Research Article

Lindblad Operator Computation for a Single-Mode Quantum Field in a Cylindrical Fibre Based on the Boltzmann Equation to Account for Random Scattering by the Phonon Lattice

Harish Parthasarathy[1], Anand Srivastava[2], Parul Garg[2], Dushyant Kumar[2], Nidarshana Pandey[2]

1. Physics and Astronomy, ECE Division, Netaji Subhas University of Technology, Delhi, India; 2. Netaji Subhas University of Technology, Delhi, India

The first and second order Boltzmann kinetic transport equations, taking into account external and internal electromagnetic interactions of the charged phonon lattice, are set up. The incident classical field component interacts with the phonon lattice and gets scattered randomly owing to the random motion of the phonons. The incident quantum field component thus interacts with both the incident classical field and the scattered classical field, thereby generating respectively non-random and random Hamiltonian perturbations to the total quantum field Hamiltonian. By using a second order Taylor expansion of the mixed state Schrödinger evolution of the quantum field, taking into account these nonrandom and random perturbations, and then forming statistical averages for the random component using the first and second order Boltzmann distributions of the particles in phase space, we are able to calculate the Lindblad operator term corresponding to the random component. We then demonstrate how to cancel out these nonrandom and Lindblad perturbations to the mixed state dynamics using counterterms, based on a Monte Carlo method involving generating a sequence of iid random Hamiltonian operators, applying Schrödinger evolution, and then forming ensemble averages making use of the strong law of large numbers.

1. The two particle classical Boltzmann equation with applications to the calculation of the scattered electromagnetic field

Let there be N identical charges, each of charge q with positions and velocities $(r_a(t), v_a(t)), a = 1, 2, \ldots, N$ at time t. We write $r(t) = (r_a(t))_{a=1}^N \in \mathbb{R}^{3N}, v(t) = (v_a(t))_{a=1}|^N \in \mathbb{R}^{3N}$. The joint probability density of these is $f(t, r, v)$ with $(r, v) \in \mathbb{R}^{6n}$. At time $t = 0$, the particles have the Gibbs equilibrium density at temperature $T = 1/k\beta$:

$$f(0, r, v) = Z(\beta)^{-1}.exp(-\beta((1/2)\sum_{a=1}^N v_a^2 + U(r))) \qquad (1)$$

where

$$U(r) = \sum_{1\le a<b\le N} U_{12}(|r_a - r_b|) + \sum_{a=1}^N U_0(r_a) \qquad (2)$$

where U_{12} is the interaction potential energy between a pair of charges and U_0 is the binding potential energy of a charge to its centre. The joint density $f(t, r, v)$ in phase space of the charges at time t is determined by their initial density

 doi.org/10.32388/Z8KOU2

$f(0, r, v)$ and their dynamical equations

$$dr_a(t)/dt = v_a(t), mdv_a(t)/dt = q(E_0(t, r_a(t)) + v_a(t) \times B(t, r_a(t))) + \sum_{b=1, b\neq a}^{N} F_{12}(r_a(t), v_a(t)|r_b(t), v_b(t)) \tag{3}$$

with $F_{12}(r, v|r', v')$ being the electromagnetic force exerted on the charge at phase coordinates (r, v) by a charge at (r', v') in the non-relativistic approximation:

$$F_{12}(r, v|r', v') = (q^2/4\pi\epsilon_0)(r - r')?|r - r'|^3 + (\mu_0 q^2/4\pi)(v \times (v' \times (r - r')))/|r - r'|^3 \tag{4}$$

The two particle Boltzmann equation is derived from the complete N particle dynamics

$$\partial_t f(t, r, v) + \sum_{a=1}^{N}(v_a, \nabla_{r_a})f(t, r, v) + (q/m)\sum_{a=1}^{N}(E_0(t, r_a) + v_a \times B_0(t, r_a), \nabla_{v_a})f(t, r, v)$$

$$+ \sum_{a,b=1,2,\ldots,N, a\neq b}(F_2(r_a, v_a|r_b, v_b), \nabla_{v_a})f(t, r, v) = 0 \tag{5}$$

by integrating over $(r_b, v_b), b = 3, 4, \ldots, N$. Doing so results in

$$\partial_t f_{12}(t, r_1, v_1, r_2, v_2) + (v_1, \nabla_{r_1})f_{12}(t, r_1, v_1, r_2, v_2) + (v_2, \nabla_{r_2})f_{12}(t, r_1, v_1, r_2, v_2)$$

$$+(q/m)\left((E_0(t, r_1) + v_1 \times B_0(t, r_1), \nabla_{v_1}) + (E_0(t, r_2) + v_2 \times B_0(t, r_2), \nabla_{v_2})\right) f_{12}(t, r_1, v_1, r_2, v_2)$$

$$+(1/m)\left[(F_{12}(r_1, v_1|r_2, v_2), \nabla_{v_1}) + (F_{21}(r_2, v_2|r_1, v_1, \nabla_{v_2})\right] f_{12}(t, r_1, v_1, r_2, v_2)$$

$$+((N-2)/m)\int [(F_{13}(r_1, v_1|r_3, v_3), \nabla_{v_1}) + (F_{23}(t, r_2, v_2|r_3, v_3), \nabla_{v_2})] f_{123}(t, r_1, v_1, r_2, v_2, r_3, v_3)d^3r_3d^3v_3 = 0 \tag{6}$$

