Research Article

Quantization of Nonlinear Transmission Line Dynamics With Noise: Some Remarks on Noise in Quantum Field and Quantum Neural Network Theories

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This paper starts with a basic formulation of a distributed parameter transmission line with memristor per unit length in series and also memristor per unit length in parallel apart from the other standard distributed parameters. The result of having memristors distributed along the line is a nonlinear effect in the form of coupling between charge and current and coupling between charge and voltage. These nonlinear effects result in the generation of an infinite number of higher harmonic components along the line even when the input voltage is of a fixed frequency or a finite superposition of voltages of different frequencies. Taking into account random white noise line loading effects, we formulate the resulting line equations based on KCL and KVL in the form of an infinite dimensional nonlinear stochastic differential equation. We study the problem of quantizing the transmission line dynamics using the well known Hudson-Parthasarathy theory of quantum stochastic differential equations derived from the Hudson-Parthasarathy noisy Schrodinger equation constructed from an appropriate transmission line Hamiltonian and Lindblad operators. We then conclude by discussing the effect of quantum stochastic noise and quantum stochastic supersymmetric noise on the action functional of a field theory and how such a model can be used to obtain corrections to the quantum effective action functional or more precisely, to the TPCP map that evolves an initially mixed state of the field to another mixed state after a finite duration. Some remarks on training and testing of quantum neural networks based on the Belavkin quantum filter are presented. This involves constructing a quantum evolution of a mixed state in such a way that the evolving pdf of a system observable will be obtained from the conditional expectation of the

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noisy quantum state given non-demolition measurements that are strongly correlated with actually measured signals.

1. Formulation of the basic distributed memristor transmission line

The distributed parameter circuit we have in mind consists of memristors and inductors distributed in series along the transmission line as well as memristors and capacitors distributed in parallel along the line. Let $R(q(t,z))i(t,z)dz$ and $L\partial_t i(t,z)dz$ denote respectively the voltage along an infinitesimal memristor and and infinitesimal inductor along the line in the length interval $[z, z+dz]$. Then, the voltage difference between the points z and $z+dz$ is given by

$$-\partial_z v(t,z)dz = R(q(t,z))i(t,z)dz + L\partial_t i(t,z)dz$$

or equivalently, after taking into account distributed white noise line loading,

$$-\partial_z v(t,z) = R(q(t,z))\partial_t q(t,z) + L\partial_t^2 q(t,z) + f(z)\partial_t B(t,z)$$

This represents the line KVL. Here $q(t,z)$ and $i(t,z) = \partial_t q(t,z)$ are respectively the charge and current along the line. Likewise, assume that along the line parallel (ie, perpendicular to its length), the current flowing between z and $z+dz$ is given by $G(-\partial_z q(t,z)dz)v(t,z)$ due to parallely distributed memristors and $C\partial_t v(t,z)dz$ due to parallely distributed capacitors. The parallel line equation is then

$$-\partial_z i(t,z)dz = G(-\partial_z q(t,z)dz)v(t,z) + C\partial_t v(t,z)dz$$

Note that the charge that flows perpendicular to the line between the points z and $z+dz$ is given by $-\partial_z q(t,z)dz$. We assume that $G(0)=0$ for consistency. Then, by Taylor expansion,

$$G(-\partial_z q(t,z)dz) = -G'(0)\partial_z q(t,z)dz + O(dz^2)$$

so the parallel equation reduces to (after accounting for line loading again)

$$-\partial_z i(t,z) = -G\partial_z q(t,z)v(t,z) + C\partial_t v(t,z) + g(z)\partial_t W(t,z)$$

where $G = -G'(0)$. Note that our argument shows that the parallely distributed memristor acts as a simple parallel resistor of value $1/(Gdz)$ in $[z, z+dz]$. Not the case for the serially distributed memristor. This equation can alternately be expressed as

$$-\partial_z\partial_t q(t,z) = -G\partial_z q(t,z)v(t,z) + C\partial_t v(t,z) + g(z)\partial_t W(t,z)$$

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Here, $B(t,z)$ and $W(t,z)$ are independent Brownian fields, ie, they are Gaussian fields having zero mean and covariance

$$\mathrm{E}(B(t,z)B(t',z')) = \min(t,t').\ \delta(z-z')$$

They can be expanded in a Fourier series along the line length:

$$B(t,z) = \sum_n B_n(t)e_n(z),\ e_n(z) = \exp(j2\pi.\ nz/d)/\sqrt{d},\ n \in \mathrm{Z}$$

Then,

$$\sum_{n,m} \mathrm{E}(B_n(t)B_m(t'))e_n(z)e_m(z') = \min(t,t').\ \sum_n e_n(z-z'), \quad 0 \le z,z' \le d$$

and this equation immediately gives us

$$\mathrm{E}(B_n(t)B_m(t')) = \delta[n+m].\ \min(t,t')$$

Since fields are real, we have

$$\bar{B}_n(t) = B_{-n}(t)$$

and hence we can express the above correlation identity as

$$\mathrm{E}(B_n(t)\bar{B}_m(t)) = \delta[n-m].\ \min(t,t'), \quad n,m \in \mathrm{Z}$$

We write

$$B_n(t) = B_n^R(t) + j.\ B_n^I(t)$$

where B_n^R and B_n^I are real processes. Then, we get

$$B_n^R(t) = B_{-n}^R(t), \quad B_n^I(t) = -B_{-n}^I(t)$$

and hence, for $n,m \ge 0$, we have

$$\mathrm{E}(B_n^R(t)B_m^R(t')) = \delta[n-m].\ \min(t,t')/2$$

$$\mathrm{E}(B_n^I(t)B_m^I(t')) = \delta[n-m].\ \min(t,t')/2$$

$$\mathrm{E}(B_n^R(t)B_m^I(t')) = 0$$

and likewise for the Gaussian field W. Now expressing

$$q(t,z) = \sum_n q_n(t)e_n(z), \quad v(t,z) = \sum_n v_n(t)e_n(z)$$

and also

$$f(z) = \sum_n f_n e_n(z), \quad g(z) = \sum_n g_n e_n(z)$$

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we obtain from the above line equations,

$$-\partial_z v(t,z) = R(q(t,z))\partial_t q(t,z) + L\partial_t^2 q(t,z) + f(z)\partial_t B(t,z)$$

$$-\partial_z \partial_t q(t,z) = -G\partial_z q(t,z)v(t,z) + C\partial_t v(t,z) + g(z)\partial_t W(t,z)$$

the corresponding Fourier series space nonlinear stochastic differential equations:

$$(-2\pi jn/d)v_n(t) = \sum_{m,n_1+...+n_m+k=n} R_m q_{n_1}(t)...q_{n_m}(t)q_k'(t) + Lq_n''(t) + \sum_m f_{n-m}B_m'(t)$$

$$(-2\pi jn/d)q_n'(t) = -G\sum_m (2\pi jm/d)q_m(t)v_{n-m}(t) + Cv_n'(t) + \sum_m g_{n-m}W_m'(t)$$

where

$$R(q) = \sum_{n\geq 0} R_n q^n$$

We now seek a perturbative algorithm for solving these equations, compute the statistics of the charge and voltage fields $q(t,z), v(t,z)$ and then to take discrete measurements along the line and using the extended Kalman filter alogrithm (EKF), to estimate the line voltage, line charge and memristor parameters $R_n, n\geq 0, G$. Defining the complex infinite dimensional vectors

$$v(t) = ((v_n^R(t), v_n^I(t))_{n\geq 0}, \quad q(t) = ((q_n^R(t), q_n^I(t)))_{n\geq 0}, \quad B(t) = ((B_n^R(t), B_n^I(t)))_{n\geq 0},$$

$$W(t) = ((W_n^R(t), W_n^I(t)))_{n\geq 0}$$

these equations can be cast in the following form:

$$q''(t) = A_1 v(t) + \sum_{n\geq 0} A_2(n)(q(t)^{\otimes n}\otimes q'(t)) + A_3 B'(t)$$

$$v'(t) = A_4 q'(t) + A_5(q(t)\otimes v(t)) + A_6 W'(t)$$

Here, $A_1, A_2(n), A_3, A_4, A_5, A_6$ are real, infinite dimensional matrices that depend upon the parameters $\theta = ((R_n, n\geq 0, L, C, G)$ that we seek to estimate. We define the state vector

$$X(t) = (v(t)^T, q(t)^T, q'(t)^T)^T$$

as well as the infinite dimensional Brownian motion vector

$$V(t) = (B(t)^T, W(t)^T)^T$$

and then the above equations can be expressed in a more convenient form:

$$X'(t) = C_1(\theta)X(t) + \sum_{n\geq 2} C_2(n|\theta)X(t)^{\otimes n} + C_3(\theta)V'(t)$$

or equivalently, in stochastic differential form

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$$dX(t) = (C_1(\theta(t))X(t) + \sum_{n\geq 2} C_2(n\,|\,\theta(t))X(t)^{\otimes n})dt + C_3(\theta(t))dV(t)$$

with

$$d\theta(t) = \sigma.\,d\epsilon(t)$$

where $\epsilon(t)$ is another standard infinitesimal Brownian motion. Thus, $[X(t)^T, \theta(t)^T]^T$ is our extended state vector. The form of the matrices $C_1, C_2(n), C_3$ can be easily written down in terms of the matrices $A_1, A_2(n), A_3, A_4, A_5, A_6$.

2. A digression on quantization of the dynamics based on the Hudson-Parthasarathy-Evans-Hudson flow method

The basics of quantum stochastic calculus are explained in[1].

It is of interest here to note how a stochastic differential equation of this kind can be quantized using the Hudson-Parthasarathy-Evans-Hudson flow method. To see how this is accomplished, we assume that the state vector X comprises of a position vector component Q and a momentum vector component P. The Hudson-Parthasarathy-noisy Schrodinger evolution for the unitary operator $U(t)$ on the tensor product of the system and bath space has the form

$$dU(t) = (-(iH + P)dt + \sum_k L_k dA_k(t) - L_k^* dA_k(t)^*)U(t), \quad P = (1/2)\sum_k L_k L_k^*$$

where A_k, A_k^* are the standard annihilation and creation processes of the Hudson-Parthasarathy quantum stochastic calculus satisfying the quantum Ito formula:

$$dA_k(t).\,dA_j(t)^* = \delta(k,j)dt,\, dA_k(t).\,dA_j(t) = 0,\, dA_k(t)^*.\,dA_j(t)^* = dA_k(t)^* dA_j(t) = 0$$

We choose as our Hamiltonian the Harmonic oscillator one:

$$H = (1/2)\sum_k (P_k^2/2 + \omega(k)^2 Q_k^2/2)$$

This Hamiltonian leads to linear Heisenberg dynamics for Q, P and hence this portion of the dynamics corresponds to only linear non-dissipative terms in the transmission line dynamics, ie, that generated by only the distributed series inductances and parallel capacitances in the transmission line. To get the linear dissipative effects generated by the series and parallel resistances, we can use Lindblad operators L_k that are linear functions of position and momentum. To get the nonlinear terms generated by memristors and resistances, capacitances and inductances that are functions of charge

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current and voltage, we can use Lindblad operators that are nonlinear functions of position and momentum. Thus, we consider Lindblad operators

$$L_k = F_k(Q, P), L_k^* = \tilde{F}_k(Q, P)$$

specifically, we can using the canonical commutation rules, define $F_k(Q, P)$ in such way that in each term, all the $Q's$ appear to the left of all the $P's$. Then in $\tilde{F}_k(Q, P) = F_k(Q, P)^*$, all the $P's$ will appear to the left of all the $Q's$. If X is any system observable, its quantum noisy Heisenberg dynam ics will be given by

$$X(t) = j_t(X) = U(t)^* XU(t) = U(t)^* (X \otimes I)U(t)$$

and by quantum Ito's formula, it satisfies

$$dj_t(X) = dU(t)^* XU(t) + U(t)^* XdU(t) + dU(t)^* XdU(t) = j_t(\theta_0(X))dt +$$

$$j_t(\theta_{1k}(X))dA_k(t) + j_t(\theta_{2k}(X))dA_k(t)^*$$

with summation over the index k being implied, where

$$\theta_0(X) = i[H, X] - (1/2)\sum_k (L_k^* L_k X + XL_k^* L_k - 2L_k^* XL_k)$$

$$= i[H, X] - (1/2)\sum_k (L_k^* [L_k, X] + [X, L_k^*]L_k)$$

$$\theta_{1k}(X) = [X, L_k], \theta_{2k}(X) = [L_k^*, X]$$

Thus,

$$\theta_0(Q_n) = P_n - (i/2)\sum_k (\frac{\partial \tilde{F}_k}{\partial P_n}. F_k - \tilde{F}_k. \frac{\partial F_k}{\partial P_n})$$

$$= P_n + G_{0n}(Q, P)$$

say.

$$\theta_{1k}(Q_n) = i\frac{\partial F_k}{\partial P_n} = G_{1nk}(Q, P)$$

$$\theta_{2k}(Q_n) = -i\frac{\partial \tilde{F}_k}{\partial P_n} = G_{1nk}(Q, P)^*$$

$$\theta_0(P_n) = -\omega(n)^2 Q_n + (i/2)\sum_k (\frac{\partial \tilde{F}_k}{\partial Q_n}. F_k - \tilde{F}_k. \frac{\partial F_k}{\partial Q_n})$$

$$= -\omega(n)^2 Q_n + K_{0n}(Q, P)$$

say.

