Research Article

How Far from the Edge Does a Population Need to Be to Survive? A Probability Model

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Let N be a natural number. We consider a population that lives on $I_N = \{-N, -N+1, \ldots, N-1, N\}$. Each individual gives birth at rate λ on each of its neighboring sites and dies at rate 1. No births are allowed from the inside of I_N to the outside or vice versa. There is no limit on the number of individuals per site and therefore on the total population. The population on the whole line (i.e., $N = +\infty$) survives with positive probability if and only if $\lambda > 1/2$. On the other hand, for any $1/2 < \lambda \le \sqrt{2}/2$, there exists a natural number N_c such that the population survives on I_N for $N \ge N_c$ but dies out for $N < N_c$. There is no limit on the number of individuals per site, so the population could grow at the center where the birth rates are maximum. Our result shows that it does not if the edge is too close.

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1. The model

A branching random walk (BRW in short) on the one-dimensional lattice $\mathbb{Z}$ evolves as follows.

- Each individual dies at rate 1.
- Let x and y in $\mathbb{Z}$ be such that $|x-y| = 1$. An individual at x gives birth to an individual at y at rate λ.
- There is no limit on the number of individuals per site.

Let $N \ge 1$ be a natural number. We are interested in a branching random walk restricted to $I_N = \{-N, -N+1, \ldots, N-1, N\}$ with the following boundary conditions. No births are allowed from the inside of I_N to the outside or vice versa. In other words, if the population is to survive, it has to survive on its own on the finite set of sites I_N.

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We will show that if $\lambda > \sqrt{2}/2$, the population on I_N survives for all $N \geq 1$. On the other hand, if $\lambda \leq 1/2$, the population dies out for all N.

When $1/2 < \lambda \leq \sqrt{2}/2$, things get more interesting. Our main result is the following.

- Let $1/2 < \lambda \leq \sqrt{2}/2$. Then there exists a natural number N_c such that if $N < N_c$, the population restricted to I_N dies out with probability 1, while if $N \geq N_c$, the population has a positive probability of surviving (i.e., there is at least one individual alive in I_N at all times).

2. Discussion

Our result may have some relevance from a theoretical ecology point of view. It is well known that habitat fragmentation is one of the main causes of species extinction; see, for instance, [1]. Fragmentation results in less contiguous space but also in lower quality of the habitat. As a result, one expects lower birth rates and higher death rates. This in turn can cause the extinction of a population. In contrast to this scenario, our model shows that even with no change in birth and death rates, the population will die out for being too close to the edge (i.e., N is too small). Note that there is no limit on the number of individuals per site, so the population could grow at the center, where the birth rates are maximum. Our result shows that it does not if the edge is too close. Moreover, in this model, the boundary is minimal (two sites at $-N$ and N). This should be helpful to the population since boundary sites are the only ones with a lower birth rate. But even with a minimal boundary, our model suggests that fragmentation can be fatal.

3. The proof

3.1. The construction

We start by giving an informal construction of the process. We first construct the process on the whole line $\mathbb{Z}$. The same construction will be used to construct the process on I_N for all $N \geq 1$. At time 0, the initial configuration is assumed to have finitely many individuals on (finitely many) sites of $\mathbb{Z}$. Every individual is assigned two Poisson processes, each with rate λ. If the individual is at $x \in \mathbb{Z}$, then at each occurrence of the first Poisson process, a new particle is born at $x - 1$. Similarly, at each occurrence of the second Poisson process, a new particle is born at $x + 1$. Every individual is also assigned an exponential random variable with rate 1. At this exponential (random) time, the individual dies. To each new individual, we again assign an exponential rate 1 random variable and two rate λ Poisson processes, and so on. All exponential random variables and Poisson processes are independent.

Let $N \geq 1$. We use the same exponential random variables and Poisson processes defined above to construct the process on I_N. The only difference with the construction on the whole line is that we suppress births from inside to outside and from outside to inside the box I_N. This allows the construction of the process on I_N for all N on the same probability space.

3.2. Monotone properties

We now use this construction to show that the process (restricted or unrestricted) is increasing in λ. Assume that $\lambda_1 < \lambda_2$. We construct the process with birth rate λ_2 and death rate 1 as indicated above. From this construction, we get the process with rate λ_1 by filtering the births. That is, every time the λ_2 process has a birth, we flip a coin. With probability λ_1/λ_2, the birth happens for the λ_1 process. With probability $1 - \lambda_1/\lambda_2$, the birth does not happen for the λ_1 process. This is a well-known procedure to obtain a rate λ_1 Poisson process from a rate λ_2 Poisson process (see Schinazi (2025)[2], for instance). This coupling shows that at every fixed time and for every site of $\mathbb{Z}$, the λ_2 process has more individuals than the λ_1 process. This is what we mean by stating that the process is increasing in λ.