In deriving this equation, we have made use of the identity

$$div_{v_1}(F_{12}(r_1, v_1|r_2, v_2)) = 0 \tag{7}$$

If we integrate (1) over (r_2, v_2), we then get the one particle Boltzmann equation:

$$\partial_t f_1(t, r_1, v_1) + (v_1, \nabla_{r_1}) + \int (v_2, \nabla_{r_2})f_{12}(t, r_1, v_1, r_2, v_2)d^3r_2d^3v_2$$

$$+(1/m)\int (F_{12}(r_1, v_1|r_2, v_2), \nabla_{v_1}) + (F_{21}(r_2, v_2|r_1, v_1, \nabla_{v_2})]f_{12}(t, r_1, v_1, r_2, v_2)d^3r_2d^3v_2$$

$$+((N-2)/m)\int (F_{12}(r_1, v_1|r_2, v_2), \nabla_{v_1})f_{12}(t, r_1, v_1, r_2, v_2)d^3r_2d^3v_2 = 0 \tag{8}$$

which simplifies to

$$\partial_t f_1(t, r_1, v_1) + (v_1, \nabla_{r_1}) + \int (v_2, \nabla_{r_2})f_{12}(t, r_1, v_1, r_2, v_2)d^3r_2d^3v_2$$

$$+((N-1)/m)\int (F_{12}(r_1, v_1|r_2, v_2), \nabla_{v_1})f_{12}(t, r_1, v_1, r_2, v_2)d^3r_2d^3v_2 = 0 \tag{9}$$

So far, no approximations have been made. Everything is exact. We now express

$$f_{12}(t, r_1, v_1, r_2, v_2) = f_1(t, r_1, v_1).\, f_1(t, r_2, v_2) + g_{12}(t, r_1, v_1, r_2, v_2) \tag{10}$$

where g_{12} is small. In shorthand notation,

$$f_{12} = f_1.\, f_2 + g_{12} \tag{11}$$

Obviously, for the consistency condition

doi.org/10.32388/Z8K0U2

$$\int f_{12} d^3 r_2 v^3 v_2 = f_1, \tag{12}$$

we require that

$$\int g_{12} d^3 r_2 d^3 v_2 = 0 \tag{13}$$

Obviously, all these results are symmetric with respect to the interchange of (r_1, v_1) and (r_2, v_2). The first-order Boltzmann equation for f_1 is obtained by substituting for f_{12} into the above equation (2) with the approximation $g_{12} = 0$. To get the second-order Boltzmann equation for f_{12}, we do not neglect g_{12} and substitute for f_{12} into (2) and then substitute for f_{123} the exact expression

$$f_{123}(t, r_1, v_1, r_2, v_2, r_3, v_3) = f_1(t, r_1, v_1).\, f(t, r_2, v_2).\, f(t, r_3, v_3) + f_1(t, r_1, v_1).\, g_{12}(t, r_2, v_2, r_3, v_3)$$
$$+ f_1(t, r_2, v_2).\, g_{12}(t, r_1, v_1, r_3, v_3) + f_1(t, r_3, v_3).\, g_{12}(t, r_1, v_1, r_2, v_2) + g_{123}(t, r_1, v_1, r_2, v_2, r_3, v_3) \tag{14}$$

and then neglect g_{123}. In this way, we obtain two nonlinear PDEs for f_1, g_{12}. Note that the above equation for f_{123} can be expressed in shorthand notation

$$f_{123} = f_1.\, f_2.\, f_3 + f_1.\, g_{23} + f_2.\, g_{13} + f_3.\, g_{12} + g_{123} \tag{15}$$

For consistency, again, we require

$$\int g_{123} d^3 r_3 d^3 v_3 = 0 \tag{16}$$

so that we obtain

$$\int f_{123} d^3 r_3 d^3 v_3 = f_{12} = f_1.\, f_2 + g_{12} \tag{17}$$

Now suppose that the space-time field $F(t, r|r_1, v_1)$ is a one-particle function and we wish to evaluate the mean and correlation of the random field $X(t, r) = \sum_{a=1}^{N} F(t, r|r_a(t), v_a(t))$. For example, the electric field produced by the N charged particles or the magnetic field produced by the same are, in the non-relativistic approximation, candidate examples of F:

$$E(t, r) = \sum_{a=1}^{N} E_0(t, r|r_a), \quad B(t, r) = \sum_{a=1}^{N} B_0(t, r|r_a, v_a) \tag{18}$$

where

$$E_0(t, r|r_a) = (q/4\pi\epsilon_0(r - r_a)/|r - r_a|^3, \quad B_0(t, r|r_a, v_a) = (\mu_0 q/4\pi) v_a \times (r - r_a)/|r - r_a|^3 \tag{19}$$