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$$\theta_{1k}(P_n) = -i\frac{\partial F_k}{\partial Q_n} = K_{1nk}(Q,P)$$

$$\theta_{2k}(Q_n) = i\frac{\partial \tilde{F}_k}{\partial Q_n} = K_{1nk}(Q,P)^*$$

The Hudson-Parthasarathy noisy Heisenberg equations are therefore given by

$$dQ_n(t) = P_n(t)dt + G_{0n}(Q(t),P(t))dt + \sum_k (G_{1nk}(Q(t),P(t))dA_k(t) + G_{1nk}(Q(t),P(t))^* dA_k(t)^*)$$

$$dP_n(t) = -\omega(n)^2 Q_n(t)dt + K_{0n}(Q(t),P(t))dt + \sum_k (K_{1nk}(Q(t),P(t))dA_k(t) + K_{1nk}(Q(t),P(t))dA_k(t)^*)$$

Defining the two classical Brownian motion processes (which do not however commute with each other)

$$B_k^R(t) = A_k(t) + A_k(t)^*, B_k^I(t) = -i(A_k(t) - A_k(t)^*)$$

and also introducing the Hermitian operators

$$S_{nk}^R(Q,P) = (1/2)(G_{1nk}(Q,P) + G_{1nk}(Q,P)^*)$$

$$S_{nk}^I(Q,P) = (i/2)(G_{1nk}(Q,P) - G_{1nk}(Q,P)^*)$$

$$T_{nk}^R(Q,P) = (1/2)(K_{1nk}(Q,P) + K_{1nk}(Q,P)^*)$$

$$T_{nk}^I(Q,P) = (i/2)(K_{1nk}(Q,P) - K_{1nk}(Q,P)^*)$$

we can express these qsde's in Hermitian form as

$$dQ_n(t) = P_n(t)dt + G_{0n}(Q(t),P(t))dt + \sum_k (S_{nk}^R(Q(t),P(t))dB_k^R(t) + S_{nk}^I(Q(t),P(t))dB_k^I(t))$$

$$dP_n(t) = -\omega(n)^2 Q_n(t)dt + K_{0n}(Q(t),P(t))dt + \sum_k (T_{1nk}^R(Q(t),P(t))dB_k^R(t) + T_{1nk}^I(Q(t),P(t))dB_k^I(t))$$

By appropriate choice of the canonical position and momentum variables in the transmission line along with the Lindblad operators, we can cast the line equations in such a quantum stochastic format, ie, we can introduce the notion of a quantum stochastic transmission line. The position variables will be the Fourier coefficient of current and the momentum variables will be those of the voltage, the purely conservative case, ie, when the line consists of only distributed inductors and capacitors. This fact can be seen from the conservative linear situation in which the line voltage and current satisfy

$$\partial_z v(t,z) + L\partial_t i(t,z) = 0, \quad \partial_z i(t,z) + C\partial_t v(t,z) = 0$$

or equivalently, in the Fourier series domain

$$(2\pi in/d)v_n(t) + Li_n^{'}(t) = 0, \quad (2\pi in/d)i_n(t) + Cv_n^{'}(t) = 0$$

so that taking real and imaginary parts,

$$-(2\pi n/d)v_n^I(t) + Li_n^{R'}(t) = 0, \quad (2\pi n/d)v_n^R(t) + Li_n^{I'}(t) = 0$$

$$-(2\pi n/d)i_n^I(t) + Cv_n^{R'}(t), \quad (2\pi n/d)i_n^R(t) + Cv_n^{I'}(t) = 0$$

which give on eliminating v_n^I, i_n^I, the following two second order linear differential equations:

$$i_n^{R''}(t) + (2\pi n/d)^2(1/LC)i_n^R(t) = 0, \quad n \ge 0$$

$$v_n^{R''}(t) + (2\pi n/d)^2(1/LC)v_n^R(t) = 0, \quad n \ge 0$$

These differential equations are those of an infinite sequence of independent harmonic oscillators with characteristic frequency of oscillation of the n^{th} mode being given by

$$\omega(n) = (2\pi n/d)(LC)^{-1/2}, \quad n \ge 0$$

This suggests to us that we can derive these differential equations from the Hamiltonian given by

$$H = H(Q,P) = (1/2)\sum_{n\ge 0}(P_{1n}^2 + \omega(n)^2 Q_{1n}^2 + P_{2n}^2 + \omega(n)^2 Q_{2n}^2),$$

$$Q = ((Q_{1n}, Q_{2n})), \quad P = ((P_{1n}, P_{2n}))$$

where

$$Q_{1n} = i_n^R, \quad Q_{2n} = v_n^R, \quad P_{1n} = i_n^{R'} = (2\pi n/dL)v_n^I, \quad P_{2n} = v_n^{R'} = (2\pi n/dC)i_n^I$$

The Hamiltonian equations are

$$Q_{kn}' = \partial H/\partial P_{kn} = P_{kn}, \quad P_{kn}' = -\partial H/\partial Q_{kn} = -\omega(n)^2 Q_{kn}, \quad k = 1,2$$

Elimination of P_{kn} from these equations gives us the above line equations:

$$Q_{kn}'' = -\omega(n)^2 Q_{kn}, \quad k = 1,2, \quad n \ge 0$$

When resistances in series and in parallel are present, this linear conservative dynamics gets damped and this situation can also be described quantum mechanically by using the Lindblad master equation for open quantum systems in the form

$$dX/dt = i[H,X] - (1/2)\sum_{k,n}(L_{kn}L_{kn}^*X + XL_{kn}L_{kn}^* - 2L_{kn}XL_{kn}^*) =$$

$$i[H,X] - (1/2)(L_{kn}[L_{kn}^*, X] + [X, L_{kn}]L_{kn}^*)$$

with H as above and

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$$L_{kn} = a(k,n)Q_{kn} + b(k,n)P_{kn}$$

with coefficients $a(k,n), b(k,n)$ depending upon the series resistance and parallel conductance per unit length. Specifically,

$$[L^*_{kn}, Q_{kn}] = -i\bar{b}(kn), \quad [Q_{kn}, L_{kn}] = ib(kn)$$

$$[L^*_{kn}, P_{kn}] = i\bar{a}(kn), \quad [P_{kn}, L_{kn}] = -ia(kn)$$

These result in the differential equations

$$Q^{'}_{kn} = P_{kn} - (1/2)(-i\bar{b}(kn)(a(kn)Q_{kn} + b(kn)P_{kn}) + ib(kn)(\bar{a}(kn))Q_{kn} + \bar{b}(kn)P_{kn})$$

$$= P_{kn} - Im(a(kn)\bar{b}(kn))Q_{kn}$$

$$P^{'}_{kn} = -\omega(n)^2 Q_{kn} - (1/2)(i\bar{a}(kn)(a(kn)Q_{kn} + b(kn)P_{kn}) - ia(kn)(\bar{a}(kn)Q_{kn} + \bar{b}(kn)P_{kn}))$$

$$= -\omega(n)^2 Q_{kn} - Im(a(kn)\bar{b}(kn))P_{kn}$$

Defining the real constants

$$\gamma(kn) = Im(a(kn)\bar{b}(kn))$$

we can express these equations as

$$Q^{'}_{kn} = P_{kn} - \gamma(kn)Q_{kn}, \quad P^{'}_{kn} = -\omega(n)^2 Q_{kn} - \gamma(kn)P(kn)$$

Now assume that $k = 1, 2, \quad n \geq 0$. Consider a line with a series distributed resistance and a parallel distributed conductance. The line equations in this case are

$$\partial_z i(t,z) + Gv(t,z) + C\partial_t v(t,z) = 0$$

$$\partial_z v(t,z) + Ri(t,z) + L\partial_t i(t,z) = 0$$

In the Fourier series domain, these translate to

$$(2\pi in/d)i_n(t) + Gv_n(t) + Cv^{'}_n(t) = 0$$

$$(2\pi in/d)v_n(t) + Ri_n(t) + Li^{'}_n(t) = 0$$

or equivalently, in terms of real and imaginary parts,

$$-(2\pi n/d)i^I_n(t) + Gv^R_n(t) + Cv^{R'}_n(t) = 0$$

$$(2\pi n/d)i^R_n(t) + Gv^I_n(t) + Cv^{I'}_n(t) = 0$$

$$-(2\pi n/d)v^I_n(t) + Ri^R_n(t) + Li^{R'}_n(t) = 0$$

$$(2\pi n/d)v^R_n(t) + Ri^I_n(t) + Li^{I'}_n(t) = 0$$

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In the presence of distributed memristors, nonlinearities are introduced into the dynamics and the resultant line equations assume the form

$$\partial_z i(t,z) + Gv(t,z) + C\partial_t v(t,z) + F_1(\partial_z q(t,z), v(t,z)) = 0$$

$$\partial_z v(t,z) + Ri(t,z) + L\partial_t i(t,z) + F_2(q(t,z), i(t,z)) = 0$$

assuming that we do not have any random line loading effects. Equivalently, in the Fourier series domain, with $v(t) = ((v_n(t)))$ and $i(t) = ((i_n(t))), q(t) = ((q_n(t)))$,

$$v'(t) = -i\alpha D.\, i(t) - (G/C)v(t) + F_1(q(t), v(t))$$

$$i'(t) = -i\beta D.\, v(t) - (R/L)i(t) + F_2(q(t), i(t))$$

where

$$D = \mathrm{diag}[2\pi n/d : n \in Z],\ \alpha = 1/C, \beta = 1/L$$

or equivalently, taking real and imaginary parts,

$$v^{R'}(t) = \alpha D i^I(t) - \gamma_1 v^R(t) + F_1^R(q^R(t), q^I(t), v^R(t), v^I(t))$$

$$v^{I'}(t) = -\alpha D i^R(t) - \gamma_1 v^I(t) + F_1^I(q^R(t), q^I(t), v^R(t), v^I(t))$$

$$i^{R'}(t) = \beta D.\, v^I(t) - \gamma_2 i^R(t) + F_2^R(q^R(t), q^I(t), i^R(t), i^I(t))$$

$$i^{I'}(t) = -\beta D.\, v^R(t) - \gamma_2 i^I(t) + F_2^I(q^R(t), q^I(t), i^R(t), i^I(t))$$

with the relationship

$$i(t) = q'(t)$$

or equivalently,

$$i^R(t) = q^{R'}(t)$$

$$i^I(t) = q^{I'}(t)$$

In principle, we can solve (a) and (c) to express (i^I, v^I) as a function of $v^{R'}, i^{R'}, v^R, i^R, q^R, q^I$. These expressions, we substitute into $(b), (d)$ to obtain second order differential equations for v^R, i^R in involving $q^R, q^I, q^{R'} = i^R, q^{I'} = i^I$. For i^I we again substitute its expression in terms of $v^{R'}, i^{R'}, v^R, i^R, q^R, q^I$ to obtain finally second order differential equations for v^R, i^R involving q^R, q^I and of course $v^{R'}, i^{R'}, v^R, i^R$ but no derivatives of q^R, q^I. These differential equations are supplemented by (e) and (f) where in the latter, we substitute for i^I, its expression in terms of $v^{R'}, i^{R'}, v^R, i^R, q^R, q^I$. The jist of these calculations is finally, a *second order* nonlinear differential equation for the "position vector"

$$Q(t) = [v^R(t), i^R(t), q^R(t), q^I]$$

This can be used as the starting point for quantization using the theory of open quantum systems or if line loading noise is included, quantization based on the HPS qsde.

