

Research Article

Multiplicity-Isolated Cores and Chromatic Symmetric Reconstruction of Trees

Zijian Zeng¹

1. Institute of Computer Science and Digital Innovation, UCSI University, Malaysia

Stanley's tree-isomorphism conjecture asks whether the chromatic symmetric function distinguishes nonisomorphic trees. We give a reconstruction criterion that permits repeated leaf-component orders. For a proper tree, collapse each leaf component to its center and record its order as a vertex weight. We prove that the chromatic symmetric function reconstructs the tree whenever every nonleaf vertex of this weighted core has a weight that occurs nowhere else in the core. Repetitions among core leaves are unrestricted. The proof uses only the leading star-basis partition and the coefficients immediately above it. As a consequence, the conjecture holds for an infinite class of diameter-six trees not covered by the condition that all leaf-component orders are distinct. We also give a canonical integer-partition model for arbitrary diameter-six trees and an exact cut-partition implementation intended for further work. The unrestricted diameter-six case remains open.

Corresponding author: Zijian Zeng, 2119516028@qq.com

1. Introduction

For a finite simple graph G , Stanley's chromatic symmetric function is

$$\mathbf{X}_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)},$$

where the sum is over all proper colorings by positive integers [1]. Stanley asked whether $\mathbf{X}_T$ distinguishes trees up to isomorphism. The conjecture remains open in general.

Small-diameter reconstruction has recently advanced through the star basis and the deletion-near-contraction relation. Aliste-Prieto, de Mier, Orellana, and Zamora proved the conjecture for proper trees of diameter at most five [2]. Gonzalez, Orellana, and Tomba then removed the properness assumption and

reconstructed every tree of diameter strictly less than six [3]. Diameter six is the next case not covered by that theorem.

The main result of this note is a sufficient condition expressed on the weighted core of a proper tree. Its point is that terminal leaf-component orders may repeat with arbitrary multiplicity. Thus the result is strictly broader than the immediate reconstruction criterion in which every part of the leading partition is distinct.

Theorem 1.1. *Let T be a proper tree. Suppose that, among the leaf components of T , the order of every component whose center is a nonleaf of the internal core occurs exactly once. Then T is determined up to isomorphism by $\mathbf{X}_T$.*

The proof is short but useful: the leading star-basis partition gives the multiset of component orders, and the next layer gives the number of core edges of every endpoint-order type. Under the hypothesis, repeated orders belong only to core leaves, so these two pieces of data determine the whole weighted core.

Theorem 1.1 applies in every diameter. In diameter six it gives an infinite class. For example, take a five-vertex path as weighted core, give its two endpoints a common weight $r \geq 2$, and give its three internal vertices pairwise distinct weights, all different from r . Expanding a weight w into a center with $w - 1$ pendant leaves produces a proper tree of diameter six covered by the theorem, although its leading partition has a repeated part.

2. Leaf components and weighted cores

An edge of a tree is a *leaf edge* if one endpoint is a leaf. The *internal core* $\mathcal{C}(T)$ is the tree induced by the nonleaf vertices of T . A tree is *proper* if every vertex of $\mathcal{C}(T)$ is adjacent in T to at least one leaf. For such a tree define

$$\omega_T(v) = 1 + \#\{u \in V(T) : u \text{ is a leaf and } uv \in E(T)\}, v \in V(\mathcal{C}(T)).$$

Every weight is at least two. Conversely, T is recovered from the strictly weighted tree $(\mathcal{C}(T), \omega_T)$ by attaching $\omega_T(v) - 1$ leaves to each core vertex v .

The connected components obtained after deleting all internal edges are stars, called the *leaf components*. The component centered at v has order $\omega_T(v)$. Thus the leaf-component partition is

$$\lambda_{LC}(T) = \{\{\omega_T(v) : v \in V(\mathcal{C}(T))\}\}.$$

Definition 2.1. The weighted core of a proper tree is *multiplicity-isolated* if

$$\deg_{\mathcal{C}(T)}(v) \geq 2 \implies \omega_T^{-1}(\omega_T(v)) = \{v\}. \quad (1)$$

Equivalently, every weight that occurs more than once occurs exclusively on leaves of $\mathcal{C}(T)$.

We use the following information recovered from the star-basis expansion.

Lemma 2.2 (Star-basis edge data). *Let T be a proper tree. The chromatic symmetric function $\mathbf{X}_T$ determines*

1. the multiset $\mathcal{W}_T = \{\{\omega_T(v) : v \in V(\mathcal{C}(T))\}\}$; and
2. for every unordered pair $\{p, q\}$, the integer

$$e_{pq}(T) = \#\{uv \in E(\mathcal{C}(T)) : \{\omega_T(u), \omega_T(v)\} = \{p, q\}\}.$$

Proof. The leading partition of the star-basis expansion is the leaf-component partition, giving (1). Since T is proper, this partition has no part of size one. Proposition 4.13(a) of [21] shows that the coefficients indexed by partitions of length one less than the leading partition recover exactly the numbers of internal edges with each unordered pair of leaf-component orders as endpoints. This gives (2). An equivalent strictly weighted D -polynomial formulation appears in Lemma 7.12 of [21]. $\square$

3. Reconstruction from multiplicity-isolated data

The graph-theoretic step is independent of symmetric functions.