We find easily, using the indistinguishability of the particles, that

$$\langle X(t, r) \rangle = N. \int F(t, r|r_1, v_1).\, f_1(t, r_1, v_1) d^3 r_1 d^3 v_1 \tag{20}$$

$$\langle X(t, r) \otimes X(t, r') \rangle = N. \int F(t, r|r_1, v_1) \otimes F(t, r'|r_1, v_1) f_1(t, r_1, v_1) d^3 r_1 d^3 v_1$$
$$+ N(N-1) \int F(t, r|r_1, v_1) \otimes F(t, r'|r_2, v_2) f_{12}(t, r_1, v_1, r_2, v_2) d^3 r_1 d^3 v_1 d^3 r_2 d^3 v_2 \tag{21}$$

Exercise: For given $R_k \in \mathbb{R}^3, k = 1, 2, \ldots, n$, express $\langle \otimes_{k=1}^{n} X(t, R_k) \rangle$ in terms of $f_{12\ldots m}, m = 1, 2, \ldots, n$

2. Basics of cylindrical wave guide field analysis

The quantum field is

doi.org/10.32388/Z8K0U2

$$E(t,r)=c(1)e(t,r)+c(1)^*\bar{e}(t,r), \quad B(t,r)=c(1)b(t,r)+c(1)^*\bar{b}(t,r), \quad r=(x,y,z) \tag{22}$$

where $[c(1),c(1)^*]=1$ and $e(t,r),b(t,r)$ are vectors in $\mathbb{R}^3$ with the z axis directed along the length of the fibre and the x,y directions spanning the cross section of the fibre. More precisely, the cross-sectional area at every point in the fibre is parallel to the xy plane. Let D denote the cross-sectional area of the fibre. For example, if the fibre is rectangular, then $D=\{(x,y): 0\le x\le a, 0\le y\le b\}$ while if it is cylindrical of radius R, then $D=\{(x,y): x^2+y^2\le R^2\}$.

The (n,m) modes in a cylindrical fibre at frequency ω are given by

$$E_{z,nm}(\omega,r)=J_n(\alpha_n(m)\rho/R).(a(1,nm)\cos(m\phi)+a(2,nm)\sin(m\phi)).\exp(-\gamma(n,m)z) \tag{23}$$

$$H_{z,nm}(\omega,r)J_n(\beta_n(m)\rho/R)(b(1,nm)\cos(m\phi)+b(2,nm)\sin(m\phi)).\exp(-\gamma(m,n)'z) \tag{24}$$

where $r=(x,y,z)$ is described in cylindrical coordinates in terms of (ρ,ϕ,z) where

$$\rho=\sqrt{x^2+y^2}, \quad \phi=\tan^{-1}(y/x) \tag{25}$$

$\alpha_n(m), m\ge 1$ are the zeroes of $J_n(x)$ while $\beta_n(m), m\ge 1$ are the zeroes of $J_n'(x)$ for each $n=0,1,2,\dots$ and

$$\gamma(n,m)=\sqrt{h(n,m)^2-\omega^2\mu\epsilon}, \gamma(n,m)'=\sqrt{h(n,m)-\omega^2\mu\epsilon} \tag{26}$$

with

$$h(n,m)=\alpha_n(m)/R, h(n,m)'=\beta_n(m)/R \tag{27}$$

$J_n(x)$ is the Bessel function of order n and it arises during the separation of variables in cylindrical coordinates while solving the Helmholtz equation that describes wave propagation. Here, the frequency ω can be arbitrary and $E_z\ne 0, H_z=0$ describes the mode TM_{nm} while $E_z=0, H_z\ne 0$ describes the mode TE_{nm}. Thus, the cutoff frequency for the TM_{nm} mode is $h(n,m)/\sqrt{\epsilon\mu}$ while that for the TE_{nm} mode is $h(n,m)'/\sqrt{\epsilon\mu}$. In case the only frequency component of the electric and magnetic fields are at frequency ω, we then have the result that the total z component of the electric and magnetic fields within the fibre are the superpositions

$$E_z(t,r)=\sum_{nm}(Re[a(1,nm)exp(j(\omega t-\gamma(n,m)z))].u_{1nm}(\rho,\phi)+Re[a(2,nm)exp(j(\omega t-\gamma(n,m)z))].u_{2nm}(\rho,\phi)]) \tag{28}$$

$$H_z(t,r)=\sum_{nm}(Re[b(1,nm)exp(j(\omega t-\gamma(n,m)'z))].v_{1nm}(\rho,\phi)+Re[b(2,nm)exp(j(\omega t-\gamma(n,m)'z))].v_{2nm}(\rho,\phi)]) \tag{29}$$

or equivalently,

$$E_z(t,r)=\sum_{nm}Re(E_{z,nm}(\omega,r).exp(j\omega t)), H_z(t,r)=\sum_{nm}Re(H_{z,nm}(\omega,r).exp(j\omega t)) \tag{30}$$

and the transverse components of the electric and magnetic fields are then given by

$$E_\perp(t,r)=\sum_{nm}Re(E_{\perp,nm}(\omega,r)exp(j\omega t)), H_\perp(t,r)=\sum_{nm}Re(H_{\perp,nm}(\omega,r).exp(j\omega t)) \tag{31}$$

where

$$E_{\perp,nm}(\omega,r)=(-\gamma(n,m)/h(n,m)^2)\nabla_\perp E_{z,nm}(\omega,r)-(j\omega\mu/h(n,m))\nabla_\perp H_{z,nm}(\omega,r)\times\hat{z} \tag{32}$$

$$H_{\perp,nm}(\omega,r)=(-\gamma(n,m)'/h(n,m))\nabla_\perp H_{z,nm}(\omega,r)+(j\omega\mu/h(n,m)^2)\nabla_\perp E_{z,nm}(\omega,r)\times\hat{z} \tag{33}$$