An alternate way to start the quantization process is to begin with the original complex form of the line differential equations

$$v'(t) = -i\alpha D.\,i(t) - \gamma_1 v(t) + F_1(q(t), v(t))$$

$$i'(t) = -i\beta D.\,v(t) - \gamma_2 i(t) + F_2(q(t), i(t))$$

From these, we derive

$$i(t) = i\alpha^{-1}D^{-1}(v'(t) + \gamma_1 v(t) - F_1(q(t), v(t)))$$

and hence

$$i\alpha^{-1}D^{-1}(v''(t) + \gamma_1 v'(t) - \partial_t F_1(q(t), v(t))) = -i\beta D.\,v(t) - \gamma_2 i\alpha^{-1}D^{-1}(v'(t) + \gamma_1 v(t) - F_1(q(t), v(t)))$$

$$+F_2(q(t), i(t))$$

This can be rearranged as

$$v''(t) + \gamma_1 v'(t) + \alpha\beta.\,D^2 v(t) = \partial_t F_1(q(t), v(t)) - \gamma_2(v'(t) + \gamma_1 v(t) - F_1(q(t), v(t))$$

$$-i\alpha.\,D.\,F_2(q(t), i(t))$$

or equivalently, as

$$v''(t) + (\gamma_1 + \gamma_2)v'(t) + (\alpha.\,\beta.\,D^2 + \gamma_1\gamma_2)v(t) =$$

$$\partial_t F_1(q(t), v(t)) + \gamma_2 F_1(q(t), v(t)) - i\alpha.\,D.\,F_2(q(t), i(t))$$

$$= F(q(t), i(t), v(t)) = F(q(t), q'(t), v(t))$$

This equation is to be supplemented with the equations

$$q'(t) = i(t) = i\alpha^{-1}D^{-1}(v'(t) + \gamma_1 v(t) - F_1(q(t), v(t)))$$

Defining

$$v(t) = Q(t),\ v'(t) = P(t),\ q'(t) = p(t)$$

these equations can be cast into the form

$$Q'(t) = P(t)$$

$$P'(t) = -\gamma.\,P(t) - \Omega^2.\,Q(t) + F(q(t), p(t), Q(t))$$

$$p'(t) = d/dt(i\alpha^{-1}D^{-1}(v'(t) + \gamma_1 v(t) - F_1(q(t), v(t)))$$

$$= i\alpha^{-1}D^{-1}(P'(t) + \gamma_1 P(t) - F_{1,1}(q(t), Q(t))p(t) - F_{1,2}(q(t), Q(t))P(t))$$

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$$= i\alpha^{-1}D^{-1}(\gamma.P(t) - \Omega^2.Q(t) + F(q(t), p(t), Q(t)) - F_{1,1}(q(t), Q(t))p(t) - F_{1,2}(q(t), Q(t))P(t))$$

where

$$\Omega^2 = \alpha.\beta.D^2 + \gamma_1\gamma_2, \gamma = \gamma_1 + \gamma_2$$

Note that the choice of $v(t)$ as the position variable $Q(t)$ and the charge $q(t)$ as another position variable is in agreement with our understanding that charge and voltage in a capacitor have their squares proportional to the electrostatic potential energy while current which is the time derivative of charge and also being proportional to the current through the capacitor with square proportional to the magnetic energy in an inductor in an LC circuit should be the momentum. The position variables $Q_1(t) = (Q(t), q(t))$ and the corresponding momentum variables $P_1(t) = (P(t), p(t))$ (More precisely, their real and imaginary parts as defined above thus satisfy dynamics of the form

$$Q_1'(t) = P_1(t),\ P_1'(t) = -\Gamma.P_1(t) - D_0.Q_1(t) + F(Q_1(t), P_1(t))$$

Here D_0 is real and symmetric but not necessarily positive definite. Γ is real but not necessarily diagonal. These equations can also be derived from the theory of open quantum systems by assuming an appropriate Hamiltonian and Lindblad operators.

3. A special case of the Lindblad equation corresponding to linear velocity damping with coefficients nonlinear functions of position

In the special case when the Lindblad operators L_{kn} are of the form

$$L_{kn} = a(kn)Q_{kn} + b(kn)P_{kn} + f_{kn}(Q), Q = ((Q_{kn})), P = ((P_{kn}))$$

ie, tbese operators are linear in the P and the Hamiltonian is of the form

$$H(Q, P) = (1/2)\sum_{kn}(P_{kn}^2 + \omega(n)^2Q_{kn}^2) + U(Q)$$

we find from the Heisenberg equation

$$dX/dt = i[H, X] - (1/2)(L_{kn}[L_{kn}^*, X] + [X, L_{kn}]L_{kn}^*)$$

that

$$Q_{kn}' = P_{kn} - \gamma(kn)Q_{kn}$$

$$P_{kn}' = -\omega(n)^2Q_{kn} - \gamma(kn)P_{kn} - (1/2)(L_{kn}[\tilde{f}_{kn}(Q), P_{kn}]$$

$$+i\bar{a}(kn)f_{kn}(Q) + [P_{kn}, f_{kn}(Q)]L_{kn}^* - ia(kn)\tilde{f}_{kn}(Q))$$

$$= -\omega(n)^2Q_{kn} - \gamma(kn)P_{kn} - (1/2)(i(a(kn)Q_{kn} + b(kn)P_{kn} + f_{kn}(Q))\partial\tilde{f}_{kn}(Q)/\partial Q_{kn}$$

$$+i\bar{a}(kn)f_{kn}(Q) - i(\partial f_{kn}(Q)/\partial Q_{kn})(\bar{a}(kn)Q_{kn} + b(kn)P_{kn} + \tilde{f}_{kn}(Q)) - ia(kn)\tilde{f}_{kn}(Q))$$

$$= -\omega(n)^2 Q_{kn} - \gamma(kn)P_{kn} - Im[(\partial f_{kn}(Q)/\partial Q_{kn})(\bar{a}(kn)Q_{kn} + b(kn)P_{kn} + \tilde{f}_{kn}(Q)) + a(kn)\tilde{f}_{kn}(Q)]$$

$$= -\omega(n)^2 Q_{kn} - \gamma_{kn}P_{kn} - h_{1kn}(Q)P_{kn} + h_{2kn}(Q)$$

with h_{1kn} being linear in f_{kn} and h_{2kn} being linear-quadratic in f_{kn}. Eliminating P from these equations results in a second order differential equation for Q having the form

$$Q_{kn}^{''} = -\omega(n)^2 Q_{kn} - (2\gamma_{kn} + \delta_{kn}(Q))Q_{kn}^{'} + \rho_{kn}(Q)$$

where $\rho_{kn}(Q), \delta_{kn}(Q)$ are given by

$$\rho_{kn}(Q) = -\gamma_{kn}^2 Q_{kn} + h_{2kn}(Q)$$

$$\delta_{kn}(Q) = h_{1kn}(Q)$$

This example illustrates how to realize linearly damped oscillations in quantum mechanics with potential being a small perturbation of the harmonic potential and with velocity damping coefficients being small perturbations of constant damping coefficients.

4. The case when Poisson noise is also present at the quantum level

We start with the HPS equation in the form

$$dU(t) = (-(iH + P)dt + \sum_k (L_{1k}dA_k - L_{2k}dA_k^* + S_k d\Lambda_k))U(t)$$

where

$$dA_k dA_j^* = \delta(k,j)dt, d\Lambda_k. d\Lambda_j = \delta(k,j)d\Lambda_k$$

$$dA_k d\Lambda_j = \delta(k,j)dA_k, d\Lambda_k dA_j^* = \delta(k,j)dA_j^*$$

Conditions for unitarity are easily derived using these quantum Ito's formula and are given by

$$P = (1/2)\sum_k L_{2k}^* L_{2k}$$

$$L_{1k} - L_{2k}^* - L_{2k}^* S_k = 0$$

$$S_k^* + S_k + S_k^* S_k = 0$$

Equivalently, these conditions are expressed by defining

$$Z_k = S_k + 1$$

as

$$Z_k^* Z_k = 1, L_{1k} = L_{2k}^* Z_k$$

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The resulting Evans-Hudson flow for

$$j_t(X) = U(t)^* XU(t)$$

are

$$dj_t(X) = j_t(\theta_0(X))dt + \sum_k j_t(\theta_{1k}(X))dA_k(t) + j_t(\theta_{2k}(X))dA_k(t)^* + j_t(\theta_{3k}(X))d\Lambda_k)$$

where

$$\theta_0(X) = i[H, X] - (1/2)\sum_k(-2L_{2k}^* XL_{2k} + L_{2k}^* L_{2k}X + XL_{2k}^* L_{2k})$$

$$= i[H, X] - (1/2)\sum_k (L_{2k}^*[L_{2k}, X] + [X, L_{2k}^*]L_{2k})$$

$$\theta_{1k}(X) = -L_{2k}^* X + XL_{1k} - L_{2k}^* XS_k + S_k^* XL_{1k}$$

$$\theta_{2k}(X) = -XL_{2k} + L_{1k}^* X - S_k^* XL_{2k} + L_{1k}^* XS_k$$

$$\theta_{3k}(X) = S_k^* X + XS_k + S_k^* XS_k = Z_k^* XZ_k - X$$

5. Quantum Belavkin filter for line voltage and line parameter estimation

For the basics of quantum filtering as first discovered by V.P.Belavkin, we refer to [2].

