Research Article

Dynamic Equations of the Discrete Particle and Incompressible Continuous Medium (Field) for Generalized Coordinates System

Shisheng Wang[1]

1. Institute for Neutron Physics and Reactor Technology (INR), Karlsruhe Institute of Technology, Karlsruhe, Germany

Using the metric tensor to define the infinitesimal displacement vector and the kinetic energy, a generalized scalar function (Lagrangian) is given. It is inherently invariant under a change of coordinates. With this scalar function and the Euler-Lagrangian procedure for a conservative system, a generalized dynamic equation of motion in covariant vector form is derived: for the discrete particle model and for the continuous flow field model. The covariant acceleration vector is explicitly given. A great advantage of this expression is that it is easy to find the conservation law in the system and easy to reflect underlying geometric symmetries if the metric tensor is independent of a specific coordinate direction; an example of the spherical coordinate is shown in the paper. The metric tensor of the spherical coordinate is independent of the azimuthal (ϕ) coordinate. For the discrete particle model, the momentum exchanges between neighboring particles (forces due to the direct interactions between the particle and its neighbors) do not need to be considered; thereby, the kinetic energy is assumed to be independent of the coordinates to neglect the influence of neighboring particles. The velocity gradient (to be exact, the gradient of kinetic energy) is ignored. The equation degenerates to Newton's second law for discrete particles, which can be located in a potential field. For a continuous flow field, the partial derivatives of the velocity with respect to the coordinates are included (the gradient of kinetic energy), so that the momentum exchanges between the particle and its neighbors are included (interactions between neighboring particles cause changes in kinetic energy and momentum). The kinetic energy gradient forms a product of a velocity gradient and velocity itself. This velocity gradient tensor can be decomposed into a symmetric part and an antisymmetric part to separate the local stretching or shrinking deformation (symmetric strain rate tensor) from the fluid parcel's local rotation (antisymmetric). There is a fine difference between this model and the Navier-

Stokes equations. The Navier-Stokes equations consider the momentum exchanges between fluid layers using a symmetric viscous stress tensor modeling, while this model is more general and adds an extra local rotational motion term to consider the interaction between velocity and vorticity, exactly to say, adding a half of the cross product of the velocity and vorticity vectors. This analysis primarily relies on a pure theoretical framework without direct validation against experimental data; further work should be conducted for fine experimental validations to support the theoretical findings. It should be emphasized that this model is only valid for a conservative system, namely without considering the irreversible process, such as viscous dissipations.

Corresponding author: Shisheng Wang, Shisheng.Wang@kit.edu

1. Introduction

The set of all possible configurations of a system forms what is called the configuration space (often a manifold). A generalized coordinate system is essentially a chart on this manifold. For example, the configuration space of a free particle in three dimensions is $\mathbb{R}^3$.

A coordinate system is used to describe the configuration space of a physical system. Unlike a Cartesian coordinate system, a generalized coordinate system sometimes is particularly useful when the system has constraints or symmetries that make non-Cartesian coordinates more natural or convenient. Generalized coordinates can be any variables, such as angles, distances, or any other parameters. For example, for a particle moving in a two-dimensional plane, one might choose polar coordinates (r, θ) rather than Cartesian coordinates (x, y). In three dimensions, the spherical coordinate system (r, θ, ϕ) is often used instead of the Cartesian coordinate system. In this paper, we will give a general dynamic equation of motion by using a metric tensor and the Euler-Lagrangian procedure. It will show the fundamental difference between the discrete particle model and the continuous medium (a field) model, and the deficiency of the widely used Navier-Stokes equations.

2. Line Element and Coordinate Velocity

A generalized coordinate system is typically denoted by q^i (with the index i running over the dimension of the coordinate system). These coordinates $\{q^1, q^2, \cdots, q^n\}$ serve as a coordinate chart of the configuration space of the system, which uniquely specifies the configuration of the system.

 doi.org/10.32388/Z8ZGYD

In a coordinate system, a local Euclidean infinitesimal displacement (the differential of the position) is written in terms of the coordinate differentials.

The squared length (or squared norm) of the infinitesimal displacement is given by the contraction

$$ds^2 = g_{ij} dq^i dq^j, \tag{1}$$

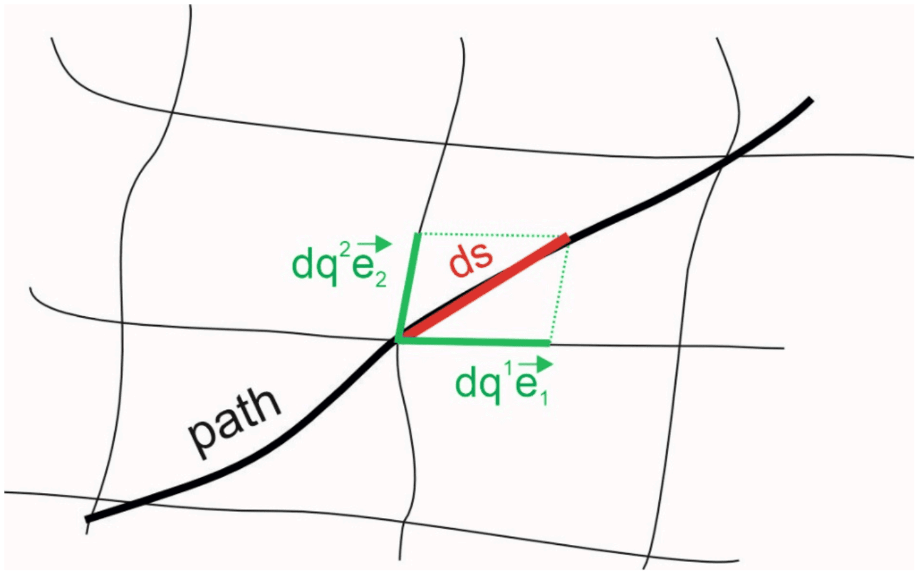

Figure 1. The length of the line element is given by $ds^2 = g_{ij} dq^i dq^j$

where g_{ij} is the metric tensor that encodes the geometric properties of the local space (or manifold). dq^i (with $i, j = 1, 2, \cdots n$) are the differentials of the generalized coordinates, as shown in Fig. 1. Here, the Einstein summation convention is used.

The metric tensor g_{ij} in a given coordinate system $\{q^1, q^2, \cdots, q^n\}$ is

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j. \tag{2}$$

It satisfies:

$$g_{ij} = g_{ji}. \tag{3}$$

This symmetry arises naturally because the metric tensor defines the inner product of two displacement vectors of eq. (1), the order of multiplication in the sum does not affect the result.

If we define the infinitesimal displacement as a column vector:

$$d\vec{q} = [dq^1, dq^2, \cdots, dq^n]^T, \tag{4}$$

doi.org/10.32388/Z8ZGYD

The square length of the infinitesimal displacement can be written as a quadratic form:

$$ds^2 = d\vec{q}^T g d\vec{q}. \tag{5}$$

If a particle moves along a path $\{q^1, q^2, \cdots, q^n\}$, then the velocity is the time derivative of the position vector. Differentiating the generalized coordinates with respect to time gives tangent velocity vectors (they are contravariant, with upper indices),

$$v^i = \dot{q}^i = \frac{dq^i}{dt}. \tag{6}$$

It represents the rate of change of the position coordinates with respect to time. This is the natural generalization of the Cartesian case to any coordinate system, where the metric tensor g_{ij} encodes the geometry of the space.