In the above expressions, for E_z, H_z, we have defined

$$u_{1nm}(\rho,\phi)=J_n(\alpha_n(m)\rho/R).cos(m\phi), u_{2nm}(\rho,\phi)=J_n(\alpha_n(m)\rho/R).sin(m\phi) \tag{34}$$

$$v_{1nm}(\rho,\phi)=J_n(\beta_n(m)\rho/R).cos(m\phi), 2_{2nm}(\rho,\phi)=J_n(\beta_n(m)\rho/R).sin(m\phi) \tag{35}$$

doi.org/10.32388/Z8KoU2

These expressions can be combined into 3-vector equations as

$$\mathbf{E}(t,r) = Re\sum_{n,m} a(nm).\,exp(j\omega t)\mathbf{u}_{nm}(\rho,\phi,z|\omega) \tag{36}$$

$$\mathbf{H}(t,r) = Re\sum_{n,m} b(nm).\,exp(j\omega t))\mathbf{v}_{nm}(\rho,\phi,z|\omega) \tag{37}$$

where $\mathbf{u}_{nm}(\rho,\phi,z|\omega)$ are complex linear combinations (with complex 3-vector valued coefficients that depend upon ω) of $\exp(-\gamma(n,m)z).\,u_{1nm}(\rho,\phi), \exp(-\gamma(n,m)z)u_{2nm}(\rho,\phi)$, $\exp(-\gamma(n,m)'z)v_{1nm}(\rho,\phi)$ and $\exp(-\gamma(n,m)'z)v_{2nm}(\rho,\phi)$ and likewise for $\mathbf{v}_{nm}|(\rho,\phi,z|\omega)$.

Now we come to the case when the two ends of the fibre at $z=0$ and $z=d$ are closed. The fibre then acts like a cylindrical cavity resonator and the possible frequencies of oscillation of the em field are forced to assume only values in a discrete sequence, known as the characteristic frequencies of oscillation of the field within the cavity resonator. This is because the boundary conditions on the fields at the beginning and end of the cavity along the z axis ($\partial_z E_z = 0$) imply that the $\exp(-\gamma(n,m)z)$ dependence of the E_z field gets replaced by a linear combination of $\exp(\pm\gamma(n,m)z)$ that is proportional to $\cos(p\pi z/d)$ for some positive integer p which means that $\gamma(n,m) = j\beta(n,m)$ so that $\pi p/d = \beta(n,m)$ or equivalently,

$$(\pi p/d)^2 = \beta(n,m)^2 = -\gamma(n,m)^2 = \omega^2\mu\epsilon - h(n,m)^2 \qquad (38)$$

implying thereby that the possible values of ω are

$$\omega = (h(n,m)^2 + (\pi p/d)^2)^{1/2} = \omega(n,m,p), n,m,p \in \mathbb{Z}_+ \qquad (39)$$

These determine the characteristic frequencies of oscillation of the TM modes. Likewise, the boundary conditions at the fibre ends imply H_z vanishes at these two ends and hence that its $\exp(-\gamma(n,m)'z)$ dependence on z gets replaced by that linear combination of $\exp(\pm\gamma(n,m)z)$ that is proportional to $\sin(p\pi z/d)$ so that

$$\omega = \omega(n,m,p)' = (h(n,m) + (\pi p/d)^2)^{1/2} \qquad (40)$$

are the characteristic frequencies of oscillation of the TE modes.

To get a single mode field, we assume that the only mode that propagates is TM_{nm} and that it is right circularly polarized. (None of the TE modes is assumed to propagate. Of course, we could repeat the whole analysis by instead assuming that the only mode that propagates is TE_{nm}, but we shall stick to the former case). Thus, we are assuming that $H_z = 0$ and

$$E_z(t,r) = 2.Re(c(1)\exp(-j\omega t + j\beta(n,m)z + jm\phi))J_n(\alpha_n(m)\rho/R) = c(1)\exp(-j\omega t)u_{1,nm}(\rho,\phi,z)$$

$$+c(1)^*\exp(j\omega t)\bar{u}_{1,nm}(\rho,\phi,z) \qquad (41)$$

where

$$u_{1,nm}(\rho,\phi,z) = \exp(j\beta(n,m)z + im\phi)J_n(\alpha_n(m)\rho/R) \qquad (42)$$

Note that $u_{1,nm}$ is an eigenfunction of $L_z = -i\partial_\phi$ with eigenvalue $+m$ and in quantum mechanics, L_z is the z-component of the angular momentum operator.

 doi.org/10.32388/Z8K0U2

3. Cylindrical wave guide analysis of the Lindblad operator computation for a single mode quantum field in an optical fibre interacting with the random scattered classical electromagnetic field caused by interaction between the phonon lattice with the incident classical field