We start with the joint unitary dynamics of the system and bath

$$dU(t) = (-i(H+P)dt + \sum_k L_t dA_k(t) - L_k^* dA_k^*(t))U(t)$$

and derive

$$dj_t(X) = j_t(\theta_0(X|\phi(t)))dt + j_t(\theta_{1k}(X|\phi(t)))dA_k(t) + j_t(\theta_{2k}(X|\phi(t)))dA_k(t)^*$$

with summation over k being understood. Here, $\phi(t)$ is the set of classical line distributed parameters satisfying the sde

$$d\phi(t) = \sigma.\, d\epsilon(t)$$

where $\epsilon(t)$ is classical Brownian motion. Assume that we take noisy measurements along the line of the voltage and current at a discrete set of points. Such noisy measurements can be derived from Belavkin's non-demolition model:

$$Y_{ok}(t) = U(t)^* Y_{ik}(t)U(t)$$

where

$$Y_{ik}(t) = c(k)A_k(t) + \bar{c}(k)A_k(t)^*$$

with k=1,2,..., p. To see how these measurements correspond to measuring line voltage and current, we calculate the differentials of the output measurements using quantum Ito's formula:

$$dY_{ok}(t) = dU^* Y_{ik}U + U^* Y_{ik}dU + dU^* Y_{ik}dU + dY_{ik} + dU^* dY_{ik}U + U^* dY_{ik}dU =$$

$$dY_{ik} + dU^* dY_{ik}U + U^* dY_{ik}dU =$$

$$= dY_{ik} - j_t(\bar{c}(k)L_k + c(k)L_k^*), k = 1, 2, \ldots p$$

If the Lindblad operators L_k are thus chosen to linear combinations of the Q_n, P_n, then in the transmission line formalism, dY_{ok} will represent noisy versions of the quantum stochastic process $-(\bar{c}(k)L_k + c(k)L_k^*)(t)$ which will be linear combinations of the Fourier series coefficients $i_n^R, i_n^I, v_n^R, v_n^I$ of the line current and line voltage. The time derivative of the line current is a linear combination of the line current and voltage and the time derivative of the line voltage is also a linear combination the line current and voltage in the Fourier series domain. It should be noted that the Fourier series coefficients of the line voltage and current are respectively given by weighted linear integrals of the same and hence if the discrete spatial points at which the line voltage and current are measured have uniform small spacings, then the Fourier series coefficients of the line voltage and current can well be approximated by the discrete Fourier transform of the line voltage and current spatial samples. Now, the standard reference probability approach of John Gough and C.Kostler can be used to determine the quantum filtering equations of Belavkin for this model:

$$\pi_t(fX) = \mathrm{E}(j_t(fX) \mid \eta_o(t))$$

$$\eta_o(t) = \sigma(Y_{ok}(s) : k = 1, 2, \ldots, p, s \leq t)$$

$$d\pi_t(fX) = F_t(fX)dt + \sum_{k=1}^{p} G_{kt}(fX)dY_{ok}(t)$$

with $F_t(X), G_{kt}(X)$ being measurable w.r.t the Abelian algebra $\eta_o(t)$ and calculated using the orthogonality principle in statistical estimation theory:

$$\mathrm{E}[(j_t(fX) - \pi_t(fX))C_t] = 0$$

where

$$j_t(fX) = f(\phi(t))j_t(X) = f(\phi(t))U(t)^* XU(t)$$

and

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$$dC_t = \sum_{k=1}^{p} g_k(t) dY_{ok}(t) C_t$$

We do not discuss the details here.

5.1. Hudson-Parthasarathy noisy Schrodinger equation for fields

Let $A_k(t) = A_t(\phi_k), k - 1, 2, \ldots$ be a countably infinite set of annihilation operator processes. Here, $\phi_k, k = 1, 2, \ldots$ is an orthonormal basis for $L^2(\mathrm{R}^4)$. If $a(\phi)$ denotes the annihilation field on $L^2(\mathrm{R}^4)$, then we can write the CCR as

$$[a(\phi), a(\psi)^*] =< \phi, \psi >$$

Then, we define the annihilation process field as

$$A_t(\phi) = a(\phi.\chi_{[0,t]}), t \geq 0, \phi \in L^2(\mathrm{R}^4)$$

It is then easy to see that

$$dA_t(\phi).dA_t(\psi)^* =< \phi_t, \psi_t > dt = (\int_{\mathrm{R}^3} \bar{\phi}(t,r).\psi(t,r) d^3r) dt$$

We also introduce the conservation field by

$$\lambda(H) = \sum < \phi_k | H | \phi_m > a(\phi_k)^* a(\phi_m)$$

where H is an operator in $L^2(\mathrm{R}^4)$. The conservation process field is then

$$\Lambda_t(H) = \lambda(H.\chi_{[0,t]})$$

where H is an operator in $L^2(\mathrm{R}^4)$ that commutes with $\chi_{[0,t]}$. In terms of the position representation,

$$\phi(x) =< x | \phi > , x \in \mathrm{R}^4, \phi \in L^2(\mathrm{R}^4)$$

we can we introduce quantum noise distributions $A_t(x), A_t(x)^*$,

$$a(\phi) = \int_{\mathrm{R}^4} a(x) \bar{\phi}(x) dx, a(\phi)^* = \int_{\mathrm{R}^4} a(x)^* \phi(x) dx$$

$$\lambda(H) = \int_{\mathrm{R}^4 \times \mathrm{R}^4} \lambda(x,y) H(x,y) dx dy$$

This is equivalent to

$$a(\phi) = a(\int |x > dx < x | \phi >) = \int a(x) < \phi | x > dx = \int a(x) \bar{\phi}(x) dx$$

$$a(\phi)^* = a(\int |x > dx < x | \phi >)^* = \int a(x)^* < x | \phi > dx = \int a(x)^* \phi(x) dx$$

$$\lambda(H) = \lambda(\int |x > dx < x | H | y > dy < y |)$$

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$$= \int \lambda(|x><y|) < x|H|y > dxdy = \int \lambda(x,y)H(x,y)dxdy$$

Thus, specifically taking $\phi \in L^2(\mathrm{R}^3)$ so that $\phi.\chi_{[0,t]} \in L^2(\mathrm{R}^4)$, we get

$$A_t(\phi) = a(\phi.\chi_{[0,t]}) = \int a(|x>\chi_{[0,t]})\bar{\phi}(x)dx = \int A_t(x)\bar{\phi}(x)dx$$

$$A_t(\phi)^* = \int A_t(x)^* \phi(x)dx$$

$$\Lambda_t(H) = \int \Lambda_t(x,y)H(x,y)dxdy$$

where

$$A_t(x) = a(|x>\chi_{[0,t]}),\ A_t(x)^* = a(|x>\chi_{[0,t]})\phi(x)dx$$

$$\Lambda_t(x,y) = \lambda(|x><y|\chi_{[0,t]})$$

Another way to express these relations is

$$a(\phi) = a(\sum_k |\phi_k><\phi_k|\phi>) = \sum_k a(|\phi_k>) < \phi|\phi_k>$$

$$= \sum_k a(\phi_k)\int \phi_k(x)\bar{\phi}(x)dx$$

$$= \int (\sum_k a(\phi_k)\bar{\phi}_k(x))\bar{\phi}(x)dx$$

implying thereby that

$$a(x) = \sum_k a(\phi_k)\bar{\phi}_k(x)$$

where $\{\phi_k\}$ is an onb for $L^2(\mathrm{R}^4)$ and $\phi \in L^2(\mathrm{R}^4)$. Likewise,

$$a(x)^* = \sum_k a(\phi_k)^* \phi_k(x)$$

and

$$\Lambda(H) = \sum_{k,m} a(\phi_k)^* a(\phi_m) < \phi_k|H|\phi_m>$$

$$= \sum_{k,m}\int a(x)^* a(y)\phi_k(x)\bar{\phi}_m(y) < \phi_k|H|\phi_m > dxdy$$

$$= \int a(x)^* a(y)\sum_{k,m} < x|\phi_k><\phi_k|H|\phi_m><y|\phi_m > dxdy$$

$$= \int a(x)^* a(y)H(x,y)dxdy$$

Thus, in the language of quantum noise field distribution theory,

$$\lambda(x,y) = a(x)^* a(y),\ x,y \in \mathrm{R}^4$$

Note that

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$$\int \bar{\phi}(x)\psi(x)dx = < \phi, \psi > = [a(\phi), a(\psi)^*]$$

$$= \int_{\mathrm{R}^4 \times \mathrm{R}^4} [a(x), a(y)^*] \bar{\phi}(x)\psi(y)dxdy$$

and hence

$$[a(x), a(y)^*] = \delta^4(x - y)$$

Likewise, for H an operator in $L^2(\mathrm{R}^4)$ and $\phi \in L^2(\mathrm{R}^4)$, we have

$$[\Lambda(H), a(\phi)]$$

$$= \sum_{km} [a(\phi_k)^* a(\phi_m), a(\phi)] < \phi_k | H | \phi_m >$$

$$= \sum_{km} [a(\phi_k)^*, a(\phi)] a(\phi_m) < \phi_k | H | \phi_m >$$

$$= -\sum_{km} < \phi | \phi_k >< \phi_k | H | \phi_m > a(\phi_m)$$

$$= -\sum_m < \phi | H | \phi_m > a(\phi_m)$$

$$= -\sum_m a(\phi_m) < H^* \phi | \phi_m > = a(\sum | \phi_m >< \phi_m | H^* \phi >) = -a(H^* \phi)$$

or

$$[a(\phi), \lambda(H)] = a(H^* \phi)$$

This gives

$$[\lambda(H), a(\phi)] =$$

$$= \int [\lambda(x, y), a(z)] H(x, y) \bar{\phi}(z) dxdydz$$

$$= -\int a(x) < H^* \phi | x > dx = \int a(x) (H^*(x, y)\phi(y))^* dxdy = -\int a(x) H(y, x) \bar{\phi}(y) dxdy$$

$$= -\int a(y) H(x, y) \bar{\phi}(z) \delta^4(z - x) dxdy$$

and therefore,

$$[a(z), \lambda(x, y)] = a(y)\delta^4(z - x)$$

This identity could also have been seen directly:

$$[a(z), \lambda(x, y)] = [a(z), a(x)^* a(y)] = [a(z), a(x)^*] a(y) = \delta^4(z - x) a(y)$$

The quantum Ito formula

$$dA_t(\phi) dA_t(\psi)^* = < \phi_t | \psi_t > dt$$

can be expressed in quantum white noise distribution notation as

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$$\int dA_t(x)dA_t(y)^* \bar{\phi}(t,x)\psi(t,y)dxdy = dt.\int \bar{\phi}(t,x).\ \psi(t,x)dx$$

so that

$$dA_t(x).\ dA_t(y)^* = \delta^3(x-y)dt,\ x,y \in \mathrm{R}^3$$

Note that for $\phi \in L^2(\mathrm{R}^4)$, we have

$$A_t(\phi) = a(\phi\chi_{[0,t]}) = \int_{[0,t]\times\mathrm{R}^3} a(s,x)\phi(s,x)dsdx$$

with $(s,x) \in \mathrm{R}^4$. In particular, if $\phi(t,x) = \phi(x), x \in \mathrm{R}^3$ (ie $\phi \in L^2(\mathrm{R}^3)$ so that $\phi\chi_{[0,t]} \in L^2(\mathrm{R}^4)$) is independent of t, we get

$$A_t(\phi) = \int_{[0,t]\times\mathrm{R}^3} a(s,x)\phi(x)dx = \int_{\mathrm{R}^3} A_t(x)\phi(x)dx$$

so that

$$A_t(x) = \int_0^t a(s,x)ds,\ x \in \mathrm{R}^3$$

or equivalently,

$$a(t,x) = dA_t(x)/dt,\ x \in \mathrm{R}^3$$

Also the relation

$$a(x)^* a(y) = \lambda(x,y), x,y \in \mathrm{R}^4$$

can be used to show that

$$d\Lambda_t(x,y) = dA_t(x)^* dA_t(y)/dt, x,y \in \mathrm{R}^3$$

as follows. We start with

$$(dA_t(x)/dt)^* (dA_t(y)/dt) = a(t,x)^* a(t,y) = \lambda((t,x),(t,y)), t \in \mathrm{R}, x,y \in \mathrm{R}^3$$

Thus,

$$dA_t(x)^* dA_t(y)/dt = dt.\ \lambda((t,x),(t,y))$$

Now, for H acting in $L^2(\mathrm{R}^3)$,

$$\Lambda_t(H) = \lambda(H.\ \chi_{[0,t]})$$

so that

$$d\Lambda_t(H)/dt = \lambda(H.\ d\chi_{[0,t]})/dt) = \lambda(H.\ \delta_t)$$

where

$$\delta_t(s) = \delta(t-s)$$

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Now,

$$\lambda(H.\delta_t) = \lambda(\int |x><x|H.\delta_t|y><y|dxdy)$$

$$= \int_{R^4\times R^4} <x|H.\delta_t|y> \lambda(|x><y|)dxdy$$

$$= \int_{R\times R^3\times R\times R^3} <t_1,x|H.\delta_t|t_2,y> \lambda(|t_1,x><t_2,y|)dt_1dxdt_2dy$$