We now turn to a monotone property in N. Let $N_1 < N_2$ be two natural numbers. By the construction above, we can simultaneously construct the processes on $\mathbb{Z}$, restricted to I_{N_1} and restricted to I_{N_2}. To get the process restricted to I_{N_2} from the one on $\mathbb{Z}$, we suppress the births from inside I_{N_2} to outside I_{N_2} and vice versa. We do the same for the process restricted to I_{N_1}. Since $N_1 < N_2$, if a birth does not happen for the process restricted to I_{N_2}, it certainly does not happen for the process restricted to I_{N_1}. However, there are births that happen in I_{N_2} but not in I_{N_1}. Hence, at every fixed time and for every site, the process restricted to I_{N_2} has more individuals than the process restricted to I_{N_1}. In this sense, the process is increasing in N.

3.3. Tom Liggett's result

We first deal with the case $\lambda < 1/2$. Consider the process on the whole line $\mathbb{Z}$. Every individual has a birth rate of 2λ and a death rate of 1. Hence, the process on $\mathbb{Z}$ survives if and only if $\lambda > 1/2$. Note that for $\lambda \leq 1/2$, the process restricted to I_N will die out for all $N \geq 1$ since (by our construction) the restricted process has fewer individuals than the unrestricted one.

Let $A_t(N)$ be the total number of individuals alive at time t for the BRW restricted to I_N. Define the critical value by

$$\lambda_c(N) = \inf\{\lambda > 0 : P_\lambda(A_t(N) \geq 1\text{~for all t}) > 0\}.$$

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Since $P_\lambda(A_t(N) \geq 1 \text{~for all t})$ is an increasing function of λ, if $\lambda > \lambda_c(N)$, the population restricted to I_N survives forever with positive probability, while the population dies out if $\lambda < \lambda_c(N)$. Actually, the population also dies out when $\lambda = \lambda_c(N)$, as will be explained below.

Tom Liggett ([3]) considered branching random walks on finite homogeneous trees. In the infinite tree, each site has $d+1$ neighbors. The finite tree $T_{d,N}$ is obtained by retaining all those sites which can be reached from the origin (i.e., a fixed site on the tree) with a path of length less than or equal to N. In the particular case $d=1$, $T_{d,N}$ is exactly I_N.

The analysis in [3] is based on the representation of the branching random walk as a (non-spatial) multi-type branching process. Such a process survives if and only if a certain known matrix has a strictly positive eigenvalue. Using this fact, the critical value $\lambda_c(N)$ is shown to be the smallest positive root of a polynomial that can be computed recursively. This allows for the explicit computation of $\lambda_c(N)$ (at least for the first values of N). In particular, we get $\lambda_c(1) = \sqrt{2}/2$. Hence, the BRW restricted to I_N will survive for $\lambda > \sqrt{2}/2$ for all $N \geq 1$. Moreover, the largest eigenvalue of the matrix mentioned above is 0 for $\lambda = \lambda_c(N)$. This shows that the BRW restricted to I_N dies out for $\lambda = \lambda_c(N)$.

In [3], it is also proved that the critical value of the branching random walk on $T_{d,N}$ has the following limit,

$$\lim_{N\to\infty} N^2 \left(2\sqrt{d}\lambda_c(N) - 1\right) = \frac{\pi^2}{2}.$$

We are interested in the case $d=1$. Not surprisingly, $\lambda_c(N)$ converges to $1/2$ (i.e., the critical value of the branching random walk on the whole line $\mathbb{Z}$). We also note that $\lambda_c(N)$ is strictly larger than 1/2 for all N. See also [4] for an analogous (but less precise) result on $\mathbb{Z}^d$ for all $d \geq 1$.

Let $1/2 < \lambda \leq \sqrt{2}/2$. Define

$$N_c = \min\{N \in \mathbb{N} : \lambda > \lambda_c(N)\}.$$

Since $\lambda_c(N)$ converges to $1/2$, the set $\{N \in \mathbb{N} : \lambda > \lambda_c(N)\}$ is not empty. By the well-ordering principle, it has a minimum, N_c. Thus, the BRW restricted to I_N with birth rate λ survives for all $N \geq N_c$.

Since $\lambda \leq \sqrt{2}/2 = \lambda_c(1)$ we see that $N_c \geq 2$. Therefore, $N_c - 1$ is a natural number such that $\lambda \leq \lambda_c(N_c - 1)$. Hence, the brw restricted to I_N with birth rate λ dies out all $N \leq N_c - 1$. This completes the proof of our result.

References

1. [^]Haddad NM, Brudvig LA, Clobert J, Davies KF, Gonzalez A, Holt RD, Lovejoy TE, Sexton JO, Austin MP, Collins CD, Cook WM, Damschen EI, Ewers RM, Foster BL, Jenkins CN, King AJ, Laurance WF, Levey DJ, Margules CR, Melbourne BA, Nicholls AO, Orrock JL, Song DX, Townshend JR (2015). "Habitat Fragmentation and Its Lasting Impact on Earth's Ecosystems." Sci Adv. **1**:e1500052.
2. [^]Schinazi RB (2024). Classical and Spatial Stochastic Processes. 3rd ed. Birkhauser.
3. [a, b, c]Liggett TM (1999). "Branching Random Walks on Finite Trees." In: Bramson M, Durrett R, editors. Perplexing Problems in Probability: Papers in Honor of Harry Kesten. Boston: Birkhauser. p. 315–330.
4. [^]Mountford T, Schinazi RB (2005). "A Note on Branching Random Walks on Finite Sets." J Appl Probab. **42**:287–294.

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