Assume that TM_{nm} mode only propagates. The frequency $\omega > h(n,m)/\sqrt{\mu\epsilon}$ (cutoff frequency of TM_{nm} mode) where $h(n,m) = \alpha_n(m)/R$ with $\alpha_n(m)$ denoting the m^{th} root of $J_n(x)$. We have

$$E_z(t,r) = E_z(t,\rho,\phi,z) = 2.Re(c(1).exp(-i\omega t)u_{nm}(\rho,\phi,z))$$

$$= c(1)exp(-i\omega t).u_{nm}(\rho,\phi,z) + c(1)^* exp(i\omega t)\bar{u}_{nm}(\rho,\phi,z) \qquad (43)$$

where

$$\gamma(n,m) = j\beta(n,m), \beta(n,m) = \sqrt{\omega^2\mu\epsilon - h(n,m)^2}, u_{nm}(\rho,\phi,z) = J_n(h(n,m)\rho).exp(jn\phi - j\beta(n,m)z) \qquad (44)$$

$H_z = 0$ is also assumed so that from Maxwell's equations,

$$E_\perp(t,r) = 2.Re(c(1)exp(-i\omega t)(-\gamma(n,m)/h(n,m)^2).\nabla_\perp u_{nm}(\rho,\phi,z)) \qquad (45)$$

$$H_\perp(t,r) = 2.Re(c(1)exp(-i\omega t)(-i\omega\epsilon/h(n,m)^2)\nabla_\perp u_{nm}(\rho,\phi,z) \times \hat{z}) \qquad (46)$$

Writing

$$\mathbf{u}_{nm}(\rho,\phi,z) = [(-\gamma(n,m)/h(n,m)^2)\nabla_\perp u_{nm}(\rho,\phi,z), u_{nm}(\rho,\phi,z)] \qquad (47)$$

and

$$\mathbf{v}_{nm}(\rho,\phi,z) = \mu.[(-i\omega\epsilon/h(n,m)^2)\nabla_\perp u_{nm}(\rho,\phi,z) \times \hat{z}, 0] \qquad (48)$$

We get the following expressions for the propagating electromagnetic field vectors within the cylindrical fibre:

$$\mathbf{E}(t,r) = 2.Re(c(1).exp(-j\omega t)\mathbf{u}_{nm}(\rho,\phi,z)) \qquad (49)$$

$$\mathbf{B}(t,r) = 2.Re(c(1).exp(-j\omega t).\mathbf{v}_{nm}(\rho,\phi,z)) \qquad (50)$$

An easy analysis shows that we can scale $u_{nm}(\rho,\phi,z)$, i.e., replace u_{nm} by $d(n,m)u_{nm}$ where $d(n,m)$ is an appropriate positive real number, so that the total energy within the fibre, assumed to have a length d is given by

$$U = \int_0^d \int_D ((\epsilon/2)|E(t,r)|^2 + (1/2\mu)|B(t,r)|^2)dxdy = \omega.c(1)^*c(1) \qquad (51)$$

or more precisely, $h\omega.c(1)^*c(1)$ where h equals Planck's constant divided by 2π. Assuming that such a scaling has been done (note that this implies that $\mathbf{u}_{nm}$ and $\mathbf{v}_{nm}$ also get scaled by the same $d(n,m)$), we obtain the following expressions for the quantum fields

$$E(t,r) = c(1)exp(-i\omega t)\mathbf{u}_{nm}(\rho,\phi,z) + c(1)^* exp(i\omega t)\bar{\mathbf{u}}_{nm}(\rho,\phi,z) \qquad (52)$$

doi.org/10.32388/Z8K0U2

$$B(t,r) = c(1)exp(-i\omega t)\mathbf{v}_{nm}(\rho,\phi,z) + c(1)^* exp(i\omega t)\overline{\mathbf{v}}_{nm}(\rho,\phi,z) \qquad (53)$$

with Hamiltonian

$$H = \omega.\, c(1)^* c(1) \qquad (54)$$

where the Bosonic CCRs are satisfied:

$$[c(1), c(1)^*] = 1 \qquad (55)$$

Note that writing $c(1,t) = c(1)exp(-i\omega t)$, we obtain the correct Heisenberg equations of motion:

$$\frac{dc(1.t)}{dt} = i\,[H, c(1,t)] = -i\omega.\, c(1,t) \tag{56}$$

and

$$\frac{dc(1,t)^*}{dt} = i\,[H, c(1,t)^*] = i\omega.\, c(1,t)^* \tag{57}$$

Note:

$$\nabla_\perp = \hat{\rho}.\,\frac{\partial}{\partial\rho} + \left(\frac{\hat{\phi}}{\rho}\right).\,\frac{\partial}{\partial\phi} = \hat{x}.\,\frac{\partial}{\partial x} + \hat{y}.\,\frac{\partial}{\partial y} \tag{58}$$

