_0}{\Omega}$ may be very large, its value is fixed so that raising the integer $n$ by one unit also requires that $\frac{E}{\hbar\Omega}$ increases by one unit. In other words, the spacing between the levels is

$$\Delta\left(\frac{E}{\hbar\Omega}\right) = 1 \qquad (51)$$

## 11.4. Energy levels of the harmonic oscillator

The lowest possible value $\left(\frac{E}{\hbar\Omega}\right)_0$ is not determined by the relation (50). For the purpose of determining the complete stack of allowed values, we again "distill" the slow dynamics out of the pulsation, in the same way as in the previous sec.10. We return to the asymptotic expansion (44) where the term of order $O\left(\frac{\Omega}{\omega_0}\right)$, in which the slow dynamics are buried, reads:

$$\left(\frac{\Omega}{\omega_0}\right) \cdot \left[\frac{\Delta s}{s_e}(t)\right]_1 = A \cos\left[\omega_0 t + \frac{E}{\hbar\Omega}\left\{\Omega t - \frac{1}{2}\sin(2\Omega t)\right\} + \varphi_0\right] - A \cos(\omega_0 t + \varphi_0) \qquad (52)$$

The phase angles of both the cosines in eq. (52) are drawn in the schematic fig. 6, with $-cos\left(\omega_0 t + \varphi_0\right) = cos\left(\omega_0 t + \varphi_0 - \pi\right)$ . Needless to say, the figure is entirely out of proportion, since in

actual fact $\frac{\omega_0}{\Omega} \frac{E}{\hbar\Omega}$. The sum of cosines in eq. (52) with their slightly differing phase-histories leads to a beat, i.e., a high frequency oscillation with a slow variation of its amplitude.
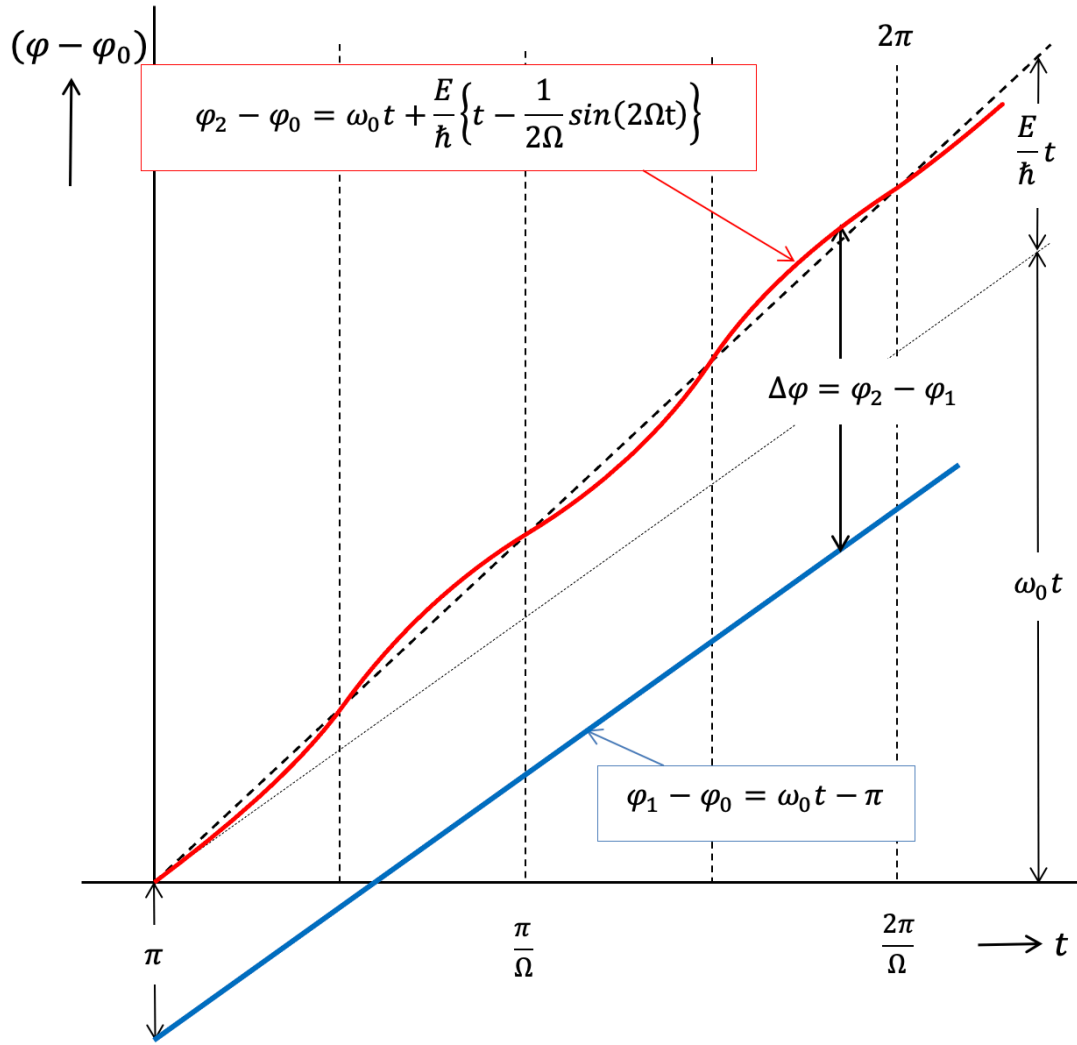


