Riemann Hypothesis on Grönwall's Function

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Abstract

Grönwall's function G is defined for all natural numbers $n > 1$ by $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ where $\sigma(n)$ is the sum of the divisors of n and log is the natural logarithm. We require the properties of extremely abundant numbers, that is to say left to right maxima of $n \mapsto G(n)$. We also use the colossally abundant and hyper abundant numbers. There are several statements equivalent to the famous Riemann hypothesis. We state that the Riemann hypothesis is true if and only if there exist infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. In addition, we prove that the Riemann hypothesis is true when there exist infinitely many hyper abundant numbers n with any parameter $u > 1$. We claim that there could be infinitely many hyper abundant numbers with any parameter $u > 1$ and thus, the Riemann hypothesis would be true.

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Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. In 1997, Ramanujan's old notes were published where it was defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [1]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \quad \text{for} \ \ (m > 1).$$

Every colossally abundant number is superabundant [2]. Let us call hyper abundant an integer n for which there exists $u > 0$ such that

$$\frac{\sigma(n)}{n \cdot (\log n)^{u}} \geq \frac{\sigma(m)}{m \cdot (\log m)^{u}} \quad \text{for} \ \ (m > 1),$$

where log is the natural logarithm. Every hyper abundant number is colossally abundant[[3], pp. 255]. In 1913, Grönwall studied the function $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ for all natural numbers $n > 1$ [4]. We have the Grönwall's Theorem:

Proposition 1.

$$\limsup_{n \to \infty} G(n) = e^{\gamma}$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [4].

Next, we have the following Robin's results:

Proposition 2. *Let $3 \leq N < N'$ be two consecutive colossally abundant numbers, then*

$$G(n) \leq \operatorname{Max}\left(G(N), G(N')\right)$$

when satisfying $N < n < N'$ [[5], Proposition 1 pp. 192].

Proposition 3. *There are infinitely many colossally abundant numbers N such that $G(N) > e^{\gamma}$ when the Riemann hypothesis is false [[5], Proposition 1 pp. 204]. There exist infinitely many colossally abundant numbers N such that $G(N) < e^{\gamma}$ [[5], Theorem 1 pp. 188], [[5], Proposition 1 pp. 204].*

Proposition 4. *Let $3 \leq N < N'$ be two consecutive colossally abundant numbers, then there exists some $\epsilon > 0$ such that [[5], Proposition 1 pp. 192]*

$$\frac{\sigma(N)}{N^{1+\epsilon}} = \frac{\sigma(N')}{N'^{1+\epsilon}}.$$

There are champion numbers (i.e. left to right maxima) of the function $n \mapsto G(n)$:

$$G(m) < G(n)$$

for all natural numbers $10080 \leq m < n$. A positive integer n is extremely abundant if either $n = 10080$, or $n > 10080$ is a champion number of the function $n \mapsto G(n)$. In 1859, Bernhard Riemann proposed his hypothesis[6]. Several analogues of the Riemann hypothesis have already been proved [6].

Proposition 5. *The Riemann hypothesis is true if and only if there exist infinitely many extremely abundant numbers*[[7], Theorem 7 pp. 6].

We use the following property for the extremely abundant numbers:

Proposition 6. *Let $N < N'$ be two consecutive colossally abundant numbers and $n > 10080$ is some extremely abundant number, then N' is also extremely abundant when satisfying $N < n < N'$*[[7], Lemma 21 pp. 12].

This is our main theorem

Theorem 1. *The Riemann hypothesis is true if and only if there exist infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$.*

The following is a key Corollary.

Corollary 1. *The Riemann hypothesis is true when there exist infinitely many hyper abundant numbers N' with any parameter $u > 1$.*

Putting all together yields a new criterion for the Riemann hypothesis. Now, we can conclude with the following result:

Theorem 2. *The Riemann hypothesis is true.*

Proof. Note also that, for all $u > 0$ [[3], pp. 254]:

$$\lim_{n \to \infty} \frac{\sigma(n)}{n \cdot (\log n)^u} = 0$$

and so, we claim that there could be infinitely many hyper abundant numbers with any parameter $u > 1$ and thus, the Riemann hypothesis would be true. □

2. Central Lemma

Lemma 1. *For two real numbers $y > x > e$:*

$$\frac{y}{x} > \frac{\log y}{\log x}.$$

Proof. We have $y = x + \varepsilon$ for $\varepsilon > 0$. We obtain that

$$
\begin{aligned}
\frac{\log y}{\log x} &= \frac{\log(x+\varepsilon)}{\log x} \\
&= \frac{\log\left(x\cdot(1+\frac{\varepsilon}{x})\right)}{\log x} \\
&= \frac{\log x + \log(1+\frac{\varepsilon}{x})}{\log x} \\
&= 1 + \frac{\log(1+\frac{\varepsilon}{x})}{\log x}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{y}{x} &= \frac{x+\varepsilon}{x} \\
&= 1 + \frac{\varepsilon}{x}.
\end{aligned}
$$