Therefore,

$$\nabla_\perp u_{nm}(\rho,\phi,z) = \exp(-j\beta(n,m)z + jn\phi)(h(n,m)J_n'(h(n,m)\rho)\hat{\rho} + J_n(h(n,m)\rho)jn.\,\hat{\phi}/\rho) \tag{59}$$

with

$$\hat{\rho} = \hat{x}.\cos(\phi) + \hat{y}.\sin(\phi) \tag{60}$$

$$\hat{\phi} = -\hat{x}.\sin(\phi) + \hat{y}.\cos(\phi) \tag{61}$$

where

$$\rho = \sqrt{x^2 + y^2}, \phi = \tan^{-1}(y/x) \tag{62}$$

Thus, in terms of column vectors of Cartesian components,

$$\mathbf{u}_{nm}(\rho,\phi,z) = d(n,m)\left[(-\gamma(n,m)/h(n,m)^2)\nabla_\perp u_{nm}(\rho,\phi,z), u_{nm}(\rho,\phi,z)\right]$$

$$= d(n,m)\exp(-j\beta(n,m)z + jn\phi)$$

$$\times \left[(-j\beta(n,m)/h(n,m)^2)(h(n,m)J_n'(h(n,m)\rho)\cos(\phi) - J_n(h(n,m)\rho)jn.\sin(\phi)/\rho),\right.$$

$$\left.(-j\beta(n,m)/h(n,m)^2)(h(n,m)J_n'(h(n,m)\rho)\sin(\phi) + J_n(h(n,m)\rho)jn.\cos(\phi)/\rho), J_n(h(n,m)\rho)\right]^T \tag{63}$$

and

$$\nabla_\perp u_{nm} \times \hat{z} = -\hat{\phi}.\,\partial_\rho u_{nm} + \hat{\rho}.\,\partial_\phi u_{nm}/\rho \tag{64}$$

so that

$$\mathbf{v}_{nm}(\rho,\phi,z) = \left[(-j\omega\mu\epsilon/h(n,m)^2)\nabla_\perp u_{nm}, 0\right] =$$

$$d(n,m)\left[(-j\omega\mu\epsilon(n,m)/h(n,m)^2)\nabla_\perp u_{nm}(\rho,\phi,z), 0\right]$$

doi.org/10.32388/Z8KOU2

$$= d(n,m)\exp(-j\beta(n,m)z + jn\phi).$$

$$\times\left[(-j\omega\mu\epsilon/h(n,m)^2(h(n,m)J_n'(h(n,m)\rho)\sin(\phi) - J_n(h(n,m)\rho)jn.\cos(\phi)/\rho),\right.$$

$$\left.(-j\omega\mu\epsilon/h(n,m)^2)(-h(n,m)J_n'(h(n,m)\rho)\cos(\phi) + J_n(h(n,m)\rho)jn.\sin(\phi)/\rho), 0\right]^T \quad (65)$$

We write $E_q(t,r), B_q(t,r)$ for $E(t,r), B(t,r)$ respectively to signify that these are the propagating quantum fields that carry information about the state/density operator which is a function of $c(1), c(1)^*$, as for example a Gibbs state

$$\rho = Z(\beta)^{-1}.\exp(-\beta\omega.c(1)^*c(1)) \quad (66)$$

Now, let $E_s(t,\rho,\phi,z)$ and $B_s(t,\rho,\phi,z)$ denote the random scattered classical electric and magnetic fields. The corresponding random interaction Hamiltonian operator between this scattered field and the quantum TM_{nm} field is given by

$$\delta H_s(t) = \epsilon.\int_C (E_s(t,r),)E_q(t,r))d^3r + \mu^{-1}\int_C (B_s(t,r), B_q(t,r))d^3r \quad (67)$$

$$= c(1)h(t) + c(1)^*\bar{h}(t) \quad (67)$$

where $h(t)$ are the complex valued functions of time given by the formulas

$$h(t) = \exp(-i\omega t)[\epsilon.\int_C (E_s(t,\rho,\phi,z), \mathbf{u}_{nm}(\rho,\phi,z))\rho.d\rho.d\phi.dz +$$

$$\mu^{-1}\int_C (B_s(t,\rho,\phi,z), \mathbf{v}_{nm}(\rho,\phi,z))\rho.d\rho.d\phi.dz] \quad (68)$$

where

$$E_s(t,\rho,\phi,z) = \sum_{a=1}^{N} E_s(t,\rho,\phi,z|r_a(t), v_a(t)) \quad (69)$$

$$B_s(t,\rho,\phi,z) = \sum_{a=1}^{N} B_s(t,\rho,\phi,z|r_a(t), v_a(t)) \quad (70)$$

with $(r_a(t), v_a(t))$ for any given a having the one particle Boltzmann probability density $f_1(t,r,v)$ and for any $a \neq b$, $(r_a(t), v_a(t)), (r_b(t), v_b(t))$ having the two particle joint Boltzmann probability density $f_{12}(t, r_1, v_1, r_2, v_2)$. We note that $E_s(t,\rho,\phi,z|r_a(t), v_a(t))$ and $B_s(t,\rho,\phi,z|r_a(t), v_a(t))$ are respectively the electric and magnetic fields in the non-relativistic approximation produced at the point (ρ,ϕ,z) at time t by a charged particle of charge q located at $r_a(t)$ and moving with velocity $v_a(t)$. Equivalently, we can write

$$h(t) = \sum_{a=1}^{N} h(t|r_a(t), v_a(t)) \quad (71)$$

where

$$h(t|r_a(t), v_a(t)) =$$

$$\exp(-i\omega t)[\epsilon.\int_C (E_s(t,\rho,\phi,z|r_a(t), v_a(t)), \mathbf{u}_{nm}(\rho,\phi,z))\rho.d\rho.d\phi.dz$$

$$+\mu^{-1}\int_C (B_s(t,\rho,\phi,z|r_a(t), v_a(t))), \mathbf{v}_{nm}(\rho,\phi,z))\rho.d\rho.d\phi.dz] \quad (72)$$