Dividing the line element, eq. (1), by dt^2, we can obtain the square of the speed (the magnitude of the velocity) in index notation and in quadratic form:

$$\left(\frac{ds}{dt}\right)^2 = \|\vec{v}\|^2 = g_{ij} v^i v^j = \vec{v}^T g \vec{v}. \tag{7}$$

Assuming the mass of the particle is m, the generalized expression for the kinetic energy, T, of the particle per unit mass, using the metric tensor, is:

$$2T = \langle \vec{v}, \vec{v} \rangle = g_{ij} v^j v^i = v_i v^i. \tag{8}$$

It is the trace of the momentum tensor, e.g., in 3D Cartesian coordinates, $2T = Tr(v^i v^j)$, and the trace is an invariant scalar function under a change of coordinates. Here $\langle \vec{v}, \vec{v} \rangle$ represents the inner product of contravariant velocity (tangent) vectors. The covariant (dual) velocity is defined by (lowering the index):

$$v_i = g_{ij} v^j. \tag{9}$$

In Lagrangian mechanics, the generalized covariant momentum, per unit mass, conjugate to a coordinate q^i is defined as (since the symmetric property of the metric tensor of eq. (3)):

$$p_i = \frac{\partial T}{\partial v^i} = g_{ij} v^j = v_i. \tag{10}$$

In a conservative system, consider a particle moving in a potential energy field, assuming the interaction energy of the particle (mass m) with others is U (J), the potential energy of the particle per unit mass is V (it is a scalar function):

$$V = \frac{U}{m}. \tag{11}$$

So that for a conservative system, the Lagrangian (it is a scalar function) per unit mass is

$$L = T - V = \frac{1}{2} g_{ij} v^i v^j - V = \frac{1}{2} \vec{v}^T g \vec{v} - V, \tag{12}$$

where T is the particle's kinetic energy (which depends on the velocities) and V is the specific potential energy that the particle holds. Its value always equals the total energy minus the kinetic energy in conservative systems through energy exchange but varies from position to position; its value depends implicitly on the velocity, since the kinetic energies vary with the velocities in different positions in a continuous field. When the velocity equals zero (so that the kinetic energy equals zero), then the Lagrangian function will reach a (minus) minimal value. e.g., at the stagnation point in a flow field, the pressure (potential energy) achieves the maximal value (the stagnation pressure), and the Lagrangian function reaches the (minus) minimal value. In conservative systems, the force is derived from the negative gradient of the potential energy. Thus, the form $T - V$ captures the idea that the motion of the system is driven by the dynamic balance between kinetic energy (energy of motion, varying from position to position) and potential energy (stored energy due to position, varying also from position to position). The difference $T - V$ effectively measures how "dynamically active" the system is in moving from one state to another.

It should be emphasized that a scalar function (the kinetic energy and the potential energy) is inherently invariant under any coordinate transformation; its intrinsic value does not depend on the choice of coordinates.

3. Equation of the Motion

The Euler–Lagrange equation, in a conservative system, for a generalized coordinate q^i is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0. \tag{13}$$

Computing the derivatives:

$$\frac{\partial L}{\partial v^i} = \frac{\partial T}{\partial v^i} = g_{ij} v^j = v_i. \tag{14}$$

Thus, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{dv_i}{dt} = g_{ij} v^j_{,t} + g_{ij,t} v^j, \tag{15}$$

and

doi.org/10.32388/Z8ZGYD

$$\frac{\partial V}{\partial q^i} = V_{,i}. \tag{16}$$

Using the chain rule:

$$\frac{\partial(2T)}{\partial q^i} = g_{lm,i}v^l v^m + g_{lm}v^l_{,i}v^m + g_{lm}v^l v^m_{,i}. \tag{17}$$

Since the symmetry of the metric tensor, eq. (3), and the commutative property of multiplication, the last two terms are equal to each other, then eq. (17) can be written as:

$$\frac{\partial T}{\partial q^i} = \frac{1}{2}g_{lm,i}v^l v^m + g_{lm}v^l_{,i}v^m. \tag{18}$$

That means the partial derivatives of the kinetic energy with respect to coordinates can be split into two parts: partial derivatives of the tensor metric ($\frac{1}{2}g_{lm,i}v^l v^m$) and the velocity field ($g_{lm}v^l_{,i}v^m$).

Substituting equations (15), (16), and (18) into eq. (13), re-arranging the terms, we have the equation of motion for coordinate q^i, in index notation:

$$g_{ij}v^j_{,t} + g_{ij,t}v^j - \frac{1}{2}g_{lm,i}v^l v^m = -V_{,i} + g_{lm}v^l_{,i}v^m. \tag{19}$$

Or using the definition of the covariant momentums of eq. (10):

$$\dot{p}_i - \frac{1}{2}g_{lm,i}v^l v^m = -V_{,i} + g_{lm}v^l_{,i}v^m. \tag{20}$$

With the indices *l and m* of the metric tensor, g_{lm}, running over the numbers of coordinates, $\{q^1, q^2, \cdots, q^n\}$.

Here, the comma "," indicates partial differentiation, and the index *i* represents the coordinate q^i.

$$g_{ij,i} = \partial_i g_{ij} = \frac{\partial g_{ij}}{\partial q^i}. \tag{21}$$

For example,

$$g_{ij,x} = \partial_x g_{ij} = \frac{\partial g_{ij}}{\partial x}. \tag{22}$$

The LHS of eq. (19) is the covariant derivative of velocity (material derivative), namely the covariant acceleration tensor:

$$a_i = g_{ij}v^j_{,t} + g_{ij,t}v^j - \frac{1}{2}g_{lm,i}v^l v^m. \tag{23}$$

doi.org/10.32388/Z8ZGYD

4. Discrete Particle Modelling

In contrast to a continuum model, where the properties, like velocity, and pressure, are described as continuous functions over space and time. In the discrete particle model, each particle is considered separately, with its own position, velocity, and properties. The velocity is considered to be independent of the position, and the partial derivatives of the velocity with respect to coordinates are equal to zero, thus, the last term on the RHS of eq. (19), $g_{lm}v^l_i v^m$, is ignored. The interaction energy with other particles is explicitly modeled as a specific potential energy, (e.g., gravitational or electrostatic potential energy).

Thus, the equation of motion is simplified to be

$$g_{ij}v^j_t + g_{ij,t}v^j - \frac{1}{2}g_{lm,i}v^l v^m = -V_i. \tag{24}$$

4.1. Cartesian coordinate system

In Cartesian coordinates (x, y, z), the metric is

$$ds^2 = dx^2 + dy^2 + dy^2. \tag{25}$$

Namely, in a Cartesian coordinate system, the metric tensor is considered "trivial", this means:

$$\begin{aligned} g_{ij} &= \delta_{ij}, \\ g_{ij,t} &= 0, \\ g_{lm,i} &= 0. \end{aligned} \tag{26}$$

Because in Euclidean space, it is constant and describes a flat space.

Then, the equation of motion reads:

$$v_{i,t} = a_i = -V_{,i}. \tag{27}$$

where, a_i is the material (total) derivative of velocity (covariant acceleration), and V_i is the gradient of the specific (with the sense of *per unit mass*) potential energy.

Recalling the incompressible Euler equations [1] with constant and uniform density for an incompressible flow field, Euler has simply extended this discrete particle equation into a continuous field; thus, eq. (27) has been re-written explicitly as:

$$v_{i,t} = D_t\vec{v} = \frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\nabla\right)\vec{v} = -\nabla p. \tag{28}$$

Here $V_{,i}$ is replaced by the gradient of the specific pressure field (with the sense of per unit density).

The equation, (27) or (28), describes the motion of an ideal fluid parcel in a field where the density remains the same throughout space and time. We will see in the continuous fluid model that, actually, it ignores the momentum exchanges between neighboring particles (between fluid layers). These equations are directly derived from the general equations, eq. (19), under the incompressibility assumption and ignoring the partial derivative of velocity with respect to the coordinates, for the Cartesian coordinate system. It must be emphasized here that the full covariant acceleration term ($D_t\vec{v}$) is given by the total (material) derivative, which contains an unsteady term (looking at how $\vec{v}$ changes with time at a fixed position) and a convective term (describing the velocity change with respect to positions in the velocity field).

The momentum equation states that fluid acceleration is only due to pressure gradients, where viscous effects are negligible, or more precisely, the term $g_{lm}v^l_i v^m$ is ignored. In other words, the neighboring fluid elements (such as the nearby velocity field) have no influence on the researched fluid element in the discrete flow particle model. Since the metric tensor for the Cartesian coordinate system is "trivial," the mathematical expression is very simple. This is just Newton's second law of motion for the discrete particles. In contrast to the discrete particle model, the field model has its special characteristics due to the interactions between neighboring particles. It must be noticed that the convective term in the incompressible Euler equations still includes a (minus) rotational (vorticity field) motion. In reality, it just is a part contribution from the velocity gradient tensor, mathematically, the projection of the velocity gradient tensor onto the velocity vector. We will discuss this issue in detail in the continuous medium modelling for Cartesian coordinate systems.

Here we will give another example of a "nontrivial" metric tensor.