Now,

$$<t_1,x|H\delta_t|t_2,y>= H(x,y)<t_1|\delta_t|t_2>= H(x,y)\int \delta(t_1-s)\delta(t-s)\delta(t_2-s)ds$$

$$= H(x,y)\delta(t_1-t).\delta(t_2-t), x,y \in R^3$$

Therefore,

$$\lambda(H.\delta_t) = \int \lambda(|t_1,x><t_2,y|)H(x,y)\delta(t_1-t)\delta(t_2-t)dt_1dt_2dxdy$$

$$= \int_{R^3\times R^3} H(x,y)\lambda(|t,x><t,y|)dxdy$$

This results in

$$d\Lambda_t(x,y)/dt = \lambda((t,x),(t,y)), t \in R, x,y \in R^3$$

On the other hand, we've seen above that

$$dt\lambda((t,x),(t,y)) = dt.a(t,x)^*a(t,y) = dA_t(x)^*dA_t(y)/dt$$

Therefore, we conclude that

$$d\Lambda_t(x,y) = dt.d\Lambda_t(x,y)/dt = dt.\lambda((t,x),(t,y)) = dA_t(x)^*dA_t(y)/dt$$

Note that in particular, we have

$$\Lambda_t(x,y) = \int_0^t \lambda((s,x),(s,y))ds$$

Now consider the quantum Lindblad noise term in the HPS equation with countably infinte degrees of freedom:

$$dW(t) = \sum_k (L(k)dA_t(\phi_k) - M(k)^*dA_t(\phi_k)^*) + \sum_{k,j} S(k,j)d\Lambda_t(|\phi_k><\phi_j|))$$

where now $\phi_k \in L^2(R^3)$ so that $\phi_k.\chi_{[0,t]} \in L^2(R^4)$. As noted above, we can write

$$A_t(\phi_k) = a(\phi_k.\chi_{[0,t]}) = \int_{[0,t]\times R^3} a(s,x)\bar{\phi}_k(x)dsdx = \int_{R^3} A_t(x)\bar{\phi}_k(x)dx$$

$$A_t(\phi_k)^* = \int_{R^3} A_t(x)^*\phi_k(x)dx$$

$$\Lambda_t(|\phi_k><\phi_j|) = \lambda(|\phi_k><\phi_j|\chi_{[0,t]})$$

$$= \int_{R^3\times R^3} \lambda(|x><y|\chi_{[0,t]})\phi_k(x)\bar{\phi}_j(y)dxdy$$

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$$= \int_{R^3 \times R^3} \Lambda_t(x, y) \phi_k(x) \bar{\phi}_j(y) dxdy$$

Thus, defining the "Lindblad operator fields"

$$L(x) = \sum_k L(k) \bar{\phi}_k(x), L(x)^* = \sum_k L(k)^* \phi_k(x), x \in R^3$$

$$S(x, y) = \sum_{k,j} S(k, j) \phi_k(x) \phi_j(y), x, y \in R^3$$

we can express the Lindblad noise term in the language of quantum noise field processes as

$$dW(t) = \int_{R^3} (L(x) dA_t(x) - M(x)^* dA_t(x)^*) d^3x + \int_{R^3 \times R^3} S(x, y) d\Lambda_t(x, y) dxdy$$

Along the lines indicated by Timothy Eyre, we can also think of constructing supersymmetric quantum noise. Specifically, if $\phi_k, k = 1, 2, \ldots$ is an orthonormal basis for $L^2(R^3)$, then we choose an integer $r > 0$ and define the operator H in $L^2(R^3)$ relative to this basis as

$$H = diag[0_r, I]$$

ie,

$$H | \phi_k >= 0, k = 1, 2, \ldots, r, H | \phi_k >= | \phi_k > , k = r + 1, r + 2, \ldots$$

Equivalently,

$$H = \sum_{k > r} | \phi_k >< \phi_k |$$

In other words, H is the orthogonal projection in $L^2(R^3)$ onto the subspace spanned by $| \phi_k > , k > r$. Define

$$G(t) = (-1)^{\Lambda_t(H)} = exp(i\pi\lambda(H\chi_{[0,t]})) = \Gamma(exp(i\pi H. \chi_{[0,t]}))$$

in the language of Weyl operators or more precisely, the projective unitary Weyl representation defined by its action on exponential vectors. We then define the supersymmetric noise processes

$$d\xi^a_b(t) = G(t)^{\sigma(a,b)} d\Lambda^a_b(t), a, b = 0, 1, 2, \ldots$$

where $A_0(t) = t, A_a(t) = A_t(\phi_a) = a(\phi_a \chi_{[0,t]}), a \geq 1$

$$\Lambda^a_b(t) = dA_b(t)^* dA_a(t) / dt$$

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6. Quantum neural network for synthesizing the line voltage and current probability density by fine tuning of weights

First we make a small modification to the parallel component of the memristor line model. We assume that the parallel memristors along the line each has a spatial width of Δ which is very small, ie, comparable to dz. This means that the current through the parallel section between the spatial points z and $z+dz$ is given by

$$dz.G(-\partial_z q(t,z).\Delta)v(t,z) \approx G(0)v(t,z)dz - G'(0)\Delta.\partial_z q(t,z)v(t,z)dz$$

We assume that $G'(0)$ is very large of the order of $1/\Delta$, so that $G'(0)\Delta$ is a finite constant which we denote by $-G_1$. We denote $G(0)$ by G_0. Thus, taking into account line loading, we can express the line equations as

$$\partial_z v(t,z) + R_0 i(t,z) + L\partial_t i(t,z) + R_1(q(t,z))i(t,z) = f_1(z)dB_1(t)/dt$$

$$\partial_z i(t,z) + G_0 v(t,z) + C\partial_t v(t,z) + G_1\partial_z q(t,z)v(t,z) = f_2(z)dB_2(t)/dt$$

In terms of the real and imaginary parts of the Fourier series components of the line voltage and current and line charge, these equations assume the form

$$Dv(t) + Ldi(t)/dt + R_0 i(t) = -F_1(q(t),i(t)) + f_1 dB_1(t)/dt$$

$$Di(t) + Gv(t) + Cdv(t)/dt = -F_2(q(t),v(t)) + f_2 dB_2(t)/dt$$

or equivalently, in Ito stochastic differential form,

$$di(t) = -((D/L)v(t) + (R_0/L)i(t) + F_1(q(t),i(t)))dt + f_1 dB_1(t)$$

$$dv(t) = -((D/C)i(t) + (G/C)v(t) + F_2(q(t),v(t)))dt + f_2 dB_2(t)$$

where F_1/L has been denoted by F_1 and F_2/C has been denoted by F_2. These equations are to be supplemented with

$$dq(t) = i(t)dt$$

In short, the state vector

$\xi(t) = [v(t)^T, q(t)^T, i(t)^T]^T$

satisfies an Ito stochastic differential equation of the form

$$d\xi(t) = (A_0\xi(t) + \delta.F(\xi(t)))dt + H.dB(t)$$

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where the constant matrix A_0 is built out of $D/L, R_0/L, D/C, G/C$ and the nonlinearity $\delta. F$ is built out of F_1, F_2, namely the nonlinearities that characterize the series and parallel memristors. We are assuming that the effect of the series and parallel memristors is small and hence we can introduce a small perturbation parameter δ that characterizes the degree of this smallness. Finally, the constant matrix H is built out of the matrices f_1, f_2 which in turn are built using the Fourier series coefficients of the functions $f_1(z), f_2(z)$. The pdf $f(t, \xi) = f(t, v, q, i)$ satisfies the Fokker-Planck or forward Kolmogorov equation

$$\partial_t f(t, \xi) = -\nabla_\xi^T[(A_0\xi + \delta. F(\xi))f(t, \xi)] + (1/2)Tr(HH^T\nabla_\xi\nabla_\xi^T f(t, \xi))$$

The aim is to design the potential field of a multidimensional Schrodinger equation for which the magnitude square of the wave function will track this pdf. More generally, we can use the Hamiltonian and the Lindblad operators of an open quantum system as our weights to be adapted so that the evolving mixed state that defines a probability density for an appropriate observable tracks this desired line pdf. This can be achieved by using the Belvakin filter whose output is the conditional expectation of an evolving observable given the measured output process derived from the measurement model

$$dY_{ok}(t) = j_t(-\bar{c}(k)L_k - c(k)L_k^*) + dY_{ik}(t)$$

with the $c(k), L_k$ selected so that it corresponds to the measuremnent of the line voltage and current of the quantum system (ie a quantum transmission line) at a finite discrete set of points. The Belavkin filter in the state formalism would then give the evolving conditional expectation of the quantum mixed state given these output measurements.

After this training stage, suppose we take another line on which some additional small disturbance has occurred. Then, we extract out an extra signal from this line that is correlated with this disturbance and model this extra signal by a measurement noise corrupted version of a function of the line voltage and current and design an extended Belavkin filter to provide fine tuned estimates of the Hamiltonian and Lindblad parameters taking into account this additional measurement. In other words, during the training stage, we estimate the state given noisy measurements of the line voltage and current and during the testing state, we fine tune the state estimate by constructing a further conditional expectation given signal measurements that are strongly correlated with the additional attack/disturbance on the system.

7. The effect of quantum stochastic noise fields in the computation of quantum effective action and also in the computation of the evolution of the state of a quantum system

We note that most of the classical action functionals of free Boson and Fermion fields can be cast in the form

$$S[A] = \int A(x)K(x,y)A(y)d^4xd^4y$$

or if there are vector or spinor indices in the field,

$$S[A] = \int A_\mu(x)K^{\mu\nu}(x,y)A_\nu(y)d^4xd^4y$$

For example, consider the Maxwell photon field in a background gravitational field:

$$S[A] = (-1/4)\int F^{\mu\nu}(x)F_{\mu\nu}(x)\sqrt{-g(x)}d^4x$$

$$= (-1/4)\int g^{\mu\alpha}g^{\nu\beta}\sqrt{-g}F_{\mu\nu}F_{\alpha\beta}d^4x$$

where

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

$A_\mu(x)$ are taken as the position fields. Note that we can always add a gauge fixing term to this action without affecting the physics:

$$S[A] \to S[A] + a.\int(\partial_\mu(A^\mu\sqrt{-g}))^2(-g)^{-1/2}d^4x$$

Note that the above gauge fixing term is indeed a scalar according to general relativity because

$$(\partial_\mu(A^\mu\sqrt{-g}))^2(-g)^{-1/2} = (A^\mu_{:\mu})^2\sqrt{-g}$$

We have in the absence of the gauge fixing term,

$$S[A] = \int(P^{\mu\nu\alpha\beta}(A_{\nu,\mu}A_{\beta,\alpha} - A_{\mu,\nu}A_{\beta,\alpha})d^4x$$

$$= \int(P^{\mu\nu\alpha\beta} - P^{\nu\mu\alpha\beta})A_{\nu,\mu}A_{\beta,\alpha}d^4x$$

where

$$P^{\mu\nu\alpha\beta}(x) = (-1/2)g^{\mu\alpha}g^{\nu\beta}\sqrt{-g}(x)$$

We define

$$Q^{\mu\nu\alpha\beta} = P^{\mu\nu\alpha\beta} - P^{\nu\mu\alpha\beta}$$

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Using integration by parts, we have

$$S[A] = \int Q^{\mu\nu\alpha\beta} A_{\nu,\mu} A_{\beta,\alpha} d^4x$$

$$-\int A_\nu(x)\partial_\mu^x(Q^{\mu\nu\alpha\beta}(x)\partial_\alpha^x\delta^4(x-y))A_\beta(y)d^4xd^4y$$

$$= \int A_\nu(x)K^{\nu\beta}(x,y)A_\beta(y)d^4xd^4y$$

where

$$K^{\nu\beta}(x,y) = -\partial_\mu^x(Q^{\mu\nu\alpha\beta}(x)\partial_\alpha^x\delta^4(x-y))$$