**Figure 6.** Phase angle $\varphi_2$ (red) of the approximated pulsation eq.(49), compared with phase angle $\varphi_1$ (blue) of the unperturbed oscillation, $\cos(\omega_0 t + \varphi_0 - \pi)$

In fig.6, at time $t = 0$, the phase difference $\Delta\varphi$ between the two cosines equals $\Delta\varphi = \pi$, and the two cosines cancel each other. As time progresses, the two phases diverge further in value, and eventually will show a phase difference of $\Delta\varphi = 2\pi$, at which point in time the cosines add up and the beat has a maximum. The distance between the two extremes of the beat is determined by a growth of $\Delta\varphi$ by an amount $\pi$. The time interval between the maxima and minima of the beat thus depends on the growth rate $\frac{d}{dt}(\Delta\varphi)$ where

$$\Delta\varphi = \varphi_2 - \varphi_1 = \frac{E}{\hbar}t - \frac{E}{\hbar\Omega}\frac{1}{2}\sin(2\Omega\text{t}) + \pi \qquad (53)$$

The growth rate $\frac{d}{dt}(\Delta\varphi)$ thus depends on the value of $\frac{E}{\hbar}$ . We can conclude that there must be a non–zero minimum value of $\frac{E}{\hbar}$ . Since the repetition time of the physical system is the period $T = \frac{2\pi}{\Omega}$ (see fig.5), for very small values of $\frac{E}{\hbar}$ , one would find a minimum of the beat pattern in one physical cycle, and the next maximum in a later cycle. This would violate the physical requirement of periodicity. The situation of the smallest allowed value of $\frac{E}{\hbar}$ is the one depicted in fig.6. Counting from $t = 0$ , the phase difference $\Delta\varphi = \varphi_2 - \varphi_1$ has increased by an amount $\pi$ at the time $t = T = \frac{2\pi}{\Omega}$ :

$$[\Delta\varphi]_{\Omega t=2\pi} - [\Delta\varphi]_{\Omega t=0} = \pi \qquad (54)$$

Substituting eq. (53) this means that the lowest energy level is given by

$$\left(\frac{E}{\hbar\Omega}\right)_0 = \frac{1}{2} \qquad (55)$$

Repeatability with the period $T = \frac{2\pi}{\Omega}$ is assured if, using the requirement (50), $\frac{\omega_0}{\Omega}$ has a half–integer value. Combining eq. (55) with the earlier result about the spacing between the allowed levels $\Delta\left(\frac{E}{\hbar\Omega}\right) = 1$ yields for the complete sequence of allowed energy levels:

$$\left(\frac{E}{\hbar\Omega}\right)_k = \frac{2k+1}{2} \; (k = 0, 1, 2, \ldots) \qquad (56)$$

This result is in agreement with the results found in conventional QM by solving Schrödinger's equation. An analysis of the stability of the droplet's motion at the so-called "allowed" energy levels $E_k$ has been given in ref.3, and will not be copied here.