4.2. Spherical coordinate system

In a spherical coordinate system, the coordinate velocity is expressed as:

$$\vec{v} = \begin{bmatrix} \dot{r} & \dot{\theta} & \dot{\phi} \end{bmatrix}^T. \tag{29}$$

The metric tensor for the spherical coordinate system is

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}. \tag{30}$$

Computing the derivatives of the metric tensor with respect to time, we have

doi.org/10.32388/Z8ZGYD

$$g_{ij,t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2r\dot{r} & 0 \\ 0 & 0 & 2r\dot{r}\sin^2\theta + 2r^2\sin\theta\cos\theta\dot{\theta} \end{bmatrix}. \tag{31}$$

Partial derivatives with respect to coordinate *r* read:

$$g_{ij,r} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2r & 0 \\ 0 & 0 & 2r\sin^2\theta \end{bmatrix}. \tag{32}$$

With respect to coordinate θ, they are:

$$g_{ij,\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2r^2\sin\theta\cos\theta \end{bmatrix}. \tag{33}$$

and with respect to coordinate ϕ are:

$$g_{ij,\phi} = 0. \tag{34}$$

Using these partial derivatives of the metric tensor, we have the following expressions:

$$\frac{1}{2}g_{lm,r}v^l v^m = \frac{1}{2}\vec{v}^T g_{,r}\vec{v} = r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2, \tag{35}$$

$$\frac{1}{2}g_{lm,\theta}v^l v^m = \frac{1}{2}\vec{v}^T g_{,\theta}\vec{v} = r^2\sin\theta\cos\theta\dot{\phi}^2, \tag{36}$$

and

$$\frac{1}{2}g_{lm,\phi}v^l v^m = \frac{1}{2}\vec{v}^T g_{,\phi}\vec{v} = 0. \tag{37}$$

Substituting these relations into eq. (24), we get the equation of motion for the spherical coordinate system, in column vector form:

$$\begin{bmatrix} \ddot{r} \\ r^2\ddot{\theta} \\ r^2\sin^2\theta\ddot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 2r\dot{r}\dot{\theta} \\ \left(2r\dot{r}\sin^2\theta + 2r^2\sin\theta\cos\theta\dot{\theta}\right)\dot{\phi} \end{bmatrix} - \begin{bmatrix} r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 \\ r^2\sin\theta\cos\theta\dot{\phi}^2 \\ 0 \end{bmatrix} = -\begin{bmatrix} \partial_r V \\ \partial_\theta V \\ \partial_\phi V \end{bmatrix} \tag{38}$$

Here, the covariant acceleration (material derivative) vector can be written compactly:

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 \\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2 \\ r^2\sin^2\theta\ddot{\phi} + \left(2r\dot{r}\sin^2\theta + 2r^2\sin\theta\cos\theta\dot{\theta}\right)\dot{\phi} \end{bmatrix}. \tag{39}$$

We can also write the equation of motion for individual coordinate directions, separately, for detailed discussion.

- The radial coordinate equation is:

doi.org/10.32388/Z8ZGYD

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = -\partial_r V. \tag{40}$$

- The polar (θ) coordinate equation is:

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2 = -\partial_\theta V. \tag{41}$$

- The azimuthal (ϕ) coordinate equation is:

$$r^2\sin^2\theta\ddot{\phi} + \left(2r\dot{r}\sin^2\theta + 2r^2\sin\theta\cos\theta\dot{\theta}\right)\dot{\phi} = -\partial_\phi V. \tag{42}$$

The metric tensor is independent of the angle of ϕ, as expressed by eq. (37). If the potential energy V is symmetric about the azimuthal angle ϕ, in other words, if the potential energy is independent of the angle ϕ, then eq. (42) becomes

$$\frac{d}{dt}\left(g_{\phi\phi}\dot{\phi}\right) = \frac{d}{dt}\left(r^2\sin^2\theta\dot{\phi}\right) = 0. \tag{43}$$

This equation is recognized as the conservation of angular momentum along the azimuthal angle ϕ, (rotation about the z-axis):

$$g_{\phi\phi}\dot{\phi} = r^2\sin^2\theta\dot{\phi} = constant. \tag{44}$$

5. Continuum Medium Modelling – Incompressible Fluid

In continuum medium modeling, the medium is analyzed without considering individual particles, making it suitable for fields like fluid mechanics to consider the momentum exchanges between neighboring particles. Properties, like velocities, are described as continuous functions over space and time. The partial derivatives of the velocity with respect to the coordinates should be included to consider the momentum exchanges between neighboring particles; namely, the last term of eq. (19), $g_{lm}v^l{}_{,i}v^m$ cannot be ignored.

5.1. Cartesian coordinate system

In fluid dynamics, the potential energy is represented by the pressure, namely, $V = p$, with the sense of per unit density in the framework of this paper. As mentioned before, it depends implicitly on the velocities and varies from position to position.

Thus, in index notation, the equation of motion reads:

$$v_{i,t} = -p_{,i} + g_{lm}v^l{}_{,i}v^m. \tag{45}$$

For the sake of simplicity, in the Cartesian coordinate system, the coordinate velocity is expressed to be:

$$\vec{v} = [u \quad v \quad w]^T. \tag{46}$$

Then, the last term on the RHS of eq. (45) can be written in matrix form:

$$g_{lm} v^l v^m{}_{,i} = \begin{bmatrix} \partial_x u & \partial_x v & \partial_x w \\ \partial_y u & \partial_y v & \partial_y w \\ \partial_z u & \partial_z v & \partial_z w \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \left(\vec{\partial} \otimes \vec{v}\right) \vec{v}. \tag{47}$$

The term $\vec{\partial} \otimes \vec{v}$ forms the transpose of the velocity gradient tensor $\nabla \vec{v}$:

$$\vec{\partial} \otimes \vec{v} = \left(\nabla \vec{v}\right)^T. \tag{48}$$

If it is rewritten explicitly in vector form, it reads:

$$D_t \vec{v} = -\nabla p + \left(\nabla \vec{v}\right)^T \vec{v}. \tag{49}$$

It is clear that the momentum equation depends not only on the velocity gradient, $\left(\nabla \vec{v}\right)^T$, (thereby through momentum exchanges between neighboring particles due to the relative motions) but also on the local velocity vector, $\vec{v}$.

It should be emphasized here that the velocity gradient is how the velocity of a fluid (or any moving medium) relatively changes with position. It represents the relative velocity difference between the researched particle and the fluid elements nearby. It gives us only the local information about the relative deformation and rotation between the fluid layers and has nothing to do with the initial velocity value of the researched particle; namely, the field is assumed to be smooth, and the field has no singularity when the velocity gradient exists. This is exactly the case for an incompressible fluid. In other words, the velocity gradient is always a real matrix according to the definition. This is the fundamental difference between compressible and incompressible fluids (where the wave speed is assumed to be infinite, $c = \infty$). For compressible fluids, where the wave speed is finite, there exists a density singularity when the flow velocity is equal to the wave speed, so that the field is not smooth. The density depends not only on the wave speed but also on the local flow velocity, $\rho_{mov} = \gamma \rho_0$, where γ is the Lorentz factor, ρ_0 is the mass density in the "rest frame of v=0". When the flow velocity approaches the wave speed, the density becomes infinitely great. Details can be found in the reference [2].

5.1.1. Decomposition of $\left(\nabla \vec{v}\right)^T$ into symmetric and antisymmetric parts

In general, any tensor $\left(\nabla \vec{v}\right)^T$ can be split into a symmetric tensor $\bar{\bar{S}}$ and an antisymmetric tensor $\bar{\bar{\Omega}}$ via

$$\bar{\bar{S}} = \frac{1}{2}\left(\nabla \vec{v}^T + \nabla \vec{v}\right), \quad \bar{\bar{\Omega}} = \frac{1}{2}\left(\nabla \vec{v}^T - \nabla \vec{v}\right). \tag{50}$$

Substituting this decomposition into eq. (49):

doi.org/10.32388/Z8ZGYD

$$D_t\vec{v} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \bullet \nabla)\,\vec{v} = -\nabla p + \left(\bar{\bar{S}} + \bar{\bar{\Omega}}\right)\vec{v} = -\nabla p + \bar{\bar{S}}\,\vec{v} + \bar{\bar{\Omega}}\,\vec{v}. \tag{51}$$

This decomposition clearly separates the local stretching and deformation (symmetric part) from the local rotation (antisymmetric part) of the fluid motion in Cartesian coordinates.

Recalling the spectral decomposition theorem, we know that any real symmetric matrix can be diagonalized by an orthogonal matrix: All eigenvalues of a real symmetric matrix are real; there exists an orthonormal basis of eigenvectors. Based on the spectral theorem, any symmetric matrix $\bar{\bar{S}}$ can be written as

$$\bar{\bar{S}} = \overrightarrow{\mathbf{e}}\;\bar{\bar{\Lambda}}\;\overrightarrow{\mathbf{e}}^{\mathbf{T}} = \lambda_1 \mathbf{e_1} \mathbf{e_1^T} + \lambda_2 \mathbf{e_2} \mathbf{e_2^T} + \lambda_3 \mathbf{e_3} \mathbf{e_3^T}. \tag{52}$$

where λ_1, λ_2, λ_3 are eigenvalues, $\bar{\bar{\Lambda}} = diag(\lambda_1, \lambda_2, \lambda_3)$, and $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ form an orthonormal basis of eigenvectors, $\overrightarrow{\mathbf{e}} = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. Thus, the matrix vector product $\bar{\bar{S}}\;\overrightarrow{v}$ becomes

$$\bar{\bar{S}}\;\overrightarrow{v} = \lambda_1 \left(\mathbf{e_1} \bullet \overrightarrow{v}\right) \mathbf{e_1} + \lambda_2 \left(\mathbf{e_2} \bullet \overrightarrow{v}\right) \mathbf{e_2} + \lambda_3 \left(\mathbf{e_3} \bullet \overrightarrow{v}\right) \mathbf{e_3}. \tag{53}$$

This expression shows that the effect of $\bar{\bar{S}}$ on the velocity vector $\overrightarrow{v}$ is to project $\overrightarrow{v}$ onto each eigenvector, e.g. $\left(\mathbf{e_1} \bullet \overrightarrow{v}\right)$, then scale that projection by the corresponding eigenvalue, and sum the contributions. Physically, it represents the stretching or shrinking of the fluid element along the eigenvector directions of S, see Fig. 2(a).