If the above gauge fixing term is taken into account, then we can still express the action in the above form but with

$$K^{\nu\beta}(x,y) = -\partial_\mu^x(Q^{\mu\nu\alpha\beta}(x)\partial_\alpha^x\delta^4(x-y)) - a.\, g^{\mu\nu}(x)\sqrt{-g(x)}\partial_\mu^x[\sqrt{-g(x)}\partial_\alpha^x(g^{\alpha\beta}(x)\sqrt{-g(x)}\delta^4(x-y))(-g(x))^{-1/2})]$$

Now consider adding quantum white noise terms in the position field to this electromagnetic action: Let $a_\mu(x), x \in R^4$ denote the quantum annihilation white noise field and $a_\mu(x)^*, x \in R^4$ the corresponding quantum creation white noise field. They satisfy the CCR

$$[a_\mu(x), a_\nu(y)^*] = \eta_{\mu\nu}\delta^4(x-y)$$

assuming that the space-time manifold is flat. In order to define the quantum white noise fields in curved space-time, we note that if we introduce the tetrad $V_\mu^a(x)$ of the metric field, then we have

$$\eta_{ab}V_\mu^a(x)V_\nu^b(x) = g_{\mu\nu}(x)$$

and hence defining

$$b_\mu(x) = a_k(x)V_\mu^k(x)$$

and then

$$[b_\mu(x), b_\nu(y)^*] = \eta_{km}V_\mu^k(x)V_\nu^m(y)\delta^4(x-y) = g_{\mu\nu}(x)\delta^4(x-y), x, y \in R^4$$

Note that this equation is still not diffeomorphic invariant. In order to make it so, we modify the noisy CCR to

$$[b_\mu(x), b_\nu(y)^*] = f(x)g_{\mu\nu}(x)\delta^4(x-y)$$

If under a space-time coordinate diffeomorphism $x^\mu \to \bar{x}^\mu$, we define with

$$T_\mu^\alpha = \partial\bar{x}^\alpha/\partial x^\mu$$

then we get assuming that $f(x)$ transforms to $\bar{f}(\bar{x})$,

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$$f(x)g_{\mu\nu}(x)\delta^4 5(x-y) = [b_\mu(x), b_\nu(y)^*] = [T_\mu^\alpha(x)\bar{b}_\alpha(\bar{x}), T_\nu^\beta(y)\bar{b}_\beta(\bar{y})^*] =$$

$$T_\mu^\alpha(x)T_\nu^\beta(y)\bar{f}(\bar{x})\bar{g}_{\mu\nu}(\bar{x})\delta^4(\bar{x}-\bar{y})$$

8. Adding Bosonic and Fermionic noise to a general Hamiltonian system with an example from quantum electrodynamics

The Hamiltonian can be expressed as

$$H = H_1(Q,P) + H_2(q,p) + H_3(Q,P,q,p)$$

where (Q,P) are position and momentum variables of the Bosonic system, (q,p) are position and momentum variables for the Fermionic system and H_3 is the interaction Hamiltonian between the Bosonic and Fermionic systems. As an example, we consider the Lagrangian functional in qed:

$$L = (1/2)\int(-(curlA)^2 + (\partial_t A)^2)d^3x + \int\bar{\psi}.(i\gamma.\partial - m)\psi d^3x + e\int\bar{\psi}(\gamma.A)\psi d^3x$$

A is the Bosonic position field, ψ is the Fermionic position field,

$$P = \delta L/\delta\partial_t A = \partial_t A$$

is the Bosonic momentum field,

$$p = \delta L/\delta\partial_t\psi = i\psi^*$$

is the Fermionic momentum field and the total Hamiltonian can be expressed as

$$H = \int P.\partial_t A d^3x + \int p.\partial_t\psi.d^3x - L$$

$$= (1/2)\int(P^2 + (curlA)^2)d^3x + \int\psi^*((\alpha, -i\nabla) + \beta m)\psi + e\int\psi^*(\alpha, A)\psi.d^3x$$

$$= (1/2)\int(P^2 + (curlA)^2)d^3x - i\int p^T.((\alpha, -i\nabla) + \beta m)\psi.d^3x - ie\int p^T(\alpha, A)\psi.d^3x$$

Here, A is the magnetic vector potential 3-vector. Writing $Q = A, q = \psi$, we can equivalently express this as

$$H = (1/2)\int(P^2 + (curlQ)^2)d^3x - i\int p^T.((\alpha, -i\nabla) + \beta m)q.d^3x - ie\int p^T(\alpha, Q)q.d^3x$$

We can thus identify the Bosonic, Fermionic and interaction Hamiltonians respectively as

$$H_1(Q,P) = (1/2)\int(P^2 + (curlQ)^2)d^3x$$

$$H_2(q,p) = Im(\int p^T.((\alpha, -i\nabla) + \beta m)q.d^3x)$$

$$H_3(Q,P,q,p) = H_3(Q,q,p) = e.Im(\int p^T(\alpha, Q)q.d^3x)$$

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Note that Q, P, q, p are to be regarded as Hermitian operators in an appropriate tensor product of Boson and Fermion Fock space. They satisfy respectively the CCR and CAR

$$[Q(t,x), P(t,x')] = i\delta^3(x-x'), \{q(t,x), p(t,x')\} = i\delta^3(x-x')$$

$$[Q(t,x), Q(t,x')] = [P(t,x), P(t,x')] = 0, \{q(t,x), q(t,x')\} = \{p(t,x), p(t,x')\} = 0$$

$$[Q(t,x), q(t,x')] = [Q(t,x), p(t,x')] = [P(t,x), q(t,x')] = [P(t,x), p(t,x')] = 0$$

While calculating the amplitudes for various scattering, absorption and emission processes of photons, electrons and positrons using the Feynman path integral, we must use the Bosonic path integral w.r.t Q and the Fermionic Berezin path integral w.r.t q.

Remark: More precisely, the Fermionic position fields q are to be taken as the real and imaginary parts of the wave operator field ψ and likewise, the corresponding Fermionic momentum fields p are to be taken as the real and imaginary parts of $i\psi^*$ or equivalently, as the imaginary and real parts of ψ respectively. It follows therefore that this is a constrained Hamiltonian problem with constraints given by

$$p_R = q_I, p_I = q_R$$

Of course, these constraints are compatible with the CAR

$$\{q_R(t,x), p_R(t,x')\} = i\delta^3(x-x'), \{q_I(t,x), p_I(t,x')\} = i\delta^3(x-x')$$

Now, given the total Bosonic and Fermionic Hamiltonian as $H(Q, P, q, p)$, we can add quantum noise to this Hamiltonian by replacing Q_a with $c(a,b)dA_b(t)/dt + \bar{c}(a,b)dA_b^*(t)/dt$ and q_a with $d(a,b)dJ_a(t)/dt + \bar{d}(a,b)dJ_a^*(t)/dt$ where $A_a(t)$ are the Bosonic annihilation processes $A_a^*(t)$ are the creation processes, and $dJ_a(t) = (-1)^{\Lambda(t)}dA_a(t)$ are the Fermionic annihilation processes with $\Lambda(t) = \lambda(I.\chi_{[0,t]})$. These processes satisfy respectively the CCR and CAR:

$$[A_a(t), A_b(s)^*] = min(t,s)\delta(a,b), \{J_a(t), J_b(s)^*\} = min(t,s)\delta(a,b),$$

$$[A_a(t), A_b(s)] = [A_a(t), J_b(s)] = [A_a(t)^*, J_b(s)] = 0$$

If instead, we deal with quantum field theories, wherein $Q_a(t,x), P_a(t,x), q_a(t,x), q_a(t,x)$ are the Bosonic position momentum and Fermionic position and momentum fields, then we would have to add quantum noise to this theory in the following form:

$$Q_a(t,x) \to Q_a(t,x) + c_a(t,x)dA_a(t,x)/dt + \bar{c}_a(t,x)dA_a(t,x)^*/dt$$

$$q_a(t,x) \to q_a(t,x) + d_a(t,x)dJ_a(t,x)/dt + \bar{d}_a(t,x)dJ_a(t,x)^*/dt$$

Note that the conservation process field is given by

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$$d\Lambda_t(a,x;b,y) = dA_b(t,x)^* dA_a(t,y)/dt, a, b \geq 1, x, y \in \mathbb{R}^3, a, b \geq 1$$

The quantum Ito formulas read

$$dA_a(t,x)dA_b(t,y)^* = dt.\delta(a,b)\delta^3(x-y)$$

so that

$$d\Lambda_t(a,x;b,y)dA_c(t,z)^* = dA_b(t,y)^* dA_a(t,x)dA_c(t,z)^*/dt = dA_b(t,y)^* \delta(a,c)\delta^3(x-z)$$

or equivalently, in terms of $\phi, \psi, \chi \in L^2(\mathbb{R}^3)$,

$$d\Lambda_t(a,\phi;b,\psi)dA_c(t,\chi)^* = dA_b(t,y)^* \delta(a,c)\delta^3(x-z)\psi(y)\bar{\phi}(x)\chi(z)d^3xd^3z$$

$$= \delta(a,c) < \phi|\chi > . dA_b(t,\psi)$$

9. Some versions of supersymmetric quantum stochastic processes

Consider first the generalized noise processes $\Lambda_b^a(t), a, b = 0, 1, \ldots, N$ where $\Lambda_0^0(t) = t, \Lambda_0^a(t) = A_a(t), \Lambda_a^0(t) = A_a(t)^*$ and $\Lambda_b^a(t) = \lambda(|e_b >< e_a|\chi_{[0,t]}), a, b = 1, 2, \ldots, .$ where $|e_a >, a = 1, 2, \ldots,$ is an onb for $L^2(\mathbb{R}^3)$. More generally, with $x, y \in \mathbb{R}^4$, we have

$$\lambda(x,y) = \lambda(|x >< y|) = \sum_{n,m} < \phi_n|x >< y|\phi_m > a(\phi_n)^* a(\phi_m)$$

$$= \sum_{n,m} a(\phi_n)^* a(\phi_m)\bar{\phi}_n(x)\phi_m(y)$$

$$= a(\sum_n |\phi_n >< \phi_n|x >)^* . a(\sum_m |\phi_m >< \phi_m|y >) = a(|x >)^* a(|y >) = a(x)^* a(y)$$

where $|\phi_n >, n = 1, 2, \ldots$ is an onb for $L^2(\mathbb{R}^4)$.

Now if H is an operator in $L^2(\mathbb{R}^3)$, we have

$$\Lambda_t(H) = \lambda(H_t) = \lambda(H.\chi_{[0,t]})$$

$$= \lambda(\sum_{a,b\geq 1} |e_a >< e_a|H|e_b >< e_b|\chi_{[0,t]})$$

$$= \sum_{a,b\geq 1} \lambda(|e_a >< e_b|\chi_{[0,t]}) < e_a|H|e_b >$$

$$= \sum_{a,b\geq 1} \Lambda_a^b(t)H(a,b)$$

where now

$$H(a,b) =< e_a|H|e_b >, a, b, \geq 1$$

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If we extend this definition by setting $H(0,0) = c, H(a,0) = m_a, H(0,b) = \bar{n}_b, a, b \geq 1$ and put $e_0 = 1$, then we can formally write

$$\lambda(H.\chi_{[0,t]}) = \lambda(\sum_{a,b\geq 0} H(a,b)|e_a><e_b|.\chi_{[0,t]})$$

$$= H(0,0)t + \sum_a H(a,0)A_a(t)^* + \sum_b H(0,b)A_b(t) + \sum_{a,b\geq 1} H(a,b)\Lambda_a^b(t)$$

$$= ct + A_t(m)^* + A_t(n) + \Lambda_t(H)$$

where $H = ((H(a,b)))_{a,b\geq 1}$.