It must again be stressed that the quantisation of $E$ is not the same as a quantisation of the total energy of the droplet. This total energy comprises contributions not only by the average velocity, but also by the potential energy in the elongation "spring", as well as by the pulsations and the velocity fluctuations. The total energy of the droplet need not even be quantised at all.

## 11.5. Planck's Energy- frequency relation

The present paper does not deal with the equations of motion applicable to energy loss or energy capture due to radiation. The latter situation will be the subject of another paper. Even so, eq. (49) can give some information about the transition from one of the allowed energy levels to another, for instance, from $E_k$ to the adjacent lower level $E_{k-1}$ , where according to the above $\frac{E_k}{\hbar\Omega} - \frac{E_{k-1}}{\hbar\Omega} = 1$ . Let us hypothesize that there is a gradual transition between these levels, meaning that the mode of motion at level $E_k$ in

some -in the present paper not further specified- way "extinguishes" whilst at the same time the mode $E_{k-1}$ grows. Both levels are accompanied by different modes of oscillation $\frac{\Delta s}{s_e}(t)$, in accordance with eq. (49), so that during the short transition time there will be a superposition of these oscillations. The superposition of oscillations leads to a beat, as is evidenced by inspecting the phase difference between the two modes:

$$\varphi_k - \varphi_{k-1} = \frac{E_k - E_{k-1}}{\hbar\Omega}\left\{\Omega t - \frac{1}{2}sin\left(2\Omega t\right)\right\} = \Omega t - \frac{1}{2}sin\left(2\Omega t\right) \qquad (57)$$

The accompanying beat pattern will be near-harmonic, with periodicity $\frac{2\pi}{\Omega}$ . Although the beat pattern will contain more harmonics, the dominant frequency in it is

$$\omega_{transition} = \Omega = \frac{E_k - E_{k-1}}{\hbar} \qquad (58)$$

In the case of an energy jump $E_k \rightarrow E_{k-2}$ (so that $\Delta E = 2\hbar\Omega$), the dominant beat frequency observable in the field is $\omega_{transition} = 2\Omega = \frac{E_k - E_{k-2}}{\hbar}$ . In general, we find for all the possible energy jumps:

$$\omega_{transition} = \frac{\Delta E}{\hbar} \qquad (59)$$

Eq. (59) is the same relation as was postulated in 1913 by Bohr to determine the colour of the light emitted by heated gases or, in reverse, the spectral line absorbed from a ray of light passing through a gas. The derivation by Em.QM clears up one of the mysteries, already noted by Einstein, concerning light emission: the frequency of the emitted light is not related to the vibration of a charge in the emitter, despite the anticipation from Maxwell's equations. According to the droplet theory of Em.QM, the observed light has the frequency of a beat, which is associated with two vibration modes of a charge. The "roots" of the emission can thus be found in the vibrations of a charge, fully in accordance with Maxwell's laws.

If the energy $\Delta E$ lost by the charge is equated to the energy transmitted by the field, the above-written equations immediately lead to:

$$E_{field} = \hbar.\,\omega_{transition} = h.\,\nu \qquad (60)$$

where $\nu$ is the beat frequency in Hz. Eq. (60) is in agreement with Planck's formula for the radiation by emitters. Note that the above does not cover Einstein's extended interpretation of the formula, implying that eq. (60) is a characteristic of the field instead of the transmitter. How the concept of the "photon" is interpreted in Em.QM will be the subject of a future publication, where an update of the theory of ref.3 will be given.

## 11.6. Energy width

The above-shown derivations concern the so-called central frequency of the Power Spectral Diagram of the chaotic signal $\frac{\Delta s}{s_e}(t)$. The so-determined energy levels will be called the "nominal" levels. Due to the actual spread of frequencies around the central frequency, the allowed levels are not sharply defined discrete lines but instead are a little diffuse. In QM, this uncertainty of the allowed levels is sometimes called the "energy width" (see e.g. ref. 15) and is attributed to Heisenberg's uncertainty principle. In Em.QM, it is a consequence of the chaotic motion.