When an antisymmetric matrix is applied to a vector, it tells us how this vector changes direction under a small rotation. The matrix is often denoted by $\left[\overrightarrow{\Omega}\right]_{\times}$. The operation $\bar{\bar{\Omega}}\;\overrightarrow{v}$ becomes a vector cross product:

$$\bar{\bar{\Omega}}\;\overrightarrow{v} = \overrightarrow{\Omega} \times \overrightarrow{v}. \tag{54}$$

The velocity vector $\overrightarrow{v}$ undergoes a tiny (infinitesimal) rotation about an axis defined by the vector $\overrightarrow{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T$. In other words, $\bar{\bar{\Omega}}\;\overrightarrow{v}$ represents the instantaneous change (or "twist") of $\overrightarrow{v}$ due to the underlying rotational motion described by $\overrightarrow{\Omega}$, as is shown by Fig. 2(b).

doi.org/10.32388/Z8ZGYD

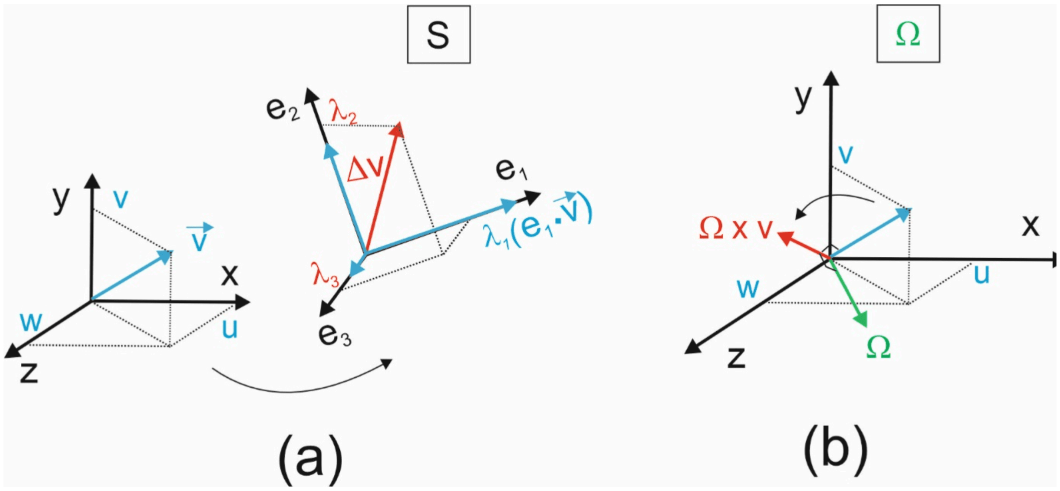

Figure 2. (a) The symmetric velocity gradient is expressed by an orthonormal basis of eigenvectors, the velocity deforms along the eigenvector directions, (b) the antisymmetric part twists the $\vec{v}$ to $\vec{\Omega} \times \vec{v}$.

The symmetric and antisymmetric parts can be written explicitly in matrix forms:

$$2\bar{\bar{S}} = \begin{bmatrix} \partial_x u & (\partial_x v + \partial_y u) & (\partial_x w + \partial_z u) \\ (\partial_y u + \partial_x v) & \partial_y v & (\partial_y w + \partial_z v) \\ (\partial_z u + \partial_x w) & (\partial_z v + \partial_y w) & \partial_z w \end{bmatrix}. \tag{55}$$

and

$$2\bar{\bar{\Omega}} = \begin{bmatrix} 0 & (\partial_x v - \partial_y u) & -(\partial_z u - \partial_x w) \\ -(\partial_y u - \partial_x v) & 0 & (\partial_y w - \partial_z v) \\ (\partial_z u - \partial_x w) & -(\partial_y w - \partial_z v) & 0 \end{bmatrix}. \tag{56}$$

Vorticity is a vector field that is twice the angular velocity of a fluid particle.

$$\vec{\omega} = 2\vec{\Omega}. \tag{57}$$

Thus, eq. (56) can be re-written as:

$$2\bar{\bar{\Omega}} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}. \tag{58}$$

where

$$\vec{\omega} = \nabla \times \vec{v}. \tag{59}$$

We may also express this in index notation:

$$\omega_i = \varepsilon_{ijk}\partial_j v^k, \quad 2\Omega_{ij} = \varepsilon_{ijk}\omega_k. \tag{60}$$

where ε_{ijk} is the three-dimensional Levi-Civita tensor.

The incompressible Navier-Stokes equation results from the assumptions on the Cauchy stress tensor and Stokes's hypothesis for viscous flow (stress constitutive equation used for incompressible viscous fluids). For an incompressible flow, the viscous stress tensor in the Navier-Stokes equation for a Newtonian fluid is typically modeled as

$$\bar{\bar{\tau}} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 2\mu\bar{\bar{S}}. \tag{61}$$

Here $\bar{\bar{S}}$ is the symmetric strain rate tensor, which is expressed by eq. (55).

Substituting eq. (61) into eq. (51), finally the incompressible momentum equation in vector form reads:

$$D_t\overrightarrow{v} = \frac{\partial\overrightarrow{v}}{\partial t} + \left(\overrightarrow{v}\cdot\nabla\right)\overrightarrow{v} = -\nabla p + \bar{\bar{\tau}}\ \overrightarrow{v} + \bar{\bar{\Omega}}\ \overrightarrow{v}. \tag{62}$$

It can also be expressed by the vorticity vector field

$$\frac{\partial\overrightarrow{v}}{\partial t} + \left(\overrightarrow{v}\cdot\nabla\right)\overrightarrow{v} = -\nabla p + \bar{\bar{\tau}}\ \overrightarrow{v} + \frac{1}{2}\overrightarrow{v}\times\overrightarrow{\omega}. \tag{63}$$

Recalling the Navier–Stokes equations [3], the total stress tensor is typically given by

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}. \tag{64}$$

The divergence of the total stress tensor, $\nabla\cdot\bar{\bar{\sigma}}$, contributes to the momentum equation; then, the incompressible Navier-Stokes equation is written as

$$\frac{\partial\overrightarrow{v}}{\partial t} + \left(\overrightarrow{v}\cdot\nabla\right)\overrightarrow{v} = -\nabla p + \nabla\cdot\bar{\bar{\tau}}. \tag{65}$$

Comparing eq. (62) with (65), we can observe the fundamental difference between both equations: the Navier–Stokes equations have molded the last two terms on the RHS of eq. (62) as a divergence of the symmetric viscous stress tensor

$$\bar{\bar{S}}\ \overrightarrow{v} + \bar{\bar{\Omega}}\ \overrightarrow{v} = \nabla\cdot\bar{\bar{\tau}}. \tag{66}$$

It is clear that the Navier-Stokes equations model the shear stress as a symmetric tensor; meanwhile, it is independent of the velocity itself, see eq. (61), and ignores the local rotation (antisymmetric part) of the fluid motion, $\bar{\bar{\Omega}}\ \overrightarrow{v} = \frac{1}{2}\overrightarrow{v}\times\overrightarrow{\omega}$. In reality, we should apply eq. (51) to model the fluid flow, instead of using eq. (65). The time evolution of the fluid velocity field depends not only on the velocity gradient, $\nabla\overrightarrow{v}$, but also on the value of the local velocity, $\overrightarrow{v}$. The term $\left(\nabla\overrightarrow{v}\right)^T\overrightarrow{v}$ represents the total momentum exchange between fluid particles due to relative motion between fluid layers for a conservative system.

The velocity gradient $\left(\nabla \overrightarrow{v}\right)^T$ includes all the information about the fluid element deformation, not only the stretching or shrinking but also the rotational motion.