Note that more generally, if H is an operator in $L^2(\mathrm{R}^4)$, we have

$$\lambda(H) = \sum_{a,b\geq 1} \lambda(|\phi_a><\phi_b|)H(a,b)$$

where now

$$H(a,b) = <\phi_a|H|\phi_b>, a, b \geq 1$$

We define for H an operator in $L^2(\mathrm{R}^3)$,

$$G(t,H) = \Gamma(\exp(H_t)) = W(0, \exp(H_t)), H_t = H.\chi_{[0,t]}$$

Note that the family $G(t,H), t \geq 0$ forms a commutative family of operators in the Boson Fock space

$$\Gamma_s(L^2(\mathrm{R}^3) \otimes L^2(\mathrm{R}_+)) = \Gamma_s(L^2(\mathrm{R}^3 \times \mathrm{R}_+))$$

and that if H is skew-Hermitian, then all the operators in this family are unitary. Let

$$\Lambda_t(K) = \sum_{a,b\geq 0} K(a,b)\Lambda_a^b(t) = K(0,0)t + K(0,b)A_b(t) + K(a,0)A_a(t)^* + \sum_{a,b\geq 1} K(a,b)\Lambda_a^b(t)$$

where $K = ((K(a,b)))_{a,b\geq 0}$ or equivalently, K corresponds to the operator $\sum_{a,b\geq 0} K(a,b)|e_a><e_b|$ in $L^2(\mathrm{R}^3)$. Now, define

$$d\xi_t(H,K) = G(t,H)d\Lambda_t(K)$$

or equivalently,

$$\xi_t(H,K) = \int_0^t G(s,H)d\Lambda_s(K), t \geq 0$$

For $s < t$, we have the obviously proven identities

$$<e(v)|d\xi_t(H,K_1).\, d\xi_s(H,K_2))|e(u)> =$$

$$<e(v)|\Gamma(H_t).\,\Gamma(H_s)|e(u)><v(s)|\exp(H)K_2|u(s)><v(t)|K_1|u(t)> dtds$$

$$<e(v)|d\xi_s(H,K_2).\, d\xi_t(H,K_1)|e(u)>$$

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$$=< e(v)|\Gamma(H_t)\Gamma(H_s)|e(u) >< v(s)|K_2. \exp(H)|u(s) >< v(t)|K_1|u(t) > dtds$$

This means that (again, keeping in mind $s < t$ and $u_0(t) = 1$), for $a, b, c, d \geq 0$,

$$< e(v)|\Gamma(H_t). d\Lambda_b^a(t). \Gamma(H_s). d\Lambda_d^c(s)|e(u) >=$$

$$u_a(t). \bar{v}_b(t)dt < e(v)|\Gamma(H_t)\Gamma(H_s)|e(u) > u_c(s)(\exp(H)^* \bar{v}(s))_d ds$$

$$=< e(v)|\Gamma(H_t)\Gamma(H_s)|e(u) > . (\exp(H)^* \bar{v}(s))_d. u_c(s). \bar{v}_b(t)u_a(t)dtds$$

and

$$< e(v)|\Gamma(H_s). d\Lambda_d^c(s). \Gamma(H_t). d\Lambda_b^a(t)|e(u) >=$$

$$< e(v)|\Gamma(H_t). \Gamma(H_s)|e(u) > . \bar{v}_d(s)(\exp(H)u(s))_c. \bar{v}_b(t)u_a(t)dtds$$

Multiplying these equations by $K_1(b, a)K_2(d, c)$ and summing over all $a, b, c, d \geq 0$, we get

$$< e(v)|\Gamma(H_t)d\Lambda_t(K_1). \Gamma(H_s)d\Lambda_s(K_2)|e(u) >=$$

$$< e(v)|\Gamma(H_t). \Gamma(H_s)|e(u) > \sum_{a,b \geq 0} (K_1(b, a)\bar{v}_b(t)u_a(t)). (\sum_{c,d \geq 0} K_2(d, c)(exp(H)^* \bar{v}(s))_d u_c(s))dtds$$

$$=< e(v)|\Gamma(H_t). \Gamma(H_s)|e(u) >< v(t)|K_1|u(t) > . < v(s)|exp(H)K_2|u(s) > dtds$$

and likewise,

$$< e(v)|\Gamma(H_s)d\Lambda_s(K_2). \Gamma(H_t)dA_t(K_1)|e(u) >$$

$$=< e(v)|\Gamma(H_t)\Gamma(H_s)|e(u) >< v(s)|K_2. exp(H)|u(s) >< v(t)|K_1|u(t) > dtds$$

Note that $u_0(t) = 1$ and $u_a(t), a \geq 1$ are complex functions on $\mathbb{R}_+$ such that $\sum_{a \geq 1}\int_0^\infty |u_a(t)|^2 dt = \| u \|^2 < \infty$. Then,

$$< e(v)|d\Lambda_t(K)|e(u) > / < e(v)|e(u) >= [K(0, 0) + \sum_{a \geq 1} K(0, a)u_a(t) + \sum_{b \geq 1} K(b, 0)\bar{v}_b(t)$$

$$+ \sum_{a,b \geq 1} K(a, b)u_a(t)\bar{u}_b(t)]dt$$

The expression on the rhs is abbreviated as

$$< v(t)|K|u(t) >$$

ie, it is the same as

$$[1, v(t)]^* K[1, u(t)^T]^T$$

where $v(t) = ((v_a(t)))_{a \geq 1}, u(t) = ((u_a(t)))_{a \geq 1}$ and $K = ((K(a, b)))_{a,b \geq 0}$.

We then deduce that for $s < t$,

$$\frac{< e(v)|\Gamma(H_t)d\Lambda_t(K_1). \Gamma(H_s)d\Lambda_s(K_2)|e(u) >}{< e(v)|\Gamma(H_s)d\Lambda_s(K_2). \Gamma(H_t)dA_t(K_1)|e(u) >}$$

$$= \frac{< v(s) | exp(H). K_2 | u(s) >}{< v(s) | K_2. exp(H) | u(s) >}$$

or equivalently,

$$\frac{< e(v) | d\xi_t(H, K_1). d\xi_s(H, K_2) | e(u) >}{< e(v) | d\xi_s(H, K_2). d\xi_t(H, K_1) | e(u) >}$$

$$= \frac{< v(s) | exp(H). K_2 | u(s) >}{< v(s) | K_2. exp(H) | u(s) >}$$

Of course, if q is any complex number, we also deduce on defining the "q commutator" between operators A, B

$$[A, B]_q = AB - qBA$$

that for $s < t$,

$$< e(v) | [d\xi_t(H, K_1), d\xi_s(H, K_2)]_q | e(u) >= dtds < e(v) | \Gamma(H_t)\Gamma(H_2) | e(u) >< v(s) | [exp(H), K_2]_q | u(s) >$$

In particular, suppose that we are able to find $q_2 = q_2(H, K_2)$ such that

$$[exp(H), K_2]_{q_2} = 0$$

then we get

$$< e(v) | [d\xi_t(H, K_1), d\xi_s(H, K_2)]_{q_2} | e(u) >= 0, s < t$$

and suppose we find $q_1 = q_1(H, K_1)$ such that

$$[exp(H), K_1]_{q_1} = 0$$

then,

$$< e(v) | [d\xi_s(H, K_2), d\xi_t(H, K_1)]_{q_1} | e(u) >= 0, s > t$$

or equivalently,

$$< e(v) | [d\xi_t(H, K_1), d\xi_s(H, K_2)]_{1/q_1} | e(u) >= 0, s > t$$

In the particular case, when $1/q_1 = q_2 = q$ for given matrices H, K_1, K_2, we get

$$< e(v) | [d\xi_t(H, K_1), d\xi_s(H, K_2)]_q | e(u) >= 0, t \neq s$$

Further, it is immediate from quantum Ito's formula that

$$d\xi_t(H, K_1). d\xi_t(H, K_2) = \Gamma(H_t)^2 d\Lambda_t(K_1). d\Lambda_t(K_2) =$$

$$= \Gamma(H_t)^2 K_1(a, b)K_2(c, d)d\Lambda_a^b(t). d\Lambda_c^d(t)$$

$$= \Gamma(H_t)^2 K_1(a, b)K_2(c, d)\epsilon_c^b d\Lambda_a^d(t)$$

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$$= \Gamma(H_t)^2(K_1.\epsilon.K_2)(a,d)d\Lambda_a^d(t) = \Gamma(H_t)^2 d\Lambda_t(K_1.\epsilon.K_2) = d\xi_t(H^2, K_1.\epsilon.K_2)$$

This immediately yields, for all $t, s > 0$

$$< e(v) | [\xi_t(H, K_1), \xi_s(H, K_2)]_q | e(u) >=$$

$$< e(v) | \xi_{min(t,s)}(H^2, K_1.\epsilon.K_2 - q.K_2.\epsilon.K_1) | e(u) >$$

so that we deduce a "q-commutator Lie algebra":

$$[\xi_t(H, K_1), \xi_s(H, K_2)]_q = \xi_{min(t,s)}(H^2, K_1.\epsilon.K_2 - q.K_2.\epsilon.K_1), t, s \geq 0$$

This identity is easily generalized further to the case when H_1, H_2 are two commuting operators in $L^2(\mathbb{R}^3)$ and q is a complex number such that

$$[exp(H_1), K_2]_q = [K_1, exp(H_2)]_q = 0$$

then

$$[\xi_t(H_1, K_1), \xi_s(H_2, K_2)]_q = \xi_{min(t,s)}(H_1 H_2, K_1.\epsilon.K_2 - q.K_2.\epsilon.K_1), t, s \geq 0$$

Now, let us consider how these results can be applied to formulate a supersymmetric quantum noisy field theory. For the basics of supersymmetric field theory and supersymmetric quantum stochastic processes, we refer to[3] and[4].

Suppose $\phi_b^a(t,x)$ form supersymmetric quantum fields with $a, b \geq 0$ such that if $\sigma(a,b) = 0$, ϕ_b^a describes a Boson field and if $\sigma(a,b) = 1$, ϕ_b^a describes a Fermionic quantum field. After adding supersymmetric quantum noise to this quantum field, we wish to write down the field equations. Specifically, assume that the action functional for this supersymmetric field has the form

$$S[\phi] = \int \bar{\phi}_b^a(t,x) M(x, y | a, b, c, d) \phi_d^c(t,y) d^3x d^3y dt$$

We have defined the quantum noise processes $\xi_t(H, K)$. We specialize to the supersymmetric case by taking $H_0 = diag[0_r, I]$ so that

$$G(t, H_0) = \Gamma(H_{0t}) = (-1)^{\Lambda_t(H_0)} = W(0, exp(i\pi H_{0t}))$$

Then, we can write for any operator H in $L^2(\mathbb{R}^3)$,

$$d\Lambda_t(H)/dt = \int < x | H | y > d\Lambda_t(|x >< y|)/dt d^3x d^3y = \int H(x,y) a(t,x)^* a(t,y) d^3x d^3y$$

$$= \int H(x,y)(dA_t(x)/dt)^*(dA_t(y)/dt) d^3x d^3y$$

In other words, we can write

$$d\Lambda_t(|x >< y|)/dt = (dA_t(x)/dt)^*(dA_t(y)/dt), x, y \in \mathbb{R}^3$$

Writing

$$G(t) = \Gamma(H_{0t}) = G(t, H_0)$$

we can define the supersymmetric processes

$$d\xi^a_b(t) = G(t)^{\sigma(a,b)} d\Lambda^a_b(t), a, b \geq 0$$

It is clear that this can be expressed as

$$d\xi^a_b(t)/dt = (d\xi_b(t)/dt). d\xi^a(t)/dt$$

where

$$d\xi^a(t) = G(t)^{\sigma(a)} dA_a(t) = G(t)^{\sigma(a)} d\Lambda^a_0(t)$$

$$d\xi_b(t) = G(t)^{\sigma(b)} dA^*_b(t) = G(t)^{\sigma(b)} d\Lambda^0_b(t) = (d\xi^a(t))^*$$

where $a, b \geq 0$. Generalizing this to quantum noisy fields, we have with $e_n(x), n \geq 1$ denoting as earlier, an onb for $L^2(\mathbb{R}^3)$ and $f_a, a = 1, 2, \ldots, N$ an onb for $\mathbb{C}^N$,

$$d\Lambda_t(b, x | a, y) = d\Lambda^a_b(t, x, y)/dt = (dA^*_b(t, x)/dt). (dA_a(t, y)/dt), x, y \in \mathbb{R}^3, a, b = 0, 1, \ldots, N$$

and

$$d\Lambda_t(f, x | g, y)/dt =< f_b | f >< g | f_a > d\Lambda^a_b(t, x, y)/dt$$

$$d\Lambda_t(f, \phi | g, \psi)/dt =< g | f_a >< f_b | f > \int < \phi | x >< y | \psi > (d\Lambda^a_b(t, x, y)/dt) d^3x d^3y$$

$$=< g | f_a >< f_b | f > \int \bar{\phi}(x)\psi(y)(d\Lambda^a_b(t, x, y)/dt) d^3x d^3y$$

$$= d\Lambda_t(g \otimes \phi | f \otimes \psi)/dt$$

$$d\xi^a_b(t, x, y)/dt = G(t)^{\sigma(a,b)} d\Lambda^a_b(t, x, y)/dt = G(t)^{\sigma(a,b)} (dA^*_b(t, x)/dt). (dA_a(t, y)/dt), a, b = 0, 1, \ldots, N$$