## 11.7. Violation of classical boundaries and tunnelling

The energy width of the allowed levels also explains why, in a potential well, the classical boundaries of the motion can be trespassed. The process is sketched in fig.7. The classical boundary of the droplet's motion coincides with the nominal energy level. The energy width of the nominal level then leads to the possibility of "violating" the classical boundary.
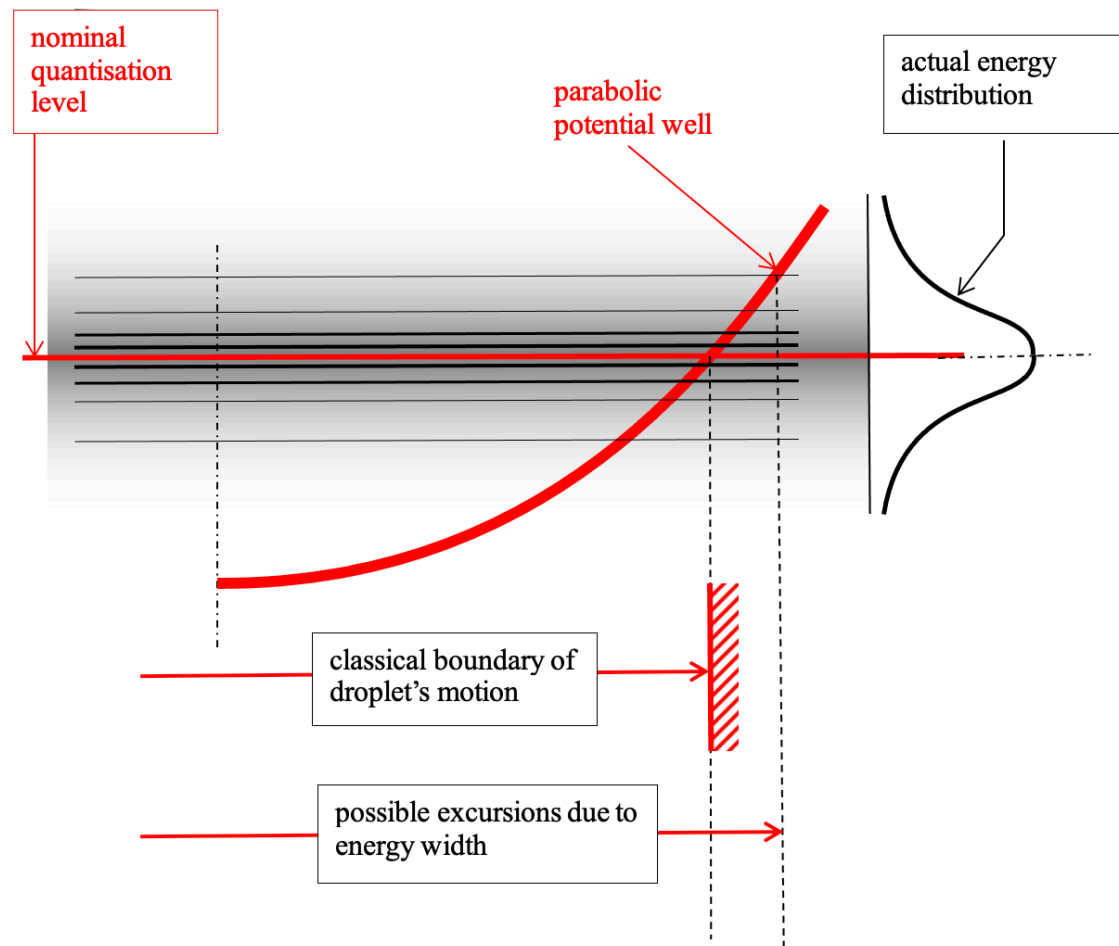
**Figure 7.** Violation of classical boundaries of the potential well

An alternative view is possible on the violation of the classical limits of motion, which also clarifies the blurring of the energy levels. In sec.5.3, it was mentioned that the total energy of the droplet remains constant throughout the changes of the droplet's motion pictured in fig.4. This is possible because the droplet has two degrees of freedom, viz. the translation $z_m(t)$ and the length variations $s(t)$ . The two degrees of freedom represent two -communicating- energy reservoirs. The kinetic energy in the pulsating and the velocity ripples may be variable, but this is compensated by changes in the elongation, the latter serving as a spring storing potential energy. In ref.3, a proof is given that the conservation of the total energy in the droplet is a general characteristic of the non-linear equations of motion, and does not depend on the particular type of motion considered.