Even if the local instantaneous value of the vorticity field is zero, for example, consider a two-dimensional flow with the component

$$\omega_z = \partial_x v - \partial_y u = 0; \qquad \partial_x v = \partial_y u \tag{67}$$

The exact equation of motion for the incompressible irrational fluid should be

$$D_t \vec{v} = \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla\right) \vec{v} = -\nabla p + \bar{\bar{S}}\, \vec{v}. \tag{68}$$

with the sense of per unit density; namely, here p represents the potential energy per unit density. In this special case, the viscous stress tensor is symmetric. Namely, the velocity gradient, $\left(\nabla \vec{v}\right)^T$, is symmetric.

5.1.2. Other variants of the dynamic equation of motion

As expressed by eq. (18), the partial derivatives of kinetic energy with respect to coordinates are split into two terms: $\frac{1}{2} g_{lm,i} v^l v^m$ and $g_{lm} v^l{}_{,i} v^m$.

For the Cartesian coordinate system, the first term is vanished. The dynamic equation of motion is simple:

$$D_t \vec{v} = \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla\right) \vec{v} = -\nabla p + \nabla T. \tag{69}$$

Recalling the vector identity:

$$\nabla T = \nabla \left(\frac{1}{2} v^2\right) = \left(\vec{v} \cdot \nabla\right) \vec{v} + \vec{v} \times \left(\nabla \times \vec{v}\right) = \left(\vec{v} \cdot \nabla\right) \vec{v} + \vec{v} \times \vec{\omega}. \tag{70}$$

Mathematically, it is an orthogonal decomposition of the kinetic energy gradient. The convective term is the directional derivative of the velocity gradient along the velocity direction; in other words, it is the projection of the velocity gradient (a tensor of type (1,1)) onto the velocity vector.

$$\left(\vec{v} \cdot \nabla\right) \vec{v} = \nabla_{\vec{v}} \left(\vec{v}\right) = \begin{bmatrix} u\partial_x u + v\partial_y u + w\partial_z u \\ u\partial_x v + v\partial_y v + w\partial_z v \\ u\partial_x w + v\partial_y w + w\partial_z w \end{bmatrix} = \left(\nabla \vec{v}\right) \vec{v}. \tag{71}$$

It is equal to the kinetic energy gradient minus the cross product of the velocity and vorticity vectors. Recalling that the result of a cross product is orthogonal to the vectors $\vec{v}$ and $\vec{\omega}$, thus, physically, eq. (70) represents an orthogonal decomposition of the kinetic energy gradient tensor: decomposition into the parallel and the orthogonal parts to the velocity vector. The parallel part forms the convective term (translational motion), while the orthogonal part ($\vec{v} \times \vec{\omega}$) forms a rotational motion perpendicular to the flow velocity.

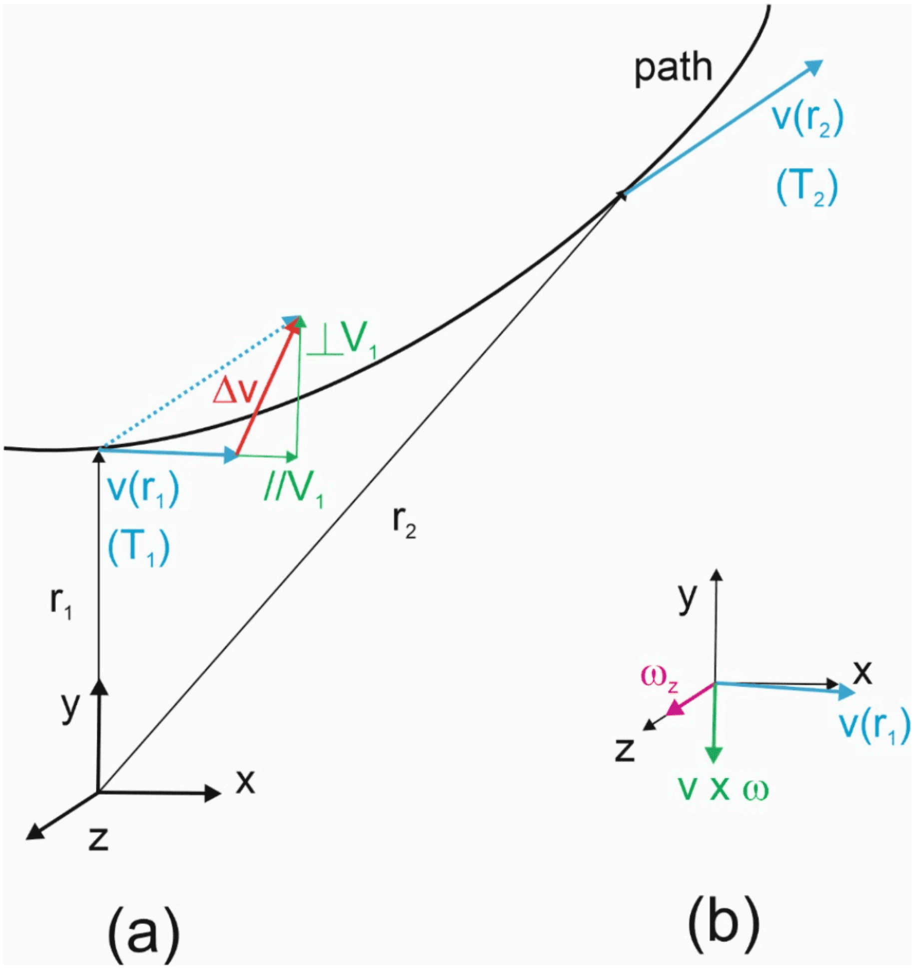

Figure 3. (a) The kinetic energy gradient is decomposed into two parts: the parallel and the orthogonal parts to the velocity vector. (b) The rotational motion ($\vec{v} \times \vec{\omega}$)).

Under the above arguments, we can see that the incompressible Euler equations, eq. (28), become

$$\frac{\partial \vec{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \frac{d\vec{r}}{dt} \times \vec{\omega} = -\nabla p. \tag{72}$$

Physically, it eliminates the rotational motion ($\vec{v} \times \vec{\omega}$) and only considers the directional derivative of the velocity field (kinetic energy gradient) along the velocity vector, $\nabla_{\vec{v}} \left(\vec{v} \right)$, as is shown by Fig. 3. It can be seen that for a steady irrotational translational flow, it just represents the work-energy theorem

doi.org/10.32388/Z8ZGYD

$dT = (-\nabla p) \cdot d\vec{r} = \vec{F}_{ext} \cdot d\vec{r}$. The term $\vec{v} \times \vec{\omega}$ is perpendicular to the velocity at all times, thus cannot change the magnitude of the kinetic energy, but forces the velocity direction to change.

With the help of the vector identity of eq. (70), we can rewrite the dynamic equation of eq. (69) as:

$$D_t\vec{v} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} = -\nabla p + \nabla\left(\frac{1}{2}v^2\right) = \nabla(T-p). \quad (73)$$

If the dynamic equation is written more compactly, it reads:

$$D_t\vec{v} - \nabla L = 0. \quad (74)$$

where L is the Lagrangian function per unit mass (with the sense per unit density).

Instead of using the convective term and the kinetic energy gradient, another variant of the dynamic equation is (the convective terms on both sides cancel each other):

$$\frac{\partial \vec{v}}{\partial t} = -\nabla p + \vec{v} \times \vec{\omega}. \quad (75)$$

This variant explicitly includes an interaction term between the vorticity field and velocity field, while the convective terms in both sides cancel each other.

If it is explicitly written in component notation:

$$\begin{cases} \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + v\omega_z - w\omega_y \\ \frac{\partial v}{\partial t} &= -\frac{\partial p}{\partial y} + w\omega_x - u\omega_z \\ \frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} + u\omega_y - v\omega_x \end{cases} \quad (76)$$

In the flow field, if at some point the instantaneous pressure gradient in one direction vanishes, e.g., in the z-direction, $\frac{\partial p}{\partial z} = 0$, the last equation becomes

$$\frac{\partial w}{\partial t} = u\omega_y - v\omega_x = (\vec{v} \times \vec{\omega})_z. \quad (77)$$

This is the induced second flow field by the interaction between the velocity and vorticity fields, which is perpendicular to the main streamwise velocity $\vec{v}$ and vorticity $\vec{\omega}$, though the local velocity is not accelerated by the pressure gradient in the z-direction. The path will still curve towards the direction of $(\vec{v} \times \vec{\omega})_z$. This is the classical 2D boundary layer approximation, where the pressure gradient in the direction normal to the plate is negligible, but there still exists an induced upward second flow velocity due to the shear flow between fluid layers, which leads the boundary layer to become thicker and thicker along the streamwise direction.