The Lindblad operators are then computed easily using the formulas

$$\langle\delta H_s(t)\rangle = c(1)\langle h(t)\rangle + c(1)^*\langle\bar{h}(t)\rangle \quad (73)$$

 doi.org/10.32388/Z8K0U2

and

$$\langle\delta H_s(t)X\delta H_s(t)\rangle$$

$$= c(1)Xc(1)\langle h(t)^2\rangle + (c(1)Xc(1)^* + c(1)^*Xc(1))\langle|h(t)|^2\rangle + (c(1)^*Xc(1)^*\langle\bar{h}(t)^2\rangle \tag{74}$$

for any quantum operator X, with

$$\langle h(t)\rangle = N\int h(t|r_1,v_1)f_1(t,r_1,v_1)d^3r_1d^3v_1 \tag{75}$$

$$\begin{aligned}&\langle h(t)^2\rangle = \\ &N\int h(t|r_1,v_1)^2 f_1(t,r_1,v_1)d^3r_1d^3v_1 + N(N-1) \\ &\int h(t|r_1,v_1)h(t|r_2,v_2)f_{12}(t,r_1,v_1,r_2,v_2)d^3r_1d^3v_1d^3r_2d^3v_2\end{aligned} \tag{76}$$

etc.

More precisely, if $\rho(t)$ is the state at time t, then the state at time $t+dt$ is given by

$$\rho(t+dt) = \rho(t) - idt[H_0,\rho(t)] - idt[\delta H_0(t),\rho(t)] - idt[\langle\delta H_s(t)\rangle,\rho(t)]$$

$$-(dt^2/2)(\langle\delta H_s(t)^2\rangle)\rho(t) + \rho(t)(\delta H_s(t)^2\rangle - 2\langle\delta H_s(t)\rho(t)\delta H_s(t)\rangle) \qquad (77)$$

where

$$\delta H_0(t) = \int_C (\epsilon(E_0(t,\rho,\phi,z),E_q(t,\rho,\phi,z))$$

$$+\mu^{-1}(B_0(t,\rho,\phi,z),B_q(t,\rho,\phi,z)))\rho.\,d\rho.\,d\phi.\,dz$$

$$= c(1)exp(-i\omega t)\int(\epsilon(E_0(t,\rho,\phi,z),\mathbf{u}_{nm}(\rho,\phi,z))$$

$$+\mu^{-1}(B_0(t,\rho,\phi,z),\mathbf{v}_{nm}(\rho,\phi,z))\rho).\,d\rho.\,d\phi.\,dz$$

$$+c(1)^*exp(i\omega t)\int\epsilon(E_0(t,\rho,\phi,z),\bar{\mathbf{u}}_{nm}(\rho,\phi,z))$$

$$+\mu^{-1}(B_0(t,\rho,\phi,z),\bar{\mathbf{v}}_{nm}(\rho,\phi,z))\rho).\,d\rho.\,d\phi.\,dz$$

$$= c(1)g(t) + c(1)^*\bar{g}(t) \qquad (78)$$

where

$$g(t) = exp(-i\omega t)\int_C(\epsilon(E_0(t,\rho,\phi,z),\mathbf{u}_{nm}(\rho,\phi,z))$$

$$+\mu^{-1}(B_0(t,\rho,\phi,z),\mathbf{v}_{nm}(\rho,\phi,z))\rho.\,d\rho.\,d\phi.\,dz \qquad (79)$$

The Lindblad terms are therefore evaluated using

$$\langle\delta H_s(t)^2\rangle =$$

$$\langle(c(1)h(t) + c(1)^*\bar{h}(t))^2\rangle = c(1)^2\langle h(t)^2\rangle + c(1)^{*2}\langle\bar{h}(t)^2\rangle + (c(1)c(1)^* + c(1)^*c(1))\langle|h(t)|^2\rangle \qquad (80)$$

Note that

$$\langle h(t)^2\rangle = N\int h(t|r_1,v_1)^2 f_1(t,r_1,v_1)d^3r_1d^3v_1$$

doi.org/10.32388/Z8KoU2

$$+N(N-1)\int h(t|r_1,v_1).h(t|r_2,v_2)f_{12}(t,r_1,v_1,r_2,v_2)d^3r_1d^3v_1d^3r_2d^3v_2 \quad (81)$$

$$\langle \bar{h}(t)^2\rangle = N\int \bar{h}(t|r_1,v_1)^2 f_1(t,r_1,v_1)d^3r_1d^3v_1$$

$$+N(N-1)\int \bar{h}(t|r_1,v_1).\bar{h}(t|r_2,v_2)f_{12}(t,r_1,v_1,r_2,v_2)d^3r_1d^3v_1d^3r_2d^3v_2 \quad (82)$$

$$\langle |h(t)|^2\rangle = N\int |h(t|r_1,v_1)|^2 f_1(t,r_1,v_1)d^3r_1d^3v_1$$

$$+N(N-1)\int h(t|r_1,v_1).\bar{h}(t|r_2,v_2)f_{12}(t,r_1,v_1,r_2,v_2)d^3r_1d^3v_1d^3r_2d^3v_2 \quad (83)$$

and finally,

$$\langle \delta H_s(t)\rho(t)\delta H_s(t)\rangle =$$

$$\langle h(t)^2\rangle c(1)\rho(t)c(1) + \langle \bar{h}(t)^2\rangle c(1)^*\rho(t)c(1)^{*2} + (c(1)\rho(t)c(1)^* + c(1)^*\rho(t)c(1))\langle |h(t)|^2\rangle \quad (84)$$