We need to show that

$$\left(1 + \frac{\log(1+\frac{\varepsilon}{x})}{\log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

which is equivalent to

$$\left(1 + \frac{\varepsilon}{x\cdot\log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

using the well-known inequality $\log(1+x) \leq x$ for $x > 0$. For $x > e$, we have

$$\frac{\varepsilon}{x} > \frac{\varepsilon}{x\cdot\log x}.$$

In conclusion, the inequality

$$\frac{y}{x} > \frac{\log y}{\log x}$$

holds on condition that $y > x > e$. □

3. Proof of Theorem 1

Proof. Suppose there are not infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. This implies that the inequality $G(N) \geq G(N')$ always holds for a sufficiently large N when $N < N'$ is a pair of consecutive colossally abundant numbers. That would mean the existence of a single colossally abundant number $N'' \geq 10080$ such that $G(n) \leq G(N'')$ for all natural numbers $n > N''$ according to Proposition 2. Certainly, the existence of such single colossally abundant number N'' is because of the Grönwall's function G would become decreasing on colossally abundant numbers starting from some single value. We use the Proposition 6 to reveal that under these preconditions, then there are not infinitely many extremely abundant numbers. This implies that the Riemann hypothesis is false as a consequence of Proposition 5. By contraposition, if the Riemann hypothesis is true, then there exist infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$.

Suppose that there exist infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. On the one hand, let's assume from these infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$, then there could be only a finite amount of these N' such that $e^{\gamma} < G(N')$. Thus, we deduce there could be only a finite amount of colossally abundant numbers N'' such that $e^{\gamma} < G(N'')$. However, when the Riemann hypothesis is false, then there are infinitely many colossally abundant numbers N'' such that $e^{\gamma} < G(N'')$ by Proposition 3. On the other hand, let's assume from these infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$, then there could be an infinite amount of these N' such that $e^{\gamma} < G(N')$.

Based on this opposite assumption, it could appear the possible scenarios:

- there would be an infinite increasing subsequence of colossally abundant numbers N_i such that $e^{\gamma} < G(N_i)$ and $G(N_i) < G(N_{i+1})$,
- or there would be a colossally abundant number N'' such that for all colossally abundant numbers $N > N''$ we have $e^{\gamma} \leq G(N)$,
- or there would be infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < e^{\gamma} < G(N')$.

However, it cannot exist an infinite increasing subsequence of colossally abundant numbers N_i such that $e^{\gamma} < G(N_i)$ and $G(N_i) < G(N_{i+1})$, by Proposition 1 and the properties of limit superior. Moreover, there cannot be a colossally abundant number N'' such that for all colossally abundant numbers $N > N''$ we have $e^{\gamma} \leq G(N)$, since this implies that there are not infinitely many colossally abundant numbers N''' such that $G(N''') < e^{\gamma}$ which is a contradiction by Proposition 3.

Furthermore, there are not infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < e^{\gamma} < G(N')$, because there exists some $\epsilon > 0$ such that $\frac{\sigma(N)}{N^{1+\epsilon}} = \frac{\sigma(N')}{N'^{1+\epsilon}}$ by Proposition 4. Certainly, we deduce that

$$\frac{G(N')}{N'^{\epsilon}} = \frac{\sigma(N')}{N'^{1+\epsilon} \cdot \log\log N'} = \frac{\sigma(N)}{N^{1+\epsilon} \cdot \log\log N'} < \frac{\sigma(N)}{N^{1+\epsilon} \cdot \log\log N} = \frac{G(N)}{N^{\epsilon}} < \frac{e^{\gamma}}{N^{\epsilon}}.$$

Finally, we obtain as contradiction that $G(N') < e^{\gamma} \leq e^{\gamma} \cdot \left(\frac{N'}{N}\right)^{\epsilon}$ under our assumption that $G(N) < e^{\gamma} < G(N')$ since

$\left(\frac{N'}{N}\right)^{\epsilon} = \frac{\left(\frac{\sigma(N')}{N'}\right)}{\left(\frac{\sigma(N)}{N}\right)} \geq 1$ holds due to every colossally abundant number is superabundant. Therefore, the Riemann hypothesis would be true when there exist infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. □

4. Proof of Corollary 1

Proof. Suppose there exists a large enough hyper abundant numbers N' with a parameter $u > 1$. We know that N' must be also a colossally abundant number. Let N be the greatest colossally abundant number such that $N < N'$, which means that N and N' is a pair of consecutive colossally abundant numbers. By definition of hyper abundant, we have

$$\frac{\sigma(N')}{N' \cdot (\log N')^{u}} \geq \frac{\sigma(N)}{N \cdot (\log N)^{u}}$$

which is the same as

$$\frac{\sigma(N') \cdot (\log N)^{u}}{N' \cdot (\log N')^{u} \cdot \log\log N} \geq \frac{\sigma(N)}{N \cdot \log\log N} = G(N).$$

Hence, it is enough to show that

$$G(N') = \frac{\sigma(N')}{N' \cdot \log\log N'} > \frac{\sigma(N') \cdot (\log N)^{u}}{N' \cdot (\log N')^{u} \cdot \log\log N}$$

which is equivalent to

$$\frac{(\log N')^{u}}{(\log N)^{u}} > \frac{\log\log N'}{\log\log N}.$$

Since $u > 1$, then we only need to show that the inequality

$$\frac{\log N'}{\log N} > \frac{\log\log N'}{\log\log N}.$$

holds on condition that $\log N' > \log N > e$ by Lemma 1. Consequently, this arbitrary large enough hyper abundant numbers N' with a parameter $u > 1$ reveals that $G(N) < G(N')$ holds on anyway. In this way, if there exist infinitely many hyper abundant numbers N' with any parameter $u > 1$, then there are infinitely many consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. Finally, the proof is complete by Theorem 1. □

5. Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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