If at some point in the field the instantaneous vorticity vector is zero, the equation becomes

 doi.org/10.32388/Z8ZGYD

$$\frac{\partial \vec{v}}{\partial t} = -\nabla p. \tag{78}$$

This is the local velocity field change with respect to time, produced by the local potential energy gradient. The velocity direction is parallel to the potential energy gradient. If we define the pressure and density relation to be (in the sense of the ideal gas and adiabatic index $\gamma = 1$):

$$p = \rho c^2. \tag{79}$$

Substituting into eq. (78), we have:

$$\frac{\partial^2 \vec{x}}{\partial t^2} = -c^2 \nabla \rho. \tag{80}$$

where $\vec{x}$ represents the displacement vector, and c is the propagation speed of sound. Mathematically, it forms a wave-like equation.

In the general case, the velocity change rate is a combination effect of both motions, described by eq. (77) and (78), as long as the velocity gradient, $(\nabla \vec{v})^T$, is not symmetric.

We know that any 3×3 antisymmetric matrix, $\underline{\underline{\omega}}$, can be associated with a unique vector $\vec{\omega}$ such that for any vector $\vec{v}$ the product $\underline{\underline{\omega}}\vec{v}$ is equivalent to the cross product $\vec{\omega} \times \vec{v}$, this matrix is denoted by $[\vec{\omega}]_\times$:

$$[\vec{\omega}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{81}$$

Assuming the speed of the wave propagation in the medium is constant, we can define the following vectors:

$$\vec{F} = -\frac{\partial \vec{v}}{c\partial t} - \nabla \left(\frac{p}{c}\right). \tag{82}$$

Thus, eq. (81) and (82) can form a 4×4 field strength tensor:

$$F_{\mu\nu} = \begin{bmatrix} 0 & -F_x & -F_y & -F_z \\ F_x & 0 & -\omega_z & \omega_y \\ F_y & \omega_z & 0 & -\omega_x \\ F_z & -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{83}$$

The wave speed and flow velocity constitute a four-vector:

$$v^\nu = [\,c \quad u \quad v \quad w\,]^T. \tag{84}$$

Finally, the equation of motion, eq. (75), can be re-written more compactly in tensor contraction form:

$$F_{\mu\nu} v^\nu = 0, \quad for\ \mu = 1, 2, 3. \tag{85}$$

6. Limitations and Future Work

While this study presents a generalized dynamic equation of motion, it is important to acknowledge certain limitations. It is valid only for conservative systems without considering irreversible processes, such as viscous dissipations. It cannot be ignored in flows with high Reynolds numbers. The analysis primarily relies on a theoretical framework without including specific fluid properties, such as the viscous coefficient, etc. Fine-tuning of the model and direct validations against experimental data are needed in future work. Additionally, the modification of the Navier-Stokes equations focuses on a specific term related to the cross-product of the velocity and vorticity vector, but further work is needed to compare these findings with existing models in computational fluid dynamics. Future research should aim to:

- Conduct numerical simulations or experimental validations to support the theoretical findings.
- Compare the proposed framework with established models to assess practical implications.
- Explore additional cases where the assumptions of the Navier-Stokes equations may or may not hold.

By addressing these aspects, the proposed approach could be further refined and contextualized within the broader field of fluid mechanics.

7. Conclusions

The general squared length of the infinitesimal displacement in the local Euclidean space is given using the metric tensor; it is a scalar function and inherently invariant under any coordinate transformation. Based on the Euler-Lagrangian method for conservative systems, a general dynamic equation of motion has been derived using the metric tensor, both for the discrete particle model and for the continuous fluid model (the classical field). The fundamental difference between the discrete particle model and the continuous fluid model is the momentum exchange terms between the neighboring particles. In the discrete particle model, the momentum exchanges between neighboring particles are ignored. However, in the continuous fluid model, the momentum exchanges between neighboring particles are expressed by the kinetic energy gradient, which gives a product of the velocity gradient tensor and the velocity itself.

The dynamic equation for the particle model degenerates automatically to a general Newton's second law of motion for a general coordinate system using the metric tensor, eq. (24). The covariant acceleration tensor is meanwhile given. It is more convenient to calculate the geodesic equations using the covariant acceleration since it is not necessary to calculate the cumbersome Christoffel symbols.

In contrast to the discrete particle model, there exists a kinetic energy gradient to consider the effect of the momentum exchanges between neighboring particles due to relative motions between fluid layers in the continuous fluid model (interaction between particles causes a change in kinetic energy from position to position in the field). Mathematically, the kinetic energy gradient is a contraction of the velocity gradient and velocity with the help of the metric tensor. It is deeply discussed in the Cartesian coordinate system due to its simple mathematical structure. Through the decomposition of the velocity gradient tensor into a symmetric part and an antisymmetric part, we finally get the equation of motion for a continuous fluid, eq. (51). Compared with the widely used Navier-Stokes equations, it is clear that the Navier-Stokes equations use a symmetric viscous stress tensor to model the momentum exchanges between a particle and its neighbors; meanwhile, half of the interaction effect between the velocity and vorticity is neglected, eq. (63), while this interaction term will produce an additional induced second flow. It is shown that the kinetic energy gradient (vector) can be orthogonally decomposed into two parts: one part is parallel to the velocity direction (translational flow), and it is represented by the convective term, and another part is perpendicular to the local instantaneous velocity, which is represented by the cross product of the velocity and the vorticity vector, that forces the particle trajectory to be curved.

Appendix A. Material Derivatives and Geodesic Equation

Because acceleration is a tensor, we are free to express it in either its contravariant (with upper indices) or covariant (with lower indices) form.

The contravariant acceleration is defined with the help of the Christoffel symbols

$$a^i = \frac{Dv^i}{Dt} = \frac{dv^i}{dt} + \Gamma^i_{jk} v^j v^k. \tag{A1}$$

In order to compute the contravariant acceleration, at first, we need to calculate the cumbersome Christoffel symbols.

Another convenient way is that, at first, we calculate the covariant form, and then, through raising the index, we can get the contravariant acceleration component.

$$a_i = \frac{Dv_i}{Dt}, \quad a^i = g^{ij} a_j. \tag{A2}$$

In index notation, the covariant acceleration (material derivative) is given by eq. (23):

doi.org/10.32388/Z8ZGYD

$$a_i = \frac{Dv_i}{Dt} = g_{ij}\frac{dv^j}{dt} + \frac{dg_{ij}}{dt}v^j - \frac{1}{2}\frac{\partial g_{lm}}{\partial q^i}v^l v^m. \tag{A3}$$

The last term encodes the geometric properties of the local coordinates. If the metric tensor is independent of the coordinate q^i, then the last term becomes zero. This is a great advantage in finding the conservation component of the generalized momentum, as expressed by eq. (43).

For the motion of a free-falling particle, without any external forces acting on it, the geodesic equation can be described by the contravariant acceleration of eq. (A1) or the covariant acceleration of eq. (A3):

$$\begin{cases} a^i = \frac{Dv^i}{Dt} = \frac{dv^i}{dt} + \Gamma^i_{jk}v^j v^k = 0 \\ a_i = g_{ij}\frac{dv^j}{dt} + \frac{dg_{ij}}{dt}v^j - \frac{1}{2}\frac{\partial g_{lm}}{\partial q^i}v^l v^m = 0 \end{cases} \tag{A4}$$

A1. Polar coordinate system

The local infinitesimal displacement in a polar coordinate (r, θ) is

$$ds^2 = dr^2 + r^2 d\theta^2. \tag{A5}$$

The tangent coordinate velocity (contravariant) is expressed as:

$$\vec{v} = \begin{bmatrix} \frac{dr}{dt} & \frac{d\theta}{dt} \end{bmatrix}^T = \begin{bmatrix} \dot{r} & \dot{\theta} \end{bmatrix}^T. \tag{A6}$$

The metric tensor in polar coordinates, thus, is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}. \tag{A7}$$

Computing the derivatives of the metric tensor with respect to time, we have

$$g_{ij,t} = \frac{d}{dt}\begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2r\dot{r} \end{bmatrix}. \tag{A8}$$

Partial derivatives with respect to coordinates *r* and θ are, respectively:

$$g_{ij,r} = \frac{\partial}{\partial r}\begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}. \tag{A9}$$

and

$$g_{ij,\theta} = \frac{\partial}{\partial \theta}\begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} = 0. \tag{A10}$$

The last term of eq. (A3), thus, is

$$\frac{1}{2}g_{lm,r}v^l v^m = \frac{1}{2}\vec{v}^T g_{,r}\vec{v} = r\dot{\theta}^2, \tag{A11}$$

and

doi.org/10.32388/Z8ZGYD

$$\frac{1}{2}g_{lm,\theta}v^l v^m = \frac{1}{2}\vec{v}^T g_{,\theta}\vec{v} = 0. \tag{A12}$$

The eq. (A10) or (A12) shows that the metric tensor is independent of the angle θ. The covariant acceleration degenerates to

$$a_\theta = \frac{d(g_{\theta j}v^j)}{dt} = g_{\theta j}\frac{dv^j}{dt} + \frac{dg_{\theta j}}{dt}v^j. \tag{A13}$$

By eq. (A3), we can get the covariant acceleration vector:

$$\begin{bmatrix} a_r \\ a_\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2r\dot{r} \end{bmatrix}\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} - \begin{bmatrix} r\dot{\theta}^2 \\ 0 \end{bmatrix}. \tag{A14}$$

In component form, it reads:

$$\begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 \\ a_\theta = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} \end{cases} \tag{A15}$$

Now, we can convert the covariant acceleration (with lower indices) to the contravariant components (with upper indices) using the inverse metric tensor.