There is a randomly occurring, mutual exchange of energy between the two dynamic modes of the droplet. If one focuses the attention solely on the translation, one will only see a fluctuation of energy in

this mode of motion. Sometimes, the energy in the translation is larger than its average value, and at other times, the energy is smaller. This Em.QM characteristic of particle behaviour corresponds with the time-energy uncertainty in usual QM, because the second energy reservoir consisting of the length variations is "invisible" in the conventional QM-theory. Sometimes, the energy fluctuations in QM are formulated as "borrowing energy from the uncertainty". In Em.QM, the loan process is a physical reality, viz. energy is borrowed by one of the degrees of freedom from the other. This explains why the translational energy of the droplet is sometimes larger than nominal, enabling the droplet to exceed the classical limits. If the potential well is capped, the same process leads to the possibility of "tunnelling", i.e. escaping from the well entirely.

## 12. Discussion and conclusions

The droplet theory of Em.QM as it has been developed until now is no more than a first exploration of the consequences of assuming a non-singular electron model. The model itself has been restricted to the bare minimum of one extra degree of freedom (the elongation of an ellipsoid) next to the back-and-forth motion within a one-dimensional potential well. The core of the mathematical analysis of the self-forces is an asymptotic expansion procedure (the so-called "max" approach) truncated after the terms of order $O\left(\frac{v}{c}\right)^3$, and the equations of motion were linearised in the parameter $k = \frac{1}{\mu\tau}\left(\frac{v}{c}\right)^2$.

Even so, the consequences of this non-singular model are intriguing. A list of achievements has already been given in the introduction and will not be repeated here. The list comprises the basic quantum mechanics expressions $E = h\nu$ and $p = \frac{h}{\lambda}$, derived by the droplet theory with a practically correct numerical value of Planck's Constant $h$. Another remarkable result is the possible explanation of why there are three charged leptons (electron, muon and tau-particle) and where the differences in their masses may come from.

The analysis of the quantum harmonic oscillator, as given in the present paper, is simpler and more transparent than in earlier publications (at least in the author's subjective opinion), and yields the correct stack of discrete energy levels. It also can explain why, apart from the short moments of transitioning between energy levels, there is no loss of energy due to radiation. What exactly happens during such a transition has not been worked out yet in full, although in ref.3, formulae for the shedding of energy have been derived, and a theory is proposed on how to view the phenomenon of "photons" within the scope of Em.QM. These subjects, including several recent updates of this theory, will be described in a future paper.

Finally, Em.QM is also able to explain how the excursions of the droplet within the parabolic well can violate the classical limits, and how tunnelling out of a capped potential well can be explained without violating the principle of conservation of energy, nor the need to invoke an uncertainty principle (which would be incompatible with the classical laws anyway).

In Feynman's opinion, any theory that cannot explain the double-slit experiment is immature, or it might even turn out to be outright wrong. The results obtained so far by the droplet theory were strictly limited to one-dimensional motion, so that the double-slit diffraction cannot yet be covered by this theory. In ref.3, a chapter is included that speculates about future extensions of the droplet theory, and the results that may reasonably be expected. It is therefore made plausible that a good chance exists that the droplet theory can correctly describe the double-slit experiment. Another reason to expect this is the fact that the oil droplet experiments by Couder et al., a two-dimensional analogue of quantum mechanics, show the expected diffraction of the path of oil droplets "walking" through a doubly slitted barrier.

If the droplet theory is in the future extended by adding more degrees of freedom, due consideration should also be given to the phenomenon of "spin". In the theory by Anderson and Brady (refs.8 and 9) concerning the experimental oil-droplet analogues of QM, spin has already been covered. Due to the fact that the theory of a droplet of charge until now was restricted to one-dimensional motion, spin could not and did not need to be included. It did not hamper the study of one-dimensional potential wells, but during further investigations of Em.QM spin will soon be essential to include.