Lemma 3.1. *Let (C, ω) be a weighted tree with at least three vertices. Assume that every nonleaf vertex has a weight of multiplicity one. Then (C, ω) is determined up to weight-preserving isomorphism by*

1. the multiplicities $m_p = |\omega^{-1}(p)|$; and
2. the numbers $e_{pq} = \#\{uv \in E(C) : \{\omega(u), \omega(v)\} = \{p, q\}\}$.

Proof. For every p with $m_p = 1$, create a single vertex v_p of weight p . For every p with $m_p > 1$, create m_p indistinguishable vertices of weight p . By hypothesis, all vertices in the latter class are leaves.

If $m_p = m_q = 1$, join v_p and v_q precisely when $e_{pq} = 1$. If $m_p > 1$ and $m_q = 1$, attach exactly e_{pq} of the weight- p leaves to v_q . There is no edge between two repeated weight classes: such an edge would join two leaves, which is impossible in a tree on at least three vertices. These rules account for every edge and every repeated-weight vertex, because

$$\sum_{q: m_q=1} e_{pq} = m_p \quad (m_p > 1).$$

The construction is forced, and choices among vertices of the same repeated weight differ only by a weight-preserving isomorphism. $\square$

Proof of Theorem 1.1.

By Lemma 2.2, $\mathbf{X}_T$ determines the input data of Lemma 3.1. Condition (1) is exactly its multiplicity hypothesis. Hence $\mathbf{X}_T$ determines the weighted core. Attaching $\omega_T(v) - 1$ leaves at every recovered core vertex determines T .

If the core has one vertex, T is a star. If it has two vertices, T is a bi-star; the endpoint weights determine it, including the equal-weight case. Thus the small cores are covered as well. $\square$

Corollary 3.2. *Stanley's tree-isomorphism conjecture holds for every proper tree of diameter six whose weighted core is multiplicity-isolated.*

Remark 3.3. Repeated terminal types are genuinely unrestricted. A repeated weight may occur on arbitrarily many leaves of the core and may be distributed among several uniquely weighted nonleaf vertices; the numbers e_{pq} recover that distribution.

4. A canonical model for diameter six

For completeness, we record a compact parametrization useful beyond Corollary 3.2. A diameter-six tree has a unique central vertex c . Every component of $T - c$, rooted at its neighbor of c , has height at most two. Such a rooted branch is encoded by an integer partition π : a part 1 denotes a leaf child of the branch root, while a part $q \geq 2$ denotes a child supporting $q - 1$ leaves. The branch order is $1 + |\pi|$, and the empty partition encodes a one-vertex branch.

Consequently, a diameter-six tree is canonically represented by a multiset $\{\{\pi_1, \dots, \pi_s\}\}$ satisfying

$$|V(T)| = 1 + \sum_{i=1}^s (1 + |\pi_i|),$$

with at least two π_i containing a part at least two. This is a bijection with unlabelled diameter-six trees.

Stanley's power-sum formula

$$\mathbf{X}_T = \sum_{A \subseteq E(T)} (-1)^{|A|} p_{\lambda(A)}$$

also gives an exact computational invariant: for a tree, the sign is determined by the length of $\lambda(A)$, so $\mathbf{X}_T$ is equivalent to the table counting edge subsets by their component-order partition. The supplementary implementation computes this table by integer dynamic programming and

independently checks small orders by direct edge-subset enumeration. These computations are diagnostic only and are not used in the proof.

5. Limitations and further questions

The unrestricted diameter-six problem is not resolved here. Two obstacles remain outside the theorem:

1. a nonproper tree produces weight-one core vertices, for which the simplest adjacency extraction changes form; and
2. if a nonleaf core weight is repeated, the aggregate numbers e_{pq} need not separate the individual vertices in that weight class.

Higher star-basis layers record contractions of several internal edges and are natural candidates for resolving the second obstruction. The canonical partition model above turns the full diameter-six question into recovery of a multiset of height-two rooted branches from exact cut-partition data.

Notes

2020 Mathematics Subject Classification: 05C05, 05E05, 05C60

Statements and Declarations

Funding

No specific funding was received for this work.

Potential Competing Interests

No potential competing interests to declare.

Data Availability

The accompanying source archive contains the exact cut-partition evaluator, the canonical diameter-six generator, and a separate finite checker for the weighted-core reconstruction lemma. No computational output is used as a premise of Theorem 1.1.

References

1. ^aStanley R (1995). "A Symmetric Function Generalization of the Chromatic Polynomial of a Graph." *Adv Math*. **111**(1):166–194. doi:[10.1006/aima.1995.1020](https://doi.org/10.1006/aima.1995.1020).
2. ^a, ^bAliste-Prieto J, de Mier A, Orellana R, Zamora J (2023). "Marked Graphs and the Chromatic Symmetric Function." *SIAM J Discrete Math*. **37**(3):1881–1919. doi:[10.1137/22M148046X](https://doi.org/10.1137/22M148046X).
3. ^a, ^bGonzalez M, Orellana R, Tomba M (2025). "On the Reconstruction of Trees from Their Chromatic Symmetric Functions." *Sém Lothar Combin*. **93B**:Article No. 78, 11 pp.. <https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2025/78.html>.

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.