The inverse metric tensor of the polar coordinate is

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}. \tag{A16}$$

Thus, in a convenient manner, we can get the contravariant acceleration components through raising the indices:

$$\begin{cases} a^r = g^{rr}a_r = \ddot{r} - r\dot{\theta}^2 \\ a^\theta = g^{\theta\theta}a_\theta = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} \end{cases} \tag{A17}$$

In this manner, we can avoid calculating the cumbersome Christoffel symbols to calculate the contravariant acceleration.

For example, at first, we should calculate the nonzero Christoffel symbols:

$$\Gamma^r_{\theta\theta} = -r,\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}. \tag{A18}$$

Substituting these Christoffel symbols into eq. (1), we have:

- Contravariant acceleration for the radial component:

$$a^r = \frac{Dv^r}{Dt} = \frac{dv^r}{dt} + \Gamma^r_{\theta\theta}\dot{\theta}^2 = \ddot{r} - r\dot{\theta}^2. \tag{A19}$$

- Contravariant acceleration for the angular component:

$$a^\theta = \frac{Dv^\theta}{Dt} = \frac{dv^\theta}{dt} + 2\Gamma^\theta_{r\theta}\dot{r}\dot{\theta} = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}. \tag{A20}$$

A2. Spherical coordinate system

For a spherical coordinate system, the covariant acceleration (material derivative) vector is given by eq. (39):

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 \\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2 \\ r^2\sin^2\theta\ddot{\phi} + \left(2r\dot{r}\sin^2\theta + 2r^2\sin\theta\cos\theta\dot{\theta}\right)\dot{\phi} \end{bmatrix}. \tag{A21}$$

The inverse metric tensor of the spherical coordinate is

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}. \tag{A22}$$

Raising the indices, we have the expressions for contravariant accelerations:

$$\begin{cases} a^r = g^{rr}a_r = a_r \\ a^\theta = g^{\theta\theta}a_\theta = \frac{1}{r^2}a_\theta \\ a^\phi = g^{\phi\phi}a_\phi = \frac{1}{r^2\sin^2\theta}a_\phi \end{cases} \tag{A23}$$

If they are written explicitly out, they are read

$$\begin{cases} a^r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 \\ a^\theta = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 \\ a^\phi = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} \end{cases} \tag{A24}$$

Appendix B. Gradient of the Velocity Vector

In classical fluid dynamics (field theory), the momentum tensor in 3D is a 3×3 symmetric matrix, e.g., in a 3D Cartesian coordinate system per unit density:

$$v^i v^j = \vec{v} \otimes \vec{v} = \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix}. \tag{B1}$$

The trace is two times the specific kinetic energy and is invariant under similarity transformations, which means it does not depend on the choice of basis or coordinate system.

$$2T = uu + vv + ww. \tag{B2}$$

The partial derivatives of the specific kinetic energy (per unit mass) with respect to coordinates give out two terms, as expressed by eq. (18): The first term is the partial derivative of the metric tensor, and the second term is the partial derivative of the velocity field. For the continuous medium (field), the velocity field is the function of positions $v^i = v^i(q^1, q^2, \cdots, q^n)$, thus, the term $g_{lm} v^l_{,i} v^m$ cannot be ignored. Of that, $v^l_{,i}$ is the partial derivative of the velocity vector with respect to coordinate q^i.

When the partial derivative operator acts on a vector field, $\vec{v}$, with contravariant components v^j, it is given by

$$v^j_{,i} = \frac{\partial v^j}{\partial q^i} = \partial_i v^j. \tag{B3}$$

In this expression, the index *j* (from v^j) remains contravariant, while the differentiation index *i* (from q^i) is covariant. The resulting object is a mixed tensor of type (1,1), and the partial derivative ∂_i serves as the covariant part.

Since the symmetry of the metric tensor, $g = g^T$, and the commutative property of multiplication, we have two possibilities to express $g_{lm} v^l v^m{}_{,i}$. A more "physical" expression will be given in Appendix C.

If the metric tensor is distributed to the velocity gradient, one possibility is the inner product of a contravariant velocity (with an upper index of *l*) and a covariant derivative (with a lower index of *l*):

$$g_{lm} v^l v^m{}_{,i} = v^l \left(g_{lm} v^m{}_{,i} \right) = v^l v_{l,i} = \left\langle v^l, v_{l,i} \right\rangle. \tag{B4}$$

Here, the covariant derivative (lowering the index by the metric tensor) is given by

$$v_{l,i} = g_{lm} v^m{}_{,i} \tag{B5}$$

Another possibility is the inner product of the natural partial derivative of the velocity (mixed type of (1,1)) and the covariant velocity vector (the metric tensor is distributed to the contravariant velocity vector, lowering the index gives the covariant velocity):

$$g_{lm} v^l v^m{}_{,i} = \left(g_{ml} v^l \right) v^m{}_{,i} = v_m v^m{}_{,i} = \left\langle v_m, v^m{}_{,i} \right\rangle. \tag{B6}$$

Here, the metric tensor is distributed to the contravariant velocity, resulting in a covariant velocity, which is given by

$$v_m = g_{ml} v^l. \tag{B7}$$

We will see that the second expression is more convenient.

In a more compact vector form, it can also be written in a quadratic form:

$$g_{lm}v^l v^m{}_{,i} = \left(\overrightarrow{v}^j\right)^T \left(g\overrightarrow{v}^j_{,i}\right) = \left(\overrightarrow{v}^j_{,i}\right)^T \left(g\overrightarrow{v}^j\right). \tag{B8}$$

where the vector is written to be a column vector.

$$\overrightarrow{v}^j = \left(v^1, v^2, \cdots, v^n\right)^T, \ \ \overrightarrow{v}^j_{,i} = \left(\partial_i v^1, \partial_i v^2, \cdots, \partial_i v^n\right)^T. \tag{B9}$$

For the "trivial" metric tensor in a Cartesian coordinate system, this term is given in the paper by eq. (47).

B1. Polar coordinate system

When the partial operator acts on a vector field $\overrightarrow{v} = \left(v^r, v^\theta\right)^T = \left(\dot{r}, \dot{\theta}\right)^T$, the gradient of the velocity is given by

$$\left(\partial\overrightarrow{v}\right)^j_r = \left(\partial_r v^r, \partial_r v^\theta\right), \ \ \left(\partial\overrightarrow{v}\right)^j_\theta = \left(\partial_\theta v^r, \partial_\theta v^\theta\right). \tag{B10}$$

In matrix form, it reads:

$$\left(\nabla\overrightarrow{v}\right)^j_i = \begin{bmatrix} \partial_r v^r & \partial_r v^\theta \\ \partial_\theta v^r & \partial_\theta v^\theta \end{bmatrix} = \overrightarrow{\partial} \bigotimes \overrightarrow{v}. \tag{B11}$$

Then, we have the covariant derivative expression (with the free lower index of *l*):

$$\begin{cases} v_{l,r} = g_{lm}v^m{}_{,r} = \left(\partial_r v^r, r^2\partial_r v^\theta\right)^T \\ v_{l,\theta} = g_{lm}v^m{}_{,\theta} = \left(\partial_\theta v^r, r^2\partial_\theta v^\theta\right)^T \end{cases} \tag{B12}$$

The inner product of the covariant derivative and contravariant vector gives

$$\begin{cases} \left(g_{lm}v^m{}_{,r}\right)v^l = v_{l,r}v^l = v^r\partial_r v^r + r^2 v^\theta\partial_r v^\theta \\ \left(g_{lm}v^m{}_{,\theta}\right)v^l = v_{l,\theta}v^l = v^r\partial_\theta v^r + r^2 v^\theta\partial_\theta v^\theta \end{cases} \tag{B13}$$