There is one –seemingly compelling - objection to the expectation that Em.QM might eventually cover the whole of QM phenomena. This is the subject of entanglement and non-local behaviour or, in Einstein's words, "spooky action at a distance". It is often thought that the classical laws will never be able to show this kind of behaviour. This opinion was contradicted by Anderson and Brady in ref.9, who gave an example of a conspiracy of the classical laws where a so-called Bell-test showed the non-local behaviour of a macroscopic system. A summary of their paper is also discussed in ref.3, where it is concluded that the droplet theory, in principle, permits non-local phenomena to occur, and how this could be explained physically. In fact, in ref.3, not only a heuristic explanation is given, but also the mathematical foundations are described (although not fully worked out) of the phenomenon of non-local effects in droplet theory. A future paper is being planned where all this is elaborated, including recent updates of the material found in ref.3.

Overseeing the not-inconsiderable achievements but also the many remaining white areas on the Em.QM-map, one must conclude that future extensions of the theory might either lead to a complete

abandonment of this field or, on the contrary, lead to something bordering on a paradigm change. Em.QM, anyway deserves much more attention than it has been given until now.

# Appendix

## *Appendix A: Ordering scheme of terms in the equation of motion*

The differential equation which has to be satisfied by a proper choice of the function $\frac{\Delta s}{s_e}(t)$ is eq. (43):

$$\frac{\ddot{\Delta s}}{s_e} + \omega^2 \frac{\Delta s}{s_e} = 0 \qquad (\text{A.1})$$

where

$$\omega = \omega_0 \left(1 + 2\frac{E}{\hbar\Omega}\frac{\Omega}{\omega_0} sin^2(\Omega t)\right) \qquad (\text{A.2})$$

A tentative approximation of the solution is the following FM–modulated near–harmonic function, eqs. (47) and (48):

$$\left(\frac{\Delta s}{s_e}\right)_{FM} = A\cos\left[\int_0^t \omega(\tau)\cdot d\tau + \varphi_0\right] \qquad (\text{A.3})$$

By back–substitution of (A.3) into (A.1), one finds that $\left(\frac{\Delta s}{s_e}\right)_{FM}$ , in reality, satisfies the following differential equation:

$$\left(\frac{\ddot{\Delta s}}{s_e}\right)_{FM} + \omega^2\left(\frac{\Delta s}{s_e}\right)_{FM} = -A\dot{\omega}\sin\left[\int_0^t \omega(\tau)\cdot d\tau + \varphi_0\right] \qquad (\text{A.4})$$

Because of the high frequency of $\frac{\Delta s}{s_e}(t)$, the two terms on the left–hand side of eq. (A.4) have the same order of magnitude. This can be made explicit by non–dimensionalising:

$$\begin{aligned} &\frac{d^2(\Delta s/s_e)_{FM}}{d(\omega_0 t)^2} + \left(1 + 4\frac{E}{\hbar\Omega}\frac{\Omega}{\omega_0}\sin^2(\Omega t)\right)\left(\frac{\Delta s}{s_e}\right)_{FM} = \\ &= -A.2\frac{E}{\hbar\Omega}\left(\frac{\Omega}{\omega_0}\right)^2 \sin(2\Omega t).\sin\left[\int_0^t \omega(\tau).d\tau + \varphi_0\right] \end{aligned} \qquad (\text{A.5})$$

The conclusion is that on the l.h.s. the perturbation term with $\frac{E}{\hbar\Omega}$ is of the order $O\left(\frac{\Omega}{\omega_0}\right)$ w.r.t. the other terms on the l.h.s., whereas the r.h.s. of the equation is of the order $O\left(\frac{\Omega}{\omega_0}\right)^2$.

## *Appendix B: The relative order of magnitude of the squeezing inertia*

In refs.3 and 4, the following relations are derived:

$$m_{em} = 2\frac{U_{es}}{c^2} \qquad \text{(B.1)}$$

$$U_{es} = \frac{1}{2} \iiint_{droplet} \sigma . \Phi_{el.static} dx. dy. dz = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{s}\alpha \qquad \text{(B.2)}$$

where $\sigma$ denotes the charge density, and $\Phi_{el.static}$ the electrostatic potential. The squeezing inertia is likewise related to an electrostatic energy:

$$m_{em}{}^* = 2\frac{U_{es}{}^*}{c^2} \qquad \text{(B.3)}$$

where $U_{es}{}^*$ denotes the electrostatic energy of a charge distribution with charge density $\sigma^* = \sigma . \left(\frac{z-z_m}{s}\right)$:

$$U_{es}{}^* = \frac{1}{2} \iiint_{droplet} . \sigma^* \Phi^*_{el.static} dx. dy. dz = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{s}\alpha^* \qquad \text{(B.4)}$$