This formula immediately suggests to us how the counter potential and counter TPCP terms are to be designed to reduce the noise in the system. The counter potential term $-i[V_0(t),\rho(t)]$ is designed so that

$$V_0(t) \approx -(\delta H_0(t) + \langle \delta H_s(t)\rangle) \quad (85)$$

while the counter TPCP term $\theta(\rho(t))$ is designed so that

$$\theta(\rho(t)) \approx (dt/2)(\langle \delta H_s(t)^2\rangle\rho(t) + \rho(t)\langle \delta H_s(t)^2\rangle - 2\langle \delta H_s(t)\rho(t)\delta H_s(t)\rangle) \quad (86)$$

Such a counter term can be implemented using a Monte-Carlo algorithm based on averaging a sequence of independent identically distributed counterterms: At time t, choose iid random matrices $X_1, X_2, \ldots, X_N$ so that

$$\langle X_i\rangle = \delta H_0(t) + \langle \delta H_s(t)\rangle, i = 1,2,\ldots,N \quad (87)$$

$$\langle \delta X_i \delta X_j\rangle = \langle \delta X_1^2\rangle\delta_{ij} = \langle \delta H_s(t)^2\rangle\delta_{ij}, i,j = 1,2,\ldots.,N \quad (88)$$

where

$$\delta X_i = X_i - \langle X_i\rangle \quad (89)$$

and implement the algorithm

$$\rho_i(t+dt) = idt[X_i,\rho(t+dt)] + (dt^2/2)[X_i,[X_i,\rho(t+dt)]], i = 1,2,\ldots,N \quad (90)$$

Taking the average of the output state over all the N ensembles then gives us the output state as

$$\rho_N(t+dt) = N^{-1}\sum_{i=1}^N \rho_i(t+dt) = idt[N^{-1}\sum_{i=1}^N X_i,\rho(t)]$$

$$+(dt^2/2)N^{-1}\sum_{i=1}^N [X_i,[X_i,\rho(t)]] \quad (91)$$

which in the limit as $N\to\infty$, by the strong law of large numbers, converges to the desired state $\rho_0(t+dt)$:

$$\lim_{N\to\infty}\rho_N(t+dt) = \rho_0(t+dt) \quad (92)$$

where $\rho(t+dt)$ is given by (a) $\rho_0(t+dt)$ is given by cancelling out both the perturbations, i.e.,

$$\rho_0(t+dt) = \rho(t) - idt[H_0,\rho(t)] \quad (93)$$

4. Conclusions

This paper presents a detailed analysis outlining the various steps involved in computing all components of the Lindblad operator that describe the generalized Schrodinger equation for the dynamics of the mixed state of a quantum electromagnetic field propagating within a cylindrical optical fibre. The incident field consists of a purely quantum component assumed to be single mode plus a purely classical component which can get pretty large. This classical component interacts with the charged random phonon lattice of atoms in the fibre, thereby causing these atoms to execute random motion whose joint probability distribution in phase space satisfies the Liouville equation. By partial integration, we derive two approximate equations from this, namely the first and second order Boltzmann equations that describe the evolution of the marginal probability distribution of a single particle and of two particles. As a result, the field scattered by the phonons acquires classical randomness whose second order statistics can be determined completely by the one and two particle Boltzmann distribution functions. We then apply standard cylindrical wave guide analysis to express a single mode quantum TM_{nm} field propagating in the fibre in terms of a single photon creation and annihilation operator and the modal eigenfunctions. The interaction Hamiltonian between this quantum single mode field and the incident classical and scattered classical field is evaluated using the standard bilinear form for field energy derived from the quadratic form for the field energy in the electromagnetic field. By applying a second order Taylor expansion to the Schrodinger evolution with this interaction Hamiltonian, we derive the Lindblad master equation for the mixed state evolution of the single mode quantum photon field. The commutator/conservative term in this equation receives contributions from the incident field and the statistical average of the scattered field, which is computed using the single particle distribution function, while the dissipative/Lindblad/TPCP term receives contributions from the statistical correlations in the scattered field, which are computed using the single and two particle distribution functions. Finally, we indicate a procedure by which both of these perturbations can be canceled out based on Monte-Carlo simulations of the random perturbing Hamiltonian.

Acknowledgements

The authors would like to thank the Department of Science and Technology, New Delhi, for funding a project related to quantum key distribution, for which the work of this paper forms a part. The authors of this paper belong to the QKD group at the NSUT, with the second author being the principal investigator.

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.