Commentary

On a Paradox of Gravitational Redshift

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Gravitational redshift refers to the increase in wavelength (or reddening) of light as it escapes from a gravitational well. Einstein first derived this phenomenon in 1907 and 1911 by combining special relativity with the equivalence principle. The result was later incorporated into the full framework of general relativity. Today, nearly all textbooks on general relativity, both introductory and advanced, include derivations of gravitational redshift based on the principle of energy conservation. After a thorough review of the most representative of these derivations, we present what appears to be a previously unnoticed paradox affecting the energy-based derivation of gravitational redshift, and we examine its implications.

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1. Introduction

In 1907, Einstein introduced the equivalence principle [1], using it to extend the effects of special relativity to systems at rest in a gravitational field by positing their equivalence to uniformly accelerated systems. In this seminal work, he was the first to derive gravitational redshift, time dilation, and light deflection.

However, Einstein's initial attempt to extend special relativity to gravitational phenomena, as he himself acknowledged, left much to be desired. Consequently, he revisited the topic in 1911, presenting a simpler derivation of time dilation, redshift, and light deflection. Here, we briefly examine this second derivation of gravitational redshift [2].

Consider two material systems, S_1 and S_2, which are at rest within a local, uniform gravitational **a** (Fig. 1). S_1 and S_2 are separated by a distance *d*. Now, envision a reference frame, K_0. This frame, K_0, is a freely falling (gravitation-free) system located near S_2, with an initial instantaneous velocity of zero relative to S_2.

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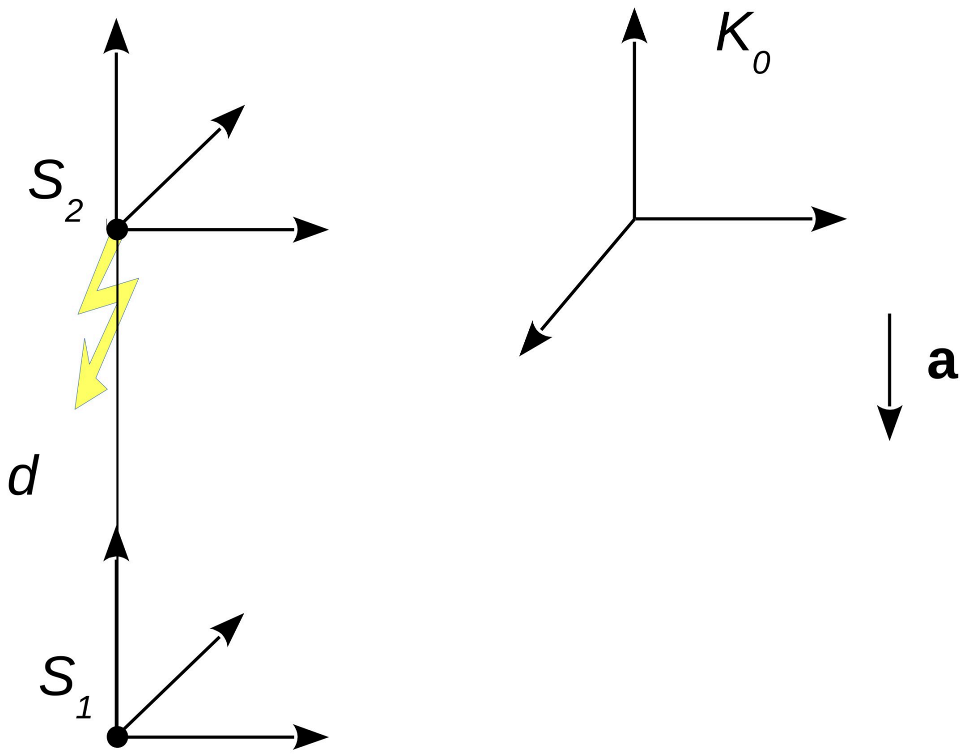

Figure 1. Material systems S_1 and S_2 remain stationary within a local, uniform gravitational field $\mathbf{a}$. The reference frame K_0 represents a free-falling system, free of gravitational effects, situated near S_2 with an initial velocity of zero relative to S_2. According to the equivalence principle, this configuration is equivalent to S_1 and S_2 accelerating upward with an acceleration of $-\mathbf{a}$, while frame K_0 remains an inertial, stationary frame.