Both $\sigma^*(z)$ and $\Phi^*_{el.static}(z)$ are anti–symmetric functions w.r.t. the "equator plane" $z = z_m$ of the droplet. The integrand contains the product $\sigma^* \Phi^*_{el.static}$ , so that the charge distribution near the plane $z = z_m$ hardly contributes to the integral $U_{es}{}^*$ . A crude estimate of the integral may be made by using an "effective" distribution consisting of two "outboard" lumps of charge, one positive and one negatively charged. In order to get a feeling for the order of magnitude of $U_{es}{}^*$ in comparison with $U_{es}$ , one could for instance assume that the total charge in each lump, equal to $\frac{1}{8}q$ , has been collected in rectangular blocks between $|z - z_m| = \frac{1}{4}s$ and the tip at $\frac{1}{2}s$ . The energy of formation of the two lumps together would then amount to $\frac{1}{8}U_{es}$ , but is somewhat less than this value due to the mutual attraction between the lumps.

Needless to say, if an assumption is made concerning the actual charge distribution within the droplet, a better estimate might be made. However, until now, no assumptions about the charge distribution have been made in the droplet theory, nor were they needed.

## Notations

Symbols recurring throughout the text are here listed. Non–recurring symbols are explained immediately underneath the expressions in which they are used.

- $a$ width of potential well (see fig.2)
- $A, B, C$ constants in apparent surface tension
- $c$ velocity of light
- $E$ energy associated with "marble–like" translational motion
- $F$ radiation resistance corresponding to outgoing waves
- $F_{external}$ force due to external electric field

- $F_{rad}$ radiation resistance due to combinations of incoming and outgoing waves

- $F_\times$ cross-coupling force

- $h$ Planck's constant

- $\hbar$ $\frac{h}{2\pi}$

- $k$ constant defined in Equation (27)

- $k$ integer counter in eq.56

- $K$ constant in "De Broglie–like" relation, defined in eq. (37), $K \cong h$

- $L$ Lagrangian, $L = T - U$

- $m$ inertial mass ("translational inertia")

- $m^*$ generalised mass in s-direction ("squeezing inertia")

- $m_{em}$ electromagnetic mass

- $m_{em}{}^*$ generalised electromagnetic mass in s-direction

- $n$ integer counter

- $p$ momentum

- $p_{av}$ average momentum without pulsation effects

- $q$ total charge of droplet

- $q_i$ generalised coördinate

- $Q_i$ generalised force

- $Q_{s,electr}$ Lagrangian generalised expansion force in s-direction due to internal repulsion

- $Q_{s,surf.tension}$ Lagrangian generalised squeezing force in s-direction due to surface tension

- $s$ length of droplet, see fig.2

- $\Delta s$ perturbation of equilibrium length ("pulsation")

- $\delta s$ virtual displacement

- $s_e$ static equilibrium length

- $t$ time

- $T$ kinetic energy

- $U$ potential energy

- $U_{es}$ electrostatic energy ("formation energy") of droplet

- $U_{es}^*$ "generalised" electrostatic energy defined in eq. (B.4)

- $v, v_m$ velocity of midpoint, as defined in fig.2

- $v_{av}$ average ("marble") velocity ignoring fast velocity ripples

- $\Delta v$ velocity perturbation induced by pulsations

- $V$ maximum velocity of classical harmonic oscillator

- $W$ total radiated energy

- $\alpha$ form factor in $m_{em}$

- $\alpha^*$ form factor in $m_{em}{}^*$

- $\beta$ constant defined in eq. (38)

- $\delta$ factor indicating the amount and sign of dissipation by radiation resistance

- $\varepsilon_0$ vacuum permittivity

- $\lambda_{beat}$ wavelength of beats

- $\mu$ mass ratio muon/common electron

- $\tau$ mass ratio tau particle/common electron

- $\varphi_{0,1,2}$ Phase angles, defined in fig.6

- $\omega$ frequency of $\Delta s(t)$ and $\Delta v(t)$

- $\omega_0$ "zero-speed frequency"

- $\Omega$ frequency of classical harmonic oscillator

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## Declarations