It can be written in matrix form:

$$\left(g_{lm}v^m{}_{,i}\right)v^l = \begin{bmatrix} \partial_r v^r & r^2\partial_r v^\theta \\ \partial_\theta v^r & r^2\partial_\theta v^\theta \end{bmatrix} \begin{bmatrix} v^r \\ v^\theta \end{bmatrix}. \tag{B14}$$

As mentioned before, a more convenient form is using the natural partial derivatives and the covariant velocity components:

$$v^m{}_{,i}\left(g_{ml}v^l\right) = v^m_{,i}v_m = \begin{bmatrix} \partial_r v^r & \partial_r v^\theta \\ \partial_\theta v^r & \partial_\theta v^\theta \end{bmatrix} \begin{bmatrix} v^r \\ r^2 v^\theta \end{bmatrix}. \tag{B15}$$

Using the velocity gradient definition of eq. (B11) for $\left(\nabla\overrightarrow{v}\right)^j_i$, eq. (B15) can be expressed to be

$$v^m_{,i}(g_{ml}v^l) = (\nabla\overrightarrow{v})^j_i v_j. \tag{B16}$$

Similarly, the velocity gradient can be decomposed into a symmetric and an antisymmetric part:

doi.org/10.32388/Z8ZGYD

$$2\bar{\bar{S}} = \begin{bmatrix} \partial_r v^r & (\partial_r v^\theta + \partial_\theta v^r) \\ (\partial_\theta v^r + \partial_r v^\theta) & \partial_\theta v^\theta \end{bmatrix}, \tag{B17}$$

and

$$2\bar{\bar{\Omega}} = \begin{bmatrix} 0 & (\partial_r v^\theta - \partial_\theta v^r) \\ -(\partial_r v^\theta - \partial_\theta v^r) & 0 \end{bmatrix}. \tag{B18}$$

B2. Spherical coordinate system

Define the velocity gradient by the partial derivative with respect to the natural coordinates (r, θ, ϕ):

$$(\nabla \vec{v})_i^j = \begin{bmatrix} \partial_r v^r & \partial_r v^\theta & \partial_r v^\phi \\ \partial_\theta v^r & \partial_\theta v^\theta & \partial_\theta v^\phi \\ \partial_\phi v^r & \partial_\phi v^\theta & \partial_\phi v^\phi \end{bmatrix} = (\vec{\partial} \bigotimes \vec{v}). \tag{B19}$$

Thus, we have

$$v^m_{,i}(g_{ml} v^l) = (\nabla \vec{v})_i^j v_j. \tag{B20}$$

In the component manner, it reads

$$(\nabla \vec{v})_i^j v_j = \begin{cases} v^r \partial_r v^r + r^2 v^\theta \partial_r v^\theta + r^2 \sin^2\theta v^\phi \partial_r v^\phi \\ v^r \partial_\theta v^r + r^2 v^\theta \partial_\theta v^\theta + r^2 \sin^2\theta v^\phi \partial_\theta v^\phi \\ v^r \partial_\phi v^r + r^2 v^\theta \partial_\phi v^\theta + r^2 \sin^2\theta v^\phi \partial_\phi v^\phi \end{cases} \tag{B21}$$

The covariant components of the velocity are given by

$$v_i = g_{ij} v^j = \begin{cases} g_{rr} v^r = v^r \\ g_{\theta\theta} v^\theta = r^2 v^\theta \\ g_{\phi\phi} v^\phi = r^2 \sin^2\theta v^\phi \end{cases} \tag{B22}$$

The symmetric part of the velocity gradient is

$$2\bar{\bar{S}} = \begin{bmatrix} \partial_r v^r & (\partial_r v^\theta + \partial_\theta v^r) & (\partial_r v^\phi + \partial_\phi v^r) \\ (\partial_\theta v^r + \partial_r v^\theta) & \partial_\theta v^\theta & (\partial_\theta v^\phi + \partial_\phi v^\theta) \\ (\partial_\phi v^r + \partial_r v^\phi) & (\partial_\phi v^\theta + \partial_\theta v^\phi) & \partial_\phi v^\phi \end{bmatrix}. \tag{B23}$$

and the antisymmetric part is:

$$2\bar{\bar{\Omega}} = \begin{bmatrix} 0 & -(\partial_\theta v^r - \partial_r v^\theta) & (\partial_r v^\phi - \partial_\phi v^r) \\ (\partial_\theta v^r - \partial_r v^\theta) & 0 & -(\partial_\phi v^\theta - \partial_\theta v^\phi) \\ -(\partial_r v^\phi - \partial_\phi v^r) & (\partial_\phi v^\theta - \partial_\theta v^\phi) & 0 \end{bmatrix}. \tag{B24}$$

Appendix C. "Physical" Decomposition of the Metric Tensor

Generalized coordinates can be any parameters, such as angles. It is common in applications to express the "physical" velocity in terms of the metric tensor.

In Riemannian geometry (local Euclidean space), a metric tensor is a symmetric bilinear form that is positive definite. This means that for any nonzero tangent vector $\overrightarrow{v}$, we have $g(\overrightarrow{v}, \overrightarrow{v}) > 0$.

Because the matrix representation of a Riemannian metric tensor is symmetric and has all positive eigenvalues, one can define its square root using techniques such as eigenvalue decomposition:

$$g = Q\,\Lambda\,Q^T, \qquad \sqrt{g} = Q\,\Lambda^{1/2}\,Q^T. \tag{C1}$$

where Q forms an orthonormal basis of eigenvectors (spectral decomposition theorem).

Thus, the square root of the metric tensor can be distributed to both parts, for example:

$$g_{lm} v^l v^m{}_{,i} = (v^m \sqrt{g})(\sqrt{g} v^m{}_{,i}). \tag{C2}$$

Using this technique, we can redefine the tensor contractions in the sense of more "physical".

Actually, based on this split technique of the metric tensor, the contraction of any two nonzero tangent vectors can be written as:

$$g(\overrightarrow{v}_1, \overrightarrow{v}_2) = (\overrightarrow{v}_1 \sqrt{g})(\sqrt{g}\,\overrightarrow{v}_2). \tag{C3}$$

C1. Polar coordinate system

The square root of the metric tensor for the polar coordinate is

$$\sqrt{g} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}. \tag{C4}$$

Then, the velocity gradient and velocity can be re-defined to be

$$\sqrt{g} v^m_{,i} = \begin{bmatrix} \partial_r v^r & r \partial_r v^\theta \\ \partial_\theta v^r & r \partial_\theta v^\theta \end{bmatrix}, \tag{C5}$$

and

$$v^m \sqrt{g} = [\,v^r \quad r v^\theta\,]^T. \tag{C6}$$

So, the coordinate (contravariant) component in the θ direction is still

$$v^\theta = \frac{d\theta}{dt}, \tag{C7}$$

while the "physical" speed in that direction is

$$r v^\theta = r \frac{d\theta}{dt}. \tag{C8}$$

The SI unit of the velocity will be $[m/s]$.

C2. Spherical coordinate system

Similarly, the square root of the metric tensor for spherical coordinate is

$$\sqrt{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{bmatrix}. \tag{C9}$$

The velocity gradient is re-defined to be

$$\sqrt{g}v^m_{,i} = \begin{bmatrix} \partial_r v^r & r\partial_r v^\theta & r\sin\theta\partial_r v^\phi \\ \partial_\theta v^r & r\partial_\theta v^\theta & r\sin\theta\partial_\theta v^\phi \\ \partial_\phi v^r & r\partial_\phi v^\theta & r\sin\theta\partial_\phi v^\phi \end{bmatrix}, \tag{C10}$$

while the physical velocity is

$$v^m\sqrt{g} = \begin{bmatrix} v^r & rv^\theta & r\sin\theta v^\phi \end{bmatrix}^T. \tag{C11}$$

In this case, the physical speed in azimuthal direct, thus, is

$$r\sin\theta v^\phi = r\sin\theta\frac{d\phi}{dt}. \tag{C12}$$

The SI unit of the velocity is $[m/s]$.

Acknowledgments

The author appreciates the constructive suggestions from the Qeios Team to improve the quality of this script.

References

1. ^*Landau LD, Lifshitz EM (1987). Fluid Mechanics. 2nd ed. Pergamon Press. p.3. ISBN: 0-08-033933-6.*
2. ^*Wang S (2024). Hamiltonian, Lagrangian, Dynamics and Singularity of the Compressible Fluid Flow. Avail able from: https://doi.org/10.32388/UNV9CL*
3. ^*Landau LD, Lifshitz EM (1987). Fluid Mechanics. 2nd ed. Pergamon Press. p.46. ISBN: 0-08-033933-6.*

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.