Suppose a ray of light with frequency ν_2 is emitted by S_2 towards S_1 when the velocity of the free-falling frame K_0 relative to S_2 and S_1 is still zero. The ray of light reaches S_1 after a time approximately equal to d/c, where c represents the speed of light. According to the equivalence principle, this scenario is physically equivalent to one in which K_0 is at rest, and S_2 and S_1 accelerate with acceleration $-\mathbf{a}$ and initial velocity equal to zero.

When the ray of light arrives at S_1, the velocity of S_1 relative to the stationary frame K_0 is equal to $v = ad/c$. Therefore, from the perspective of any observer in frame K_0, the ray of light received at S_1 has a frequency ν_1, given by the Doppler formula:

$$\nu_1 = \nu_2 \left(1 + \frac{v}{c}\right) = \nu_2 \left(1 + \frac{ad}{c^2}\right), \tag{1}$$

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where $\nu_1 = \nu_2(1+v/c)$ is the Doppler formula for $v \ll c$.

By substituting ad for the gravitational potential Φ of S_2, with that of S_1 taken as zero, and assuming that the relation deduced for a homogeneous gravitational field would also hold for other field forms, Einstein arrived at the well-known (approximated) formula for the gravitational redshift (or blueshift in this case):

$$\nu_1 = \nu_2 \left(1 + \frac{\Phi}{c^2}\right). \tag{2}$$

From this formula, Einstein also derived the gravitational time dilation formula. Suppose that, during the time interval Δt_2 (as measured by a clock at rest at S_2), S_2 emits n waves. Then, from the definition of frequency, we have $n = \nu_2 \Delta t_2$. Let S_1 receive these same n waves during the time interval Δt_1 (as measured by a clock at rest at S1). Then, again, according to the definition of frequency, we have $n = \nu_1 \Delta t_1 = \nu_2 \Delta t_2$. Hence, equation leads to the gravitational time dilation formula:

$$\Delta t_2 = \Delta t_1 \left(1 + \frac{\Phi}{c^2}\right). \tag{3}$$

All these findings were subsequently integrated into the broader framework of general relativity (GR) completed in 1916, becoming a fundamental part of it. However, GR itself does not provide an independent or novel derivation of these phenomena [3].

In modern times, nearly every textbook on GR, whether introductory or advanced, includes derivations of gravitational redshift grounded in the principle of energy conservation. In the following section, we present what we consider to be a comprehensive and authoritative selection of gravitational redshift derivations based on energy conservation. In Section 3, we present a thought experiment that appears to be novel and challenges the consistency between energy conservation and gravitational redshift, thereby leading to a paradox. Finally, in the concluding section, we evaluate the robustness of this argument and examine its implications.

2. Gravitational redshift from energy conservation

The derivation of the gravitational frequency shift from energy conservation is widely considered an alternative, independent confirmation of the phenomenon, separate from the classical derivation based on special relativity and the principle of equivalence. This approach is commonly found in many GR textbooks. In the following subsections, we will showcase a representative sample of these derivations.

2.1. Gravitational frequency shift and the 'gravitational mass' of a photon

One of the most well-known derivations of gravitational redshift is based on the following premises, which are either explicitly stated or implicitly assumed:

1. Mass can be converted into energy, and all forms of energy have mass, as expressed by the mass-energy equivalence formula $E = m_0 c^2$, where m_0 is the rest mass [1][2];
2. Inertial mass is equivalent to gravitational mass;
3. The energy of a photon with frequency ν is given by $E = h\nu$, where h is Planck's constant;
4. The principle of conservation of energy.