The quantum Ito formula then reads

$$d\Lambda^a_b(t, x, y). d\Lambda^c_d(t, x', y') = \epsilon^a_d \delta^3(y - x') d\Lambda^c_b(t, x, y')/dt$$

In particular,

$$dA_a(t, x). dA_b(t, y)^* = \delta_{ab}\delta^3(x - y)dt, a, b = 1, 2, \ldots, N$$

and,

$$d\xi^a(t, x) = G(t)^{\sigma(a)} dA_a(t, x)/dt$$

$$d\xi_b(t, x)/dt = G(t)^{\sigma(b)} dA^*_b(t, x)/dt$$

so that

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$$d\xi^a(t,x).\,d\xi_b(t,y)=\delta^a_b.\,\delta^3(x-y)dt,\ a,b=1,2,\ldots,n$$

and the supercommutation relations then read

$$d\xi^a_b(t,x,y).\,d\xi^c_d(t,x',y')-(-1)^{\sigma(a,b)\sigma(c,d)}d\xi^c_d(t,x',y').\,d\xi^a_b(t,x,y)=$$

$$\epsilon^a_d\delta^3(y-x').\,d\xi^c_b(t,x,y')-(-1)^{\sigma(a,b).\sigma(c,d)}\epsilon^c_b\delta^3(y'-x)d\xi^a_d(t,x',y)$$

$$d\xi^a_b(t,x,y)=d\xi_b(t,x).\,d\xi^a(t,y)/dt$$

or equivalently,

$$d\xi^a_b(t,x,y)/dt=(d\xi_b(t,x)/dt).\,d\xi^a(t,y)/dt=(d\xi^b(t,x)/dt)^*(d\xi^a(t,x)/dt)$$

$$=G(t)^{\sigma(b)}(dA_b(t,x)^*/dt).\,G(t)^{\sigma(a)}(dA_a(t,y)/dt)$$

$$=G(t)^{\sigma(a)+\sigma(b)}(dA_b(t,x)/dt)^*.\,(dA_a(t,y)/dt)$$

$$=G(t)^{\sigma(a,b)}d\Lambda^a_b(t,x,y)/dt$$

Now, the supersymmetric action functional

$$S_0[\phi]=\int\phi^a(t,x)^*M(x,y|a,b)\phi^b(t,y)d^3xd^3ydt$$

after taking into account supersymmetric noise gets modified to

$$S[\phi]=(S_0+\delta S)[\phi]=\int(\phi^a(t,x)^*+d\xi_a(t,x)/dt).\,M(t,x,t',y|a,b).\,(\phi^b(t',y)_+d\xi^b(t',y)/dt)d^3xd^3ydtdt'$$

$$=\int\phi^a(t,x)^*M(t,x,t',y|a,b)\phi^b(t',y)d^3xd^3ydtdt'+$$

$$+\int\phi^a(t,x)^*M(t,x,t',y|a,b)(d\xi^b(t',y)/dt)d^3xd^3ydtdt'+\int(d\xi_a(t,x)/dt)M(t,x,t',y|a,b)\phi^b(t',y)d^3xd^3ydtdt'$$

$$+(M(t,x,t',y|a,b)(d\xi_a(t,x)/dt).\,(d\xi^b(t',y)/dt)d^3xd^3ydtdt'$$

Note that $\phi^a(t,x)$ are Bosonic fields for $a=1,2,\ldots,r$ and Fermionic for $a=r+1,\ldots,N$, just as $\xi^a(t,x)$ are Bosonic fields for $a=1,2,\ldots,r$ and Fermionic for $a=r+1,\ldots,N$. Note that $\sigma(a)=0,\ a=1,2,\ldots,r,\ \sigma(a)=1,\ a=r+1,\ldots,N$ so that since $\sigma(a,b)=\sigma(a)+\sigma(b)$ modulo 2, we can write

$$((\sigma(a,b)))_{1\le a,b\le N}=diag[0_r,I_{N-r}]$$

Thus, writing

$$\phi_a(t,x)=\phi^a(t,x)^*$$

and recalling that

$$d\xi^a_b(t,x,y)/dt=(d\xi_b(t,x)/dt).\,(d\xi^a(t,y)/dt$$

we get our formula for the noisy supersymmetric action as

$$S=S_0[\phi]+\delta S[\phi]$$

where

$$S_0[\phi] = \int \phi^a(t,x)^* M(t,x,t',y|a,b)\phi^b(t',y)d^3xd^3ydtdt'$$

$$\delta S[\phi] =$$

$$+\int \phi_a(t,x)M(t,x,t',y|a,b)(d\xi^b(t',y)/dt)d^3xd^3ydtdt' + \int (d\xi_a(t,x)/dt)M(t,x,t',y|a,b)\phi^b(t',y)d^3xd^3ydtdt'$$

$$+\int M(t,x,t,y|a,b)(d\xi_a^b(t,x,y)/dt)d^3xd^3ydt$$

The fundamental problem in computing the quantum effective action for the supersymmetric fields $\phi^a(t,x), a = 1,2,...,N$ in the presence of quantum noise now gets modified to the problem of computing the TPCP evolution of a mixed state on field space in the presence of quantum noise by means of the Feynman path integral:

$$\rho_T(\phi_T, \psi_T) =< \chi(u)|\int exp(iS_T[\phi]).\rho_0(\phi_0,\psi_0).exp(-iS_T[\psi])D\phi(0,T).D\phi^*(0,T)D\psi(0,T)D\psi^*(0,T)|\chi(u)>$$

This computation in the purely Bosonic case can be evaluated in closed form based on the formula for

$$<\chi(u)|exp(a(\phi) + a(\psi)^* + \lambda(H))|\chi(u)>$$

where $\chi(u)$ is a normalized coherent state. Note that in the purely Bosonic case, the noisy action functional is given by

$$S[\phi] = \int (\phi_a(t,x) + dA_a(t,x)^*/dt).M(t,x,t',y|a,b).(\phi^b(t',y) + dA_b(t',y)/dt)dtdt'd^3xd^3y$$

$$S_0[\phi] + \delta S[\phi]$$

where

$$S_0[\phi] = \int \phi_a(t,x).M(t,x,t',y|a,b).\phi^b(t',y)dtdt'd^3xd^3y$$

$$\delta S[\phi] =$$

$$+\int \phi_a(t,x)M(t,x,t',y|a,b)(dA_b(t',y)/dt)d^3xd^3ydtdt' + \int (dA_a(t,x)/dt)M(t,x,t',y|a,b)\phi^b(t',y)d^3xd^3ydtdt'$$

$$+\int M(t,x,t,y|a,b)(d\Lambda_a^b(t,x,y)/dt)d^3xd^3ydt$$

$$= a(\psi_1) + a(\psi_2)^* + \lambda(H)$$

where

$$\psi_1^b(t',y)^* = \int M(t,x,t',y|a,b)\phi_a(t,x)d^3xdt$$

or equivalently,

$$\psi_1^b(t',y) = \int \bar{M}(t,x,t',y|a,b)\phi^a(t,x)d^3xdt = \int M(t',y,t,x|b,a)\phi^a(t,x)d^3xdt = (M\phi)^b(t',y)$$

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$$\psi_2^a(t,x) = \int M(t,x,t',y|a,b)\phi^b(t',y)d^3ydt' = (M\phi)^a(t,x)$$

or in short,

$$\psi_1 = \psi_2 = M\phi$$

ie

$$\psi_1(t,x) = \psi_2(t,x) = M\phi(t,x)$$

ie,

$$\psi_1^a(t,x) = \psi_2^a(t,x) = (M\phi)^a(t,x)$$

and finally,

$$\lambda(H) = \int M(t,x,t,y|a,b)(d\Lambda_a^b(t,x,y)/dt)d^3xd^3ydt$$

so that

$$H(t,x,a|t',y,b) = M(t,x,t,y|a,b)\delta(t-t') = H_a^b(t,x,t',y)$$

so that

$$\lambda(H) = \int H(t,x,a|t',y,b)a_a(t,x) * a_b(t',y)dt'd^3ydtd^3x =$$

$$\int H(t,x,a|t',y,b)\lambda(a,t,x|b,t',y)dt'd^3ydtd^3x$$

Remark:

$$\int M(t,x,t',y|a,b)a_a(t,x) * a_b(t',y)dtd^3xdt'd^3y$$

$$= \int_{t\neq t'} M(t,x,t',y|a,b)(dA_a(t,x)/dt) * (dA_b(t,y)/dt)dtd^3xdt'd^3y'$$

$$+\int_{t=t'} M(t,x,t',y|a,b)(d\Lambda_a^b(t,x,y)/dt)d^3xd^3ydtdt'$$

We assume that the kernel $M(t,x,t',y|a,b)$ has support at $t = t'$, ie, it is expressible as

$$M(t,x,t',y|a,b) = \sum_{r=0}^{q} M_r(t,x,y|a,b)\delta^{(r)}(t-t')$$

for some finite positive integer q. In that case, the first term

$$\int_{t\neq t'} M(t,x,t',y|a,b)(dA_a(t,x)/dt) * (dA_b(t,y)/dt)dtd^3xdt'd^3y'$$

vanishes and we are left with the second term only. Therefore, in such a case, we have

$$\int_{t=t'} M(t,x,t',y|a,b)(d\Lambda_a^b(t,x,y)/dt)d^3xd^3ydtdt' = \int M(t,x,t',y|a,b)(d\Lambda_a^b(t,x,y)/dt)d^3xd^3ydtdt'$$

Such a singularity w.r.t the time coordinate holds for almost all the known field theories like electromagnetism, scalar Klein-Gordon field, vector Boson fields, Dirac field, Yang-Mills non-Abelian gauge fields and gravity in general relativity.

10. Conclusions

We have formulated a quantum version of nonlinear transmission line theory based on open quantum systems and the quantum master equation based on a Hamiltonian for the transmission line to account for inductance and capacitance distributed parameters and Lindblad operators to account for resistive loss as well as loss, memory and nonlinear effects from distributed memristor parameters. We have then explained how random line loading in the classical sense can also be quantized in the from of quantum stochastic differential equations for the line voltage, charge and current using the formulation of Hudson and Parthasarathy. We have explained how line parameters, voltage and current can be estimated using quantum filtering theory as first formulated by Belavkin and finally polished and presented by John Gough and his colleagues. We have included a short digression on quantum neural networks on how to use the quantum master equation to track the joint probability distribution of the line voltage and current hence how to simulate the transmission line with random loading using a quantum mechanical model. Since the transmission line describes a quantum field in one space and one time dimension, a natural question to ask is how quantum noise can be incorporated in standard Bosonic, Fermionic and supersymmetric quantum field theories in one time and three space dimensions. We have provided some suggestions in this regard, once again using the Hudson-Parthasarathy quantum stochastic calculus.

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