Review of: "Periodic second-order systems and coupled forced Van der Pol oscillators"

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Potential competing interests: No potential competing interests to declare.

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The manuscript \textbf{Periodic second-order systems and coupled forced Van del Pol Oscillators}, written by Feliz Minh\'os and Sara Perestrelo, provides an analytical study of the existence and location of periodic solutions of second-order non-linear coupled planar systems, without periodicity conditions on the involved non-linearities.

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In this manuscript, we found well-known techniques previously developed in some papers and fundamental books regarding the study of periodic solutions in second-order non-linear differential equations, as well as results resembling the results from these other papers. The paper under review confirms this view. In particular, the authors mention that somehow the present paper is an extension of the same problem but for first-order non-linear coupled systems. There is nothing wrong with it: when the same approach helps to understand different problems and (or) a technique is improved to study the same problem, it allows us to understand it better and improve it.

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It is worth highlighting that the authors provide a new \textit{Nagumo-Type Condition} and a new (at least for me) concept of upper and lower solutions for non-linear coupled planar systems by establishing a new analytical approach and definitions. In my opinion, the authors should present a clear example of this so that the reader can understand the basic idea without going through all computations.

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\textbf{Major comments.} I have only one major comment for the authors concerning the proof of the main theorem (Theorem 4).

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In claim 1, the authors prove that for any pair $(z(t),w(t))$ of solutions of (18)-(2), it follows that $|z(t)|<r_1 $ and $|w(t)|<r_2 $ for all $t\in [0,T]$ independently of $\lambda, \mu \in [0,1]$. But in their argument, it is not clear why the

monotonicity condition (15) is necessary. Please provide a clearer explanation.

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In general, I recommend rewriting the proof of Theorem 4, preserving the same structure but being more clear in the arguments, for example, the conditions (15)-$(16).$

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\textbf{Minor comments.} The rest of my comments are extremely minor but desirable:

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Abbrev.: p.=page, s.=section, l.=line, l=-. counts from the end.

\begin{itemize}
\item p.2, s.1. The expression (4) is not an equation. Please correct and put the symbol = in the corresponding place.

\item p.2 s.1. The notation for the derivative with respect to t is not global. Firstly, $\chi^{\prime}$ seems the notation for the derivative with respect to t, but after in equation (5) we have $\dot{x}$.

\item p.7 s.3. l.8 exist. Replace with exists. Please check the whole document for typos/grammatical notes.

\item p.7 s.3. l.11. If $t \in [0,t]$... Replace t with t_0.

\item p.8 s.3. l.6. In this part, seems to me that the inequalities follow for all points $(t,z,\delta_1(t,z),\delta_2(t,z),z_1,z_2).$

\item Improved the quality of all the Figures.
\item p.12 s.4. l.-1. Provides $\mu \in \R$.
\end{itemize}

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\textbf{My recommendation}: Concerning the continuous attention in the study of general conditions for the existence and location of periodic solutions of second-order non-linear coupled planar systems, the paper considers a good problem for being published in this high-quality journal. However, I do not recommend the current version of the paper for publication and will suggest to the authors to reformulate the paper for later submission. The main results seem to be correct, but the paper is missing comments and explanations at some points that, in my opinion, should be included for clarity and

completeness. The authors should make an effort to help the reader to understand.

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\textbf{REVIEWER 1.}