Consider a receiver $\mathcal{R}$ placed directly above an emitter $\mathcal{E}$ at a distance d in a uniform gravitational field g. The emitter $\mathcal{E}$ releases a photon of frequency ν, and energy $E = h\nu$ towards $\mathcal{R}$. Although photons have no rest mass, for the sake of this derivation, we assume that the emitted photon has an 'effective' gravitational mass m equal to its inertial mass obtained from the mass-energy equivalence, $m = \frac{E}{c^2} = \frac{h\nu}{c^2}$ (assumptions 1, 2, and 3). As the photon ascends a height d in the gravitational field, its energy E' at the receiver $\mathcal{R}$ is lower than E. By energy conservation (assumption 4), we have:

$$E' = E - mgd, \tag{4}$$

where the potential energy mgd is the energy 'expended' by the photon while ascending the distance d.

Equation (4) can be rewritten as follows,

$$\nu' = \frac{E'}{h} = \frac{E - mgd}{h} = \frac{h\nu - \frac{h\nu}{c^2} gd}{h} = \nu\left(1 - \frac{gd}{c^2}\right), \tag{5}$$

where gd is the approximate gravitational potential difference between $\mathcal{R}$ and $\mathcal{E}$. If the positions of $\mathcal{E}$ and $\mathcal{R}$ are reversed, the sign in the equation switches from minus to plus.

An 'infinitesimal' version of this type of derivations can be found, for instance, in the book by Rindler [4].

2.2. Misner, Thorne, and Wheeler's derivation

In their seminal highly influential book [5], Misner, Thorne, and Wheeler presented the following derivation. Consider a particle of rest mass m_0 initially at rest in a gravitational field g at point A, which falls freely to point B, covering a distance d. During this descent, it gains kinetic energy $m_0 gd$, so that its total energy becomes $m_0c^2 + m_0gd$ (including rest mass energy m_0c^2 according to mass-energy equivalence).

Now, suppose the particle annihilating at point B, converting its total energy (including rest mass and kinetic energy) into a photon with equivalent energy. This photon then ascends in the gravitational field to point A. In the absence of gravitational effects, it would retain its initial energy upon reaching A. However, this scenario creates a paradox: if the photon were converted back into a particle with rest mass m, the excess energy mgd would be generated without cost, violating the principle of energy conservation.

To resolve this paradox, and since photon energy is inextricably linked to its frequency, the photon must experience a redshift as it ascends in the gravitational field.

2.3. Weinberg's derivation

Steven Weinberg [6] proposed the following derivation. Consider a scenario where a photon is emitted at point 1 by a heavy, nonrelativistic apparatus. In a locally inertial coordinate system moving with the apparatus, an observer observes a change in the apparatus's internal energy and inertial mass, related to the observed photon frequency ν_1. This relationship is expressed as $\Delta m_1 c^2 = -h\nu_1$, where h is Planck's constant.

Now, suppose the photon is absorbed at point 2 by a second heavy apparatus. In a locally inertial coordinate system moving with the second apparatus, an observer observes a change in the apparatus's inertial mass, linked to the observed photon frequency ν_2, represented by $\Delta m_2 c^2 = h\nu_2$.

The conservation of total internal and gravitational potential energy of both apparatuses requires that their sum remain constant before and after these events. This conservation leads to the equation

$$0 = \Delta m_1 c^2 + \phi_1 \Delta m_1 + \Delta m_2 c^2 + \phi_2 \Delta m_2.$$

Here, ϕ_1 and ϕ_2 are the gravitational potentials at point 1 and 2, respectively. Simplifying this equation gives

$$\begin{aligned} 0 &= -h\nu_1 + \phi_1 \frac{-h\nu_1}{c^2} + h\nu_1 + \phi_2 \frac{-h\nu_2}{c^2}, \\ 0 &= -h\nu_1 \left(1 + \frac{\phi_1}{c^2}\right) + h\nu_2 \left(1 + \frac{\phi_2}{c^2}\right), \\ h\nu_1 \left(1 + \frac{\phi_1}{c^2}\right) &= h\nu_2 \left(1 + \frac{\phi_2}{c^2}\right). \end{aligned}$$

Dividing both sides by h and solving for ν_2/ν_1

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$$\nu_2/\nu_1 = \frac{1+\frac{\phi_1}{c^2}}{1+\frac{\phi_2}{c^2}}.$$

For small differences in potential, this simplifies to

$$\frac{\nu_2}{\nu_1} \approx 1 + \frac{\phi_1 - \phi_2}{c^2}.$$

Therefore,

$$\nu_2 \approx \nu_1 \left(1 + \frac{\phi_1 - \phi_2}{c^2}\right),$$

where $\phi_1 - \phi_2$ is the gravitational potential difference between points 1 and 2.

2.4. Feynman, Leighton, and Sands's derivation

A further derivation from energy conservation is presented in another highly influential textbook, that by Feynman, Leighton, and Sands [7]. Consider an atom with a lowest energy state E_0 and a higher energy state E_1, capable of transitioning from E_1 to E_0 by emitting light. The frequency ν of the emitted light is given by $h\nu = E_1 - E_0$.

Now, suppose we have such an atom in state E_1 sitting on the floor, and we carry it to a height d. To achieve this, work is required to lift the mass $m_1 = E_1/c^2$ against the gravitational force. The work done is $\frac{E_1}{c^2} gd$.

Subsequently, the atom emits a photon and transitions to the lower energy state E_0. The mass now becomes $m_0 = E_0/c^2$. Therefore, upon returning the atom to the floor, the energy returned is $\frac{E_0}{c^2} gd$. Hence, the net work done is $\Delta U = \frac{(E_1 - E_0)}{c^2} gd$.

If the emitted photon were to descend to the floor and be absorbed, it should deliver more energy that $E_1 - E_0$. When we finish, the energy at the floor level is the energy E_0 of the atom in its lower state plus the energy E_{ph} received from the photon. According to energy conservation, that energy must be equal to the initial energy E_1 of the atom plus the net work done ΔU. Thus, $E_{ph} + E_0 = E_1 + \Delta U$ or $E_{ph} = (E_1 - E_0) + \Delta U$.

Substituting ΔU from earlier, the photon's energy becomes

$$E_{ph} = (E_1 - E_0)\left(1 + \frac{gd}{c^2}\right).$$

This energy corresponds to a frequency $\nu' = E_{ph}/h$, ultimately giving the formula for the gravitational redshift

$$\nu' = \nu\left(1 + \frac{gd}{c^2}\right),$$

where, as before, gd is the approximate gravitational potential difference between the floor and the point at height d.

3. The paradox

Here, we introduce a simple and accessible thought experiment that reveals an unexpected conflict between gravitational redshift and the principle of energy conservation [8]. The underlying rationale of this experiment has previously been employed to show that sound waves can escape any gravitational well [9]. Notably, the experiment is based on minimal and straightforward assumptions.

While it is recognized that several complexities hinder a rigorous derivation from the principle of energy conservation [10], and that the simplified redshift formula derived from this principle–as well as Einstein's original formulations–may lead to significant paradoxes if applied indiscriminately [11], the paradox we present here does not belong to those categories.

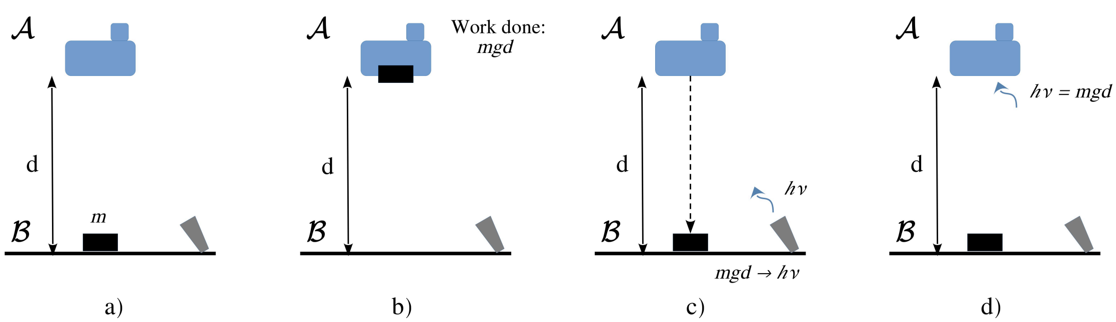

Figure 2. Pictorial representation of the thought experiment described in section 3.

Consider a body of mass m stationary at point $\mathcal{B}$ and a macroscopic apparatus stationary at point $\mathcal{A}$ at a height d above point $\mathcal{B}$ in a gravitational field g (Fig. 2). Let the apparatus do mechanical work on body m, raising it to point $\mathcal{A}$. The work done by the apparatus is equal to mgd, which is also equal to the gravitational potential energy of the body m relative to point $\mathcal{B}$.

Now, if we allow the mass m to fall from $\mathcal{A}$ to $\mathcal{B}$ and let a beacon, placed at point $\mathcal{B}$, use the kinetic energy of the mass at point $\mathcal{B}$ (which is equal to its potential energy at point $\mathcal{A}$) to create a photon with that energy, the energy of the photon must remain the same while climbing up the gravitational field back to

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point $\mathcal{A}$. The photon energy at point $\mathcal{A}$ must still equal mgd. As a matter of fact, this is demanded by the conservation of energy: by absorbing the photon, the apparatus must regain the same energy (mgd) that was expended on m at the beginning of the cycle. Therefore, owing to the Planck-Einstein relation, the photon frequency must be the same at points $\mathcal{A}$ and $\mathcal{B}$.

To emphasize the conclusion above, consider the cycle in reverse. The first step now involves the apparatus emitting a photon of energy E' (frequency ν') suitably lower than mgd. The original energy E' is such that when the photon arrives at the beacon, it becomes equal to $E_b = mgd$ ($> E'$) owing to the standard gravitational redshift (blueshift in this case). In this way, E_b is what is exactly needed to raise the mass m to the apparatus at the height d. Then, the mass is released back to the initial position, and the energy coming from that release (mgd) goes into the apparatus reservoir. At the end of the cycle, the apparatus will gain positive energy ($mgd - E' > 0$) out of nowhere.

4. Discussion

Today, gravitational redshift is widely recognized as a phenomenon with a robust theoretical foundation and substantial experimental validation, most notably through the well-known results of Pound and collaborators [12][13]. Furthermore, the practical implications of gravitational time dilation are evident in modern technology. For example, clocks on GPS satellites are adjusted prior to launch to account for the faster passage of time in orbit. It is often asserted that, without relativistic corrections, GPS-based applications, such as those used in mobile phones, would lose accuracy within hours. However, it is important to clarify that, in principle, the accuracy of the GPS system does not directly rely on the explicit inclusion of relativistic effects in the calculations. Rather, the use of signals from at least four satellites enables the system to determine the clock bias between the satellite-borne atomic clocks and the less accurate clock in the GPS receiver. Therefore, any delay due to relativistic effects would inevitably be absorbed into the determination of the clock bias through redundant satellite data.

Nevertheless, since we cannot neglect the robust theoretical and practical support for gravitational redshift, our paradox must admit a resolution. In this section, while we do not claim to provide a definitive solution, we aim to rule out what we consider to be incorrect approaches and conceptual dead ends by presenting the following three considerations.

Consideration 1: A common initial reaction to our paradox might be to question the validity of the assumptions or the logical steps in the reasoning process. However, after thorough examination, we have not found any flaws in our assumptions or inference chain. In fact, our thought experiment relies on fewer assumptions compared to the well-known proofs presented in Section 2, and the inference steps are analogous. We are reasonably confident that neither the assumptions nor the inference chain are flawed or weaker.

Consideration 2: Some may argue that the paradox is invalid because its proof lies outside the framework of general relativity. However, we have two objections to this notion. First, our thought experiment is constructed within the weak field and low velocity ($v \ll c$) approximations, where GR reduces to Newtonian mechanics. Hence, results obtained in Newtonian mechanics are applicable in this context. Second, if the validity of our paradox is questioned due to its derivation outside of GR, the same logic should apply to all thought experiments presented in Section 2, as they too were derived outside of GR. In fact, it can be demonstrated that the derivations outlined in Section 2 are logically inconsistent and exhibit at least one flaw in their deductive reasoning. This inconsistency is not related to the fact that they were derived outside the framework of GR. Readers seeking a more detailed analysis of this issue are referred to the work cited in reference [8].

Consideration 3: Another possibility is that there might be a fundamental, yet subtle, flaw in our understanding of gravitational